A SYSTEM OF GENERALIZED VARIATIONAL INCLUSION PROBLEMS INVOLVING (A, η)-MONOTONE MAPPINGS

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Abstract

This paper deals with existence and uniqueness of the solution for a system of variational inclusions with (A, η)-monotone mappings.

1 Introduction

It is well known that the resolvent operators technique plays a crucial role in computing approximate solutions of generalized variational and quasi-variational inequalities, and generalized variational and quasi-variational inclusions, which come from variational problems, optimization and control theory, operations research, complementarity problems, mathematical programming and engineering sciences; for more details see [1–23] and references therein. Fang and Huang [5] introduced the concept of an H-monotone operator and the resolvent operator associated with an H-monotone operator as a generalization of maximal monotone operators and resolvent operator associated with maximal monotone operators. Using those concepts, they studied the existence and algorithm of solutions for general variational inclusions. Recently, as an extension of H-monotone operator, Fang and Huang introduced and studied a new class of monotone operators so-called (H, η)-monotone operators and then they studied a new system of variational inclusions involving (H, η)-monotone operators in Hilbert spaces. Further, in [7] by the resolvent operator method associated with (H, η)-monotone operators due to Fang and Huang, the existence and uniqueness of solutions for a new system of variational inclusions is proved and also a new algorithm for approximating the solution of this system of variational inclusions is constructed and the convergence of iterative sequence generated by this algorithm is discussed. Verma announced the notion of the A-monotone mapping and its applications to the solvability of nonlinear variational inclusions and systems of nonlinear variational inclusions [13, 14, 16] and [17].

   More recently, Verma and Zhang respectively in [18] and [23] individually introduced the notion of (A, η)-monotone operators and G-η-monotone mappings, which

they have the same definition, also, they generalized H-monotone, (H, η)-monotone and A-monotone operators. Using the generalized resolvent operators technique, Verma [15, 18] and Zhang [23] studied the solvability of a class of nonlinear variational inclusions involving (A, η)-monotone mappings.

At the present paper, after reviewing some basic definitions and results about monotone operators and specially (A, η)-monotone operators, existence and uniqueness of the solution for a system of variational inclusions with (A, η)-monotone mappings are proved.

2   Preliminaries

To ease understanding of the material incorporated in this paper we recall some basic definitions. For details on the following notions we refer to [1, 5, 15, 23] and references therein.

Let $\mathcal{H}$ be a real Hilbert space with the norm $\|.\|$ and inner product $\langle .,. \rangle$ and let $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ is a single valued mapping with $\eta(x, y) + \eta(y, x) = 0$. The mapping η is called γ-*Lipschitz continuous* mapping, if there exists some $\gamma > 0$ such that $\|\eta(x, y)\| \leq \gamma \|x - y\|$ for all $x, y \in \mathcal{H}$. For a set-valued map $T : \mathcal{H} \multimap \mathcal{H}$ the inverse T^{-1} of T is $\{(y, x) : (x, y) \in T\}$. For a real number c, let $cT = \{(x, cy) : (x, y) \in T\}$. If T_1 and T_2 are any set-valued mappings, we define $T_1 + T_2 = \{(x, y + z) : (x, y) \in T_1, (x, z) \in T_2\}$.

Definition 2.1. [5, 15] A single valued map $A : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be

(a) η-*monotone* if $\langle A(x) - A(y), \eta(x, y) \rangle \geq 0$ for all $x, y \in \mathcal{H}$.

(b) r-*strongly η-monotone* if, there exists some constant $r > 0$ such that $\langle A(x) - A(y), \eta(x, y) \rangle \geq r \|x - y\|^2$ for all $x, y \in \mathcal{H}$.

(c) δ-*Lipschitz* if $\|A(x) - A(y)\| \leq \delta \|x - y\|$ for all $x, y \in \mathcal{H}$.

Definition 2.2. [5, 15] A set-valued map $T : \mathcal{H} \multimap \mathcal{H}$ is said to be

(a) r-*strongly monotone* if, there exists some constant $r > 0$ such that $\langle u - v, x - y \rangle \geq r \|x - y\|^2$ for all $x, y \in \mathcal{H}$ and all $u \in T(x), v \in T(y)$.

(b) r-*strongly η-monotone* if, there exists some constant $r > 0$ such that $\langle u - v, \eta(x, y) \rangle \geq r \|x - y\|^2$ for all $x, y \in \mathcal{H}$ and all $u \in T(x), v \in T(y)$.

(c) m-*relaxed monotone* if, there exists some constant $m > 0$ such that $\langle u - v, x - y \rangle \geq -m \|x - y\|^2$ for all $x, y \in \mathcal{H}$ and all $u \in T(x), v \in T(y)$.

