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ON CONVERGENCE AND DIVERGENCE OF FOURIER EXPANSIONS ASSOCIATED TO JACOBI MEASURE WITH MASS POINTS

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Abstract

We prove the failure of a.e. convergence of the Fourier expansion in terms of the orthonormal polynomials with respect to the measure $(1-x)^\alpha(1+x)^\beta dx + M\delta_{-1} + N\delta_1$, where δ_t is the delta function at a point t and $M > 0,\ N > 0$. Lebesgue norms of Koornwinder's Jacobi-type polynomials are applied to obtain a new proof of necessary conditions for mean convergence.

1 Introduction

Let $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $(\alpha, \beta > -1)$, be the Jacobi weight on the interval $[-1, 1]$. In [6] T. H. Koornwinder introduced the polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^\infty$ which are orthogonal on the interval $[-1, 1]$ with respect to the measure

$$d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\omega_{\alpha,\beta}(x)dx + M\delta_{-1} + N\delta_1,$$

where $\alpha > -1$, $\beta > -1$, and $M, N \geq 0$. They are called Koornwinder's Jacobi-type polynomials. We denote the orthonormal Koornwinder's Jacobi-type polynomial by $p_n^{(\alpha,\beta,M,N)}$, which differs from $P_n^{(\alpha,\beta,M,N)}$ by normalization constant (see [14, p. 81]). For $M = N = 0$, denoted by $p_n^{(\alpha,\beta)}$, we have the classical Jacobi orthonormal polynomials (see [13, Chapter IV]). It is known that, unlike the Jacobi orthonormal polynomials, the polynomials $p_n^{(\alpha,\beta,M,N)}$ for $M > 0$, $N > 0$ decay at the rate of $n^{-\alpha-3/2}$ and $n^{-\beta-3/2}$ at the end points 1 and -1.

We shall say that $f(x) \in L^p(d\mu)$ if $f(x)$ is measurable on the $[-1, 1]$ and $\|f\|_{L^p(d\mu)} < \infty$, where

$$\|f\|_{L^p(d\mu)} = \begin{cases} \left(\int_{-1}^1 |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \underset{-1<x<1}{esssup}\ |f(x)| & \text{if } p = \infty. \end{cases}$$

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For $f \in L^1(d\mu)$, the Fourier expansions in Koornwinder's Jacobi-type polynomials is

$$\sum_{k=0}^{\infty} \hat{f}(k) p_k^{(\alpha,\beta,M,N)}(x) \tag{1.1}$$

where the Fourier coefficients are

$$\begin{aligned} \hat{f}(k) &= \int_{-1}^{1} f(x) p_k^{(\alpha,\beta,M,N)}(x) d\mu(x) \\ &= \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x) p_k^{(\alpha,\beta,M,N)}(x) \omega_{\alpha,\beta}(x) dx \\ &\qquad + M f(-1) p_k^{(\alpha,\beta,M)}(-1) + N f(1) p_k^{(\alpha,\beta,M,N)}(1). \end{aligned} \tag{1.2}$$

The Cesàro means of order ρ of the expansion (1.1) are defined by (see [15, p. 76-77], [9])

$$\sigma_n^\rho f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^\rho}{A_n^\rho} \hat{f}(k) p_k^{(\alpha,\beta,M,N)}(x),$$

where $A_k^\rho = \binom{k+\rho}{k}$.

In 1972 Pollard [11] raised the following question: Is there an $f \in L^{4/3}(dx)$ whose Fourier-Legendre expansion diverges almost everywhere? This problem was solved by Meaney [8]. Furthermore, he proved that this is a special case of divergence result for series of Jacobi polynomials.

This paper is a continuation of [1]. We will prove that, for $\alpha > -1/2$ and $p_0 = (4\alpha+4)/(2\alpha+3)$, there are functions $f \in L^{p_0}(d\mu)$ whose Fourier expansions in terms of the $\{p_n^{(\alpha,\beta,M,N)}\}_{n=0}^{\infty}$ are divergent almost everywhere on $[-1,1]$. Moreover we show that, for $1 < p < p_0$ and $0 < \rho < 2/p - 3/2$, there are functions $f \in L^p(d\mu)$ with almost everywhere divergent Cesàro means of order ρ. We also find the necessary conditions for the convergence in $L^p(d\mu)$ norm of Fourier expansion (1.1).

In order to obtain it, previously, we need some estimates for Koornwinder's Jacobi-type orthonormal polynomials. The representation of the $p_n^{(\alpha,\beta,M,N)}$ in terms of $p_n^{(\alpha,\beta)}$, a strong asymptotic on $(-1,1)$, a Mehler-Heine type formula, Lebesgue norms of $p_n^{(\alpha,\beta,M,N)}$ are derived.

2 Estimates for Koornwinder's Jacobi-type polynomials

The goal of this section is to obtain estimates and asymptotic properties on $[-1,1]$ for the orthonormal polynomials $p_n^{(\alpha,\beta,M,N)}$. Throughout this paper positive constants are denoted by $c,\ c_1, ...$ and they may vary at every occurrence. The notation $u_n \cong v_n$ means that the sequence u_n/v_n converges to 1 and notation $u_n \sim v_n$ means $c_1 u_n \le v_n \le c_2 u_n$ for sufficiently large n.

Proposition 2.1. *The representation of the* $p_n^{(\alpha,\beta,M,N)}$ *in terms of* $p_n^{(\alpha,\beta,M,0)}$ *is*

$$p_n^{(\alpha,\beta,M,N)}(x) = A_n p_n^{(\alpha,\beta,M,0)}(x) + B_n(x-1)p_{n-1}^{(\alpha+2,\beta,4M,0)}(x), \tag{2.1}$$

where

$$A_n \cong cn^{-2\alpha-2}, \qquad B_n \cong 1. \tag{2.2}$$

Proof. Let $\{P_n^1\}_{n=0}^{\infty}$ be the orthonormal polynomials with respect to the measure (see proof of the Proposition 6 in [4])

$$(x-1)^2[\omega_{\alpha,\beta}(x)dx + M\delta_{-1}] = \omega_{\alpha+2,\beta}(x)dx + 4M\delta_{-1}.$$

Therefore $P_n^1 = p_n^{(\alpha+2,\beta,4M,0)}$. From [4, Proposition 4] it follows

$$p_n^{(\alpha,\beta,M,N)}(x) = A_n p_n^{(\alpha,\beta,M,0)}(x) + B_n(x-1)p_{n-1}^{(\alpha+2,\beta,4M,0)}(x),$$

where

$$\lim_{n\to\infty} A_n L_{n-1}(1,1) = \frac{1}{\lambda(1)+N}$$

$$\lim_{n\to\infty} B_n = \frac{N}{\lambda(1)+N}$$

$$\lambda(1) = \lim_{n\to\infty} \frac{1}{L_n(1,1)}.$$

Since (see [1, (3)] and [13, (4.5.8)])

$$L_n(1,1) = \sum_{i=0}^{n} p_i^{(\alpha,\beta,M,0)}(1)p_i^{(\alpha,\beta,M,0)}(1) \cong cn^{2\alpha+2}$$

we get (2.2). □

Combining the above proposition with [1, (7)] we obtain:

Corollary 2.1. *The representation of the* $p_n^{(\alpha,\beta,M,N)}$ *in terms of* $p_n^{(\alpha,\beta)}$ *is*

$$p_n^{(\alpha,\beta,M,N)}(x) = a_n p_n^{(\alpha,\beta)}(x) + b_n(x+1)p_{n-1}^{(\alpha,\beta+2)}(x) \\ + c_n(x-1)p_{n-1}^{(\alpha+2,\beta)}(x) + d_n(x^2-1)p_{n-2}^{(\alpha+2,\beta+2)}(x)$$

where

$$a_n \cong cn^{-2\alpha-2\beta-4}, \quad b_n \cong cn^{-2\alpha-2}, \quad c_n \cong cn^{-2\beta-2}, \quad d_n \cong 1.$$

The following proposition establishes a strong asymptotic on $(-1,1)$ for $p_n^{(\alpha,\beta,M,N)}$.

Proposition 2.2. *For* $\theta \in [\epsilon, \pi-\epsilon]$ *and* $\epsilon > 0$

$$p_n^{(\alpha,\beta,M,N)}(x) = l_n^{\alpha,\beta,M,N}(1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \\ \times \cos(k\theta+\gamma) + O(n^{-1}),$$

where $x = \cos\theta$, $k = n+(\alpha+\beta+1)/2$, $\gamma = -(\alpha+1/2)\pi/2$ *and* $\lim_{n\to\infty} l_n^{\alpha,\beta,M,N} = \sqrt{2/\pi}$

Proof. From (2.1) and [1, Lemma 1]

$$p_n^{(\alpha,\beta,M,N)}(x) = [A_n s_n^{\alpha,\beta} + B_n s_{n-1}^{\alpha+2,\beta}](1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4} \times \cos(k\theta+\gamma) + [A_n + B_n(x-1)]O(n^{-1}),$$

$\lim_{n\to\infty} s_n^{\alpha,\beta} = \sqrt{2/\pi}$. Now taking into account (2.2), the result follows. □

Next we give a Mehler-Heine type formula of the polynomials $p_n^{(\alpha,\beta,M,N)}$.

Proposition 2.3. *Uniformly on compact subsets of* $\boldsymbol{C}$

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,N)}\left(\cos\frac{z}{n}\right) = -2^{\frac{\alpha-\beta}{2}} z^{-\alpha} J_{\alpha+2}(z),$$

where J_α *is the Bessel function of order* α.

Proof. The Mehler-Heine type formula for Jacobi orthonormal polynomials $p_n^{(\alpha,\beta)}\left(\cos\frac{z}{n+j}\right)$, $j \in \mathbf{N}\cup 0$, is (see [13, Theorem 8.1.1])

$$\lim_{n\to\infty} n^{-\alpha-1/2} p_n^{(\alpha,\beta)}\left(\cos\frac{z}{n+j}\right) = 2^{-\frac{\alpha+\beta}{2}}(z/2)^{-\alpha} J_\alpha(z),$$

uniformly on compact subsets of $\mathbf{C}$. Although the above formula in [13, Theorem 8.1.1] is for $j=0$, it can be shown that this formula is also true for any fixed $j \in \mathbf{N}$.