(d) m-*relaxed η-monotone* if, there exists some constant $m > 0$ such that $\langle u - v, \eta(x, y) \rangle \geq -m \|x - y\|^2$ for all $x, y \in \mathcal{H}$ and all $u \in T(x), v \in T(y)$.

Definition 2.3. [1, 23] Suppose $A, B : \mathcal{H} \longrightarrow \mathcal{H}$ are two single valued mappings. B is said to be *s-strongly monotone with respect to* A if $\langle B(u) - B(v), A(u) - A(v) \rangle \geq s \|x - y\|^2$ for all $x, y \in \mathcal{H}$.

Definition 2.4. [1, 23] Suppose X is a nonempty set.

(a) The map $f : \mathcal{H} \times X \longrightarrow \mathcal{H}$ is said to be *s-strongly η-monotone with respect to A in first argument* if $f(., x)$ is s-strongly monotone with respect to A for all $x \in X$.

(b) The map $g : X \times \mathcal{H} \longrightarrow \mathcal{H}$ is said to be *s-strongly η-monotone with respect to A in second argument* if $g(x, .)$ is s-strongly monotone with respect to A for all $x \in X$.

Definition 2.5. [1, 23] Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces.

(a) The map $f : \mathcal{H}_1 \times X \longrightarrow \mathcal{H}_2$ is *δ-Lipschitz continuous in first argument* if $f(., x)$ is δ-Lipschitz continuous for all $x \in X$.

(b) The map $g : X \times \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is *δ-Lipschitz continuous in second argument* if $g(x, .)$ is δ-Lipschitz continuous for all $x \in X$.

3 Main results

Definition 3.1. [15, 18, 23] A set-valued map $T : \mathcal{H} \multimap \mathcal{H}$ is said to be (A, η)-*monotone mapping*, if it is m-relaxed η-monotone and $(A + \lambda T)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$.

Example 3.2. [23] Let $X = \mathbb{R}$, $T(x) = 2x$, $A(x) = x^3$ and $\eta(x, y) = y - x$, for all $x, y \in \mathcal{H}$. Then T is (A, η)-monotone, but it is not (A, η)-accretive.

Example 3.3. [23] Let $X = \mathbb{R}$, for $c > 0$,

$$T(x) = \begin{cases} [-\frac{\sqrt[4]{c^3}}{2}, \frac{\sqrt[4]{c^3}}{2}] & x = 0 \\ \{\frac{1}{x}\} & x \neq 0, \end{cases}$$

$A(x) = x^3$ and $\eta(x, y) = xy(x - y)$ for all $x, y \in \mathcal{H}$. Then T is (A, η)-monotone, but it is not A-monotone and H-monotone.

Theorem 3.4. *[23] Suppose $A : \mathcal{H} \longrightarrow \mathcal{H}$ is an r-strongly η-monotone mapping and $T : \mathcal{H} \multimap \mathcal{H}$ is (A, η)-monotone. Then $(A + \lambda T)^{-1}$ is single-valued, where $0 < \lambda < \frac{r}{m}$.*

Definition 3.5. [23] Suppose $A : \mathcal{H} \longrightarrow \mathcal{H}$ is an r-strongly η-monotone and $T : \mathcal{H} \multimap \mathcal{H}$ is (A, η)-monotone. For any λ with $0 < \lambda < \frac{r}{m}$, the *generalized resolvent operator* $R_{T,\lambda}^{A,\eta} : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by $R_{T,\lambda}^{A,\eta}(u) = (A + \lambda T)^{-1}(u)$.

Remark 3.6. For appropriate and suitable choices of A, T and η one can obtain many known resolvent operators considered in recent literature.

Theorem 3.7. *[23] Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be an r-strongly η-monotone single-valued mapping, $\eta : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ be γ-Lipschitz continuous and $T : \mathcal{H} \multimap \mathcal{H}$ be an (A, η)-monotone mapping. Then the generalized resolvent operator $R_{T,\lambda}^{A,\eta}$ is $\frac{\gamma}{r - \lambda m}$-Lipschitz continuous for all $0 < \lambda < \frac{r}{m}$.*

In the rest of this paper $\mathcal{H}_1$ and $\mathcal{H}_2$ are real Hilbert spaces, $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ are single valued mappings with $\eta_i(x, y) + \eta_i(y, x) = 0$ for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$.

Problem 3.8. *Consider single valued mappings $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$, $A_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ and $S_i : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_i$ for $i = 1, 2$. Also, consider set-valued mappings $T_i : \mathcal{H}_i \multimap \mathcal{H}_i$. Suppose T_i is an (A_i, η_i)-monotone operator for $i = 1, 2$. Our problem is finding $(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$ for which*

$$\begin{cases} 0 \in S_1(u, v) + T_1(u) \\ 0 \in S_2(u, v) + T_2(v). \end{cases}$$

Remark 3.9. By a suitable choices of $A_1, A_2, S_1, S_2, \eta_1, \eta_2, T_1$ and T_2 one can obtain many known and new classes of variational inequalities and variational inclusions as special cases of the Problem 3.8.

Theorem 3.10. *Suppose $A_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is an r_i-strongly η_i-monotone operator and $T_i : \mathcal{H}_i \multimap \mathcal{H}_i$ is an (A_i, η_i)-monotone operator for each $i = 1, 2$. Define $F : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1$, $G : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ and $Q : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \times \mathcal{H}_2$ respectively by $F(u, v) = R^{A_1, \eta_1}_{T_1, \lambda}[A_1(u) - \lambda S_1(u, v)]$, $G(u, v) = R^{A_2, \eta_2}_{T_2, \mu}[A_2(v) - \mu S_2(u, v)]$ and $Q(u, v) = (F(u, v), G(u, v))$ for all $(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$. Then the following statements are equivalent.*

(a) *$(u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a solution of* Problem 3.8,
(b) *$u = R^{A_1, \eta_1}_{T_1, \lambda}[A_1(u) - \lambda S_1(u, v)]$ and $v = R^{A_2, \eta_2}_{T_2, \mu}[A_2(v) - \mu S_2(u, v)]$,*
(c) *(u, v) is a fixed point of Q.*

Proof. It is an immediate consequence of Definition 3.5.

Lemma 3.11. *Suppose X is a nonempty set and $A : \mathcal{H} \longrightarrow \mathcal{H}$ is θ-Lipschitz continuous operator. Also, suppose that $S : \mathcal{H} \times X \longrightarrow \mathcal{H}$ is s-strongly monotone with respect to A in first argument and δ-Lipschitz continuous in first argument. Then for any $\lambda > 0$ we have*

$$\|A(u) - A(v) - \lambda(S(u, x) - S(v, x))\| \le \sqrt{\theta^2 - 2\lambda s + \lambda^2 \delta^2} \|u - v\|.$$

Proof. Clearly,

$$\|A(u) - A(v) - \lambda(S(u, x) - S(v, x))\|^2 = \langle A(u) - A(v) - \lambda(S(u, x) - S(v, x)),$$
$$A(u) - A(v) - \lambda(S(u, x) - S(v, x)) \rangle$$
$$= \|A(u) - A(v)\|^2 - 2\lambda\langle S(u, x) - S(v, x), A(u) - A(v) \rangle$$
$$+\lambda^2 \|S(u, x) - S(v, x)\|^2.$$

Since A is θ-Lipschitz continuous, S is s-strongly monotone with respect to A in first argument and δ-Lipschitz continuous in first argument, so $\|A(u) - A(v) - \lambda(S(u, x) - S(v, x))\|^2 \le (\theta^2 - 2\lambda s + \lambda^2 \delta^2)\|u - v\|^2$. Therefore, $\|A(u) - A(v) - \lambda(S(u, x) - S(v, x))\| \le \sqrt{\theta^2 - 2\lambda s + \lambda^2 \delta^2}\|u - v\|$.

Lemma 3.12. *Suppose $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is γ_1-Lipschitz continuous, $A_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is an r_1-strongly η_1-monotone and θ_1-Lipschitz continuous operator and $T_1 : \mathcal{H}_1 \multimap \mathcal{H}_1$ is an (A_1, η_1)-monotone mapping. Also, suppose that $S_1 : \mathcal{H}_1 \times$*

$\mathcal{H}_2 \longrightarrow \mathcal{H}_1$ is s_1-strongly monotone with respect to A_1 in first argument, δ_1-Lipschitz continuous in first argument and ξ_1-Lipschitz continuous in second argument. Then F defined as in Theorem 3.10 satisfies in

$$\|F(u,v) - F(u',v')\| \leq \frac{\gamma_1}{r_1 - \lambda m}\sqrt{\theta_1^2 - 2\lambda s_1 + \lambda^2 \delta_1^2}\|u - u'\| + \frac{\lambda \gamma_1 \xi_1}{r_1 - \lambda m}\|v - v'\|.$$

Proof. According to Theorem 3.7 and Lemma 3.11,

$$\|F(u,v) - F(u',v')\| = \|R_{T,\lambda}^{A;\eta}(A(u) - \lambda S(u,v)) - R_{T,\lambda}^{A;\eta}(A(u') - \lambda S(u',v'))\|$$

$$\leq \frac{\gamma_1}{r_1 - \lambda m}\|A(u) - \lambda S(u,v) - (A(u') - \lambda S(u',v'))\|$$

$$\leq \frac{\gamma_1}{r_1 - \lambda m}(\|A(u) - A(u') - \lambda(S(u,v) - S(u',v))\| + \lambda\|S(u',v) - S(u',v')\|)$$

$$\leq \frac{\gamma_1}{r_1 - \lambda m}\sqrt{\theta_1^2 - 2\lambda s_1 + \lambda^2 \delta_1^2}\|u - u'\| + \frac{\lambda \gamma_1 \xi_1}{r_1 - \lambda m}\|v - v'\|.$$

Similar to Lemma 3.11 and Lemma 3.12, one can deduce the following lemmas.

Lemma 3.13. *Suppose X is a nonempty set and $A : \mathcal{H} \longrightarrow \mathcal{H}$ is θ-Lipschitz continuous operator. Also, suppose that $S : X \times \mathcal{H} \longrightarrow \mathcal{H}$ is s-strongly monotone with respect to A in second argument and δ-Lipschitz continuous in second argument. Then for any $\mu > 0$ we have*

$$\|A(u) - A(v) - \mu(S(x,u) - S(x,v))\| \leq \sqrt{\theta^2 - 2\mu s + \mu^2 \delta^2}\|u - v\|.$$

Lemma 3.14. *Suppose $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is a γ_2-Lipschitz continuous, $A_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is an r_2-strongly η_2-monotone and θ_2-Lipschitz continuous operator and $T_2 : \mathcal{H}_2 \multimap \mathcal{H}_2$ is an (A_2, η_2)-monotone mapping. Also, suppose that $S_2 : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is s_2-strongly monotone with respect to A_2 in second argument, δ_2-Lipschitz continuous in second argument and ξ_2-Lipschitz continuous in first argument. Then G defined as in Theorem 3.10 satisfies in*

$$\|G(u,v) - G(u',v')\| \leq \frac{\mu \gamma_2 \xi_2}{r_2 - \mu m}\|u - u'\| + \frac{\gamma_2}{r_2 - \mu m}\sqrt{\theta_2^2 - 2\mu s_2 + \mu^2 \delta_2^2}\|v - v'\|.$$

Theorem 3.15. *Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces. Also, suppose for $i = 1, 2$,*

    (a) *$\eta_i : \mathcal{H}_i \times \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is γ_i-Lipschitz continuous mapping,*

    (b) *$A_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ is an r_i-strongly η_i-monotone and θ_i-Lipschitz continuous operator,*

    (c) *$T_i : \mathcal{H}_i \multimap \mathcal{H}_i$ is (A_i, η_i)-monotone operator,*

    (d) *$S_i : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_i$ is δ_i-Lipschitz continuous in i'th argument and s_i-strongly monotone with respect to A_i in i'th argument,*

(e) S_1 *is* ξ_1-*Lipschitz continuous in second argument and* S_2 *is* ξ_2-*Lipschitz continuous in first argument. If*

$$
\begin{cases}
\frac{\gamma_1}{r_1-\lambda m}\sqrt{\theta_1^2 - 2\lambda s_1 + \lambda^2 \delta_1^2} + \frac{\mu\gamma_2\xi_2}{r_2-\mu m} < 1, & 0 < \lambda < \frac{r_1}{m} \\
\frac{\lambda\gamma_1\xi_1}{r_1-\lambda m} + \frac{\gamma_2}{r_2-\mu m}\sqrt{\theta_2^2 - 2\mu s_2 + \mu^2 \delta_2^2} < 1, & 0 < \mu < \frac{r_2}{m},
\end{cases}
$$

then Problem 3.8 *has a unique solution.*

Proof. Let F, G and Q be defined as in Theorem 3.10. Equipped $\mathcal{H}_1 \times \mathcal{H}_2$ with $\|(u,v)\|_\times = \|u\| + \|v\|$ for all $(u,v) \in \mathcal{H}_1 \times \mathcal{H}_2$. It is well known that $(\mathcal{H}_1 \times \mathcal{H}_2, \|(.,.)\|_\times)$ is a Banach space. Set

$$
\begin{aligned}
k \quad := \quad & \max\Big\{ \frac{\gamma_1}{r_1 - \lambda m}\sqrt{\theta_1^2 - 2\lambda s_1 + \lambda^2 \delta_1^2} + \frac{\mu\gamma_2\xi_2}{r_2 - \mu m}, \frac{\lambda\gamma_1\xi_1}{r_1 - \lambda m} \\
& + \frac{\gamma_2}{r_2 - \mu m}\sqrt{\theta_2^2 - 2\mu s_2 + \mu^2 \delta_2^2} \Big\}.
\end{aligned}
$$

On the other hand, $\|Q(u,v) - Q(u',v')\|_\times = \|F(u,v) - F(u',v')\| + \|G(u,v) - G(u',v')\|$. It follows from Lemma 3.12 and Lemma 3.14 that $\|Q(u,v) - Q(u',v')\|_\times \leq k\|(u,v) - (u',v')\|_\times$ and hence by assumption Q is a contraction map. Now, Banach Contraction Theorem implies that Q has a unique fixed point. That Problem 3.8 has a unique solution follows from Theorem 3.10.

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