By Corollary 2.1 we have

$$\begin{aligned} n^{-\alpha-1/2} p_n^{(\alpha,\beta,M,N)}\left(\cos\frac{z}{n}\right) &= a_n n^{-\alpha-1/2} p_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) \\ &+ b_n\left(\cos\frac{z}{n}+1\right) n^{-\alpha-1/2} p_{n-1}^{(\alpha,\beta+2)}\left(\cos\frac{z}{n}\right) \\ &- 2c_n \sin^2\frac{z}{2n}\; n^{-\alpha-1/2} p_{n-1}^{(\alpha+2,\beta)}\left(\cos\frac{z}{n}\right) \\ &- d_n \sin^2\frac{z}{n}\; n^{-\alpha-1/2} p_{n-2}^{(\alpha+2,\beta+2)}\left(\cos\frac{z}{n}\right). \end{aligned}$$

Now, using the estimates for the coefficients a_n, b_n, c_n and d_n, the result follows. □

The proofs of main results are based on following proposition.

Proposition 2.4. *Let* $\alpha \geq -1/2$ *and* $M, N > 0$. *For* $1 \leq q < \infty$

$$\int_0^1 (1-x)^\alpha |p_n^{(\alpha,\beta,M,N)}(x)|^q dx \sim \begin{cases} c & \text{if } 2\alpha > q\alpha - 2 + q/2, \\ \log n & \text{if } 2\alpha = q\alpha - 2 + q/2, \\ n^{q\alpha+q/2-2\alpha-2} & \text{if } 2\alpha < q\alpha - 2 + q/2. \end{cases}$$

Proof. The upper estimates has been proved in [1, Theorem 1]. In order to prove the lower estimate, we follow the same line as in [13, Theorem 7.34] (see also [1, Theorem 2]), by using the Proposition 2.3 and [12, Lemma 2.1]. □

By using this proposition, [6, (2.5)] and [1, (3),(4)], we obtain:

Corollary 2.2. *Let* $\alpha \geq \beta \geq -1/2$ *and* $\alpha > -1/2$. *For* $q_0 = \frac{4\alpha+4}{2\alpha+1}$

$$||p_n^{(\alpha,\beta,M,N)}(x)||_{L^q(d\mu)} \sim \begin{cases} c & \text{if } 1 \leq q < q_0, \\ (\log n)^{\frac{1}{q}} & \text{if } q = q_0, \\ n^{\alpha+1/2-2(\alpha+1)/q} & \text{if } q_0 < q < \infty. \end{cases}$$

3 Divergence almost everywhere

Suppose that the expansion (1.1) converges on a subset E of positive measure in $[-1,1]$. Then

$$c_n(f)p_n^{(\alpha,\beta,M,N)}(x) \to 0, \qquad x \in E. \tag{3.1}$$

From Egorov's theorem it follows that there is a subset $E_1 \subset E$ of positive measure E such that (3.1) holds uniformly for $x \in E_1$. Therefore, from Proposition 2.2, we have

$$n^{-\delta}c_n(f)\left(\cos(k\theta+\gamma)+O(n^{-1})\right) \to 0$$

uniformly for $x = \cos\theta \in E_1$. By a variant of the Cantor-Lebesgue Theorem, cf. [9, Subsection 1.5], this implies

$$c_n(f) \to 0. \tag{3.2}$$

Now we are in position to prove our first main result

Theorem 3.1. *Let* $\alpha > -1/2$ *and* $\beta > -1$. *There is a function* f *in* $L^{p_0}(d\mu)$, *supported in* $[0,1]$, *such that its Fourier expansion (1.1) diverges for almost every* $x \in [-1,1]$.

Proof. For every function $f \in L^1(d\mu)$ the Fourier coefficients (1.2) can be written as

$$c_n(f) = c_n'(f) + Mf(-1)p_n^{(\alpha,\beta,M,N)}(-1) + Nf(1)p_n^{(\alpha,\beta,M,N)}(1), \tag{3.3}$$

where

$$c_n'(f) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{1} f(x)p_n^{(\alpha,\beta,M,N)}(x)\omega_{\alpha,\beta}(x)dx.$$

The uniform boundedness principle and Proposition 2.4 yields the existence of functions $f \in L^{p_0}(d\mu)$, supported on $[0,1]$, such that the linear functional $c_n'(f)$ satisfies

$$\frac{c_n'(f)}{(\log n)^{\frac{1}{2q_0}}} \to \infty, \qquad \text{as } n \to \infty.$$

Hence, from (3.3) and [1, (3),(4)], we obtain

$$\frac{c_n(f)}{(\log n)^{\frac{1}{2q_0}}} \to \infty, \qquad \text{as } n \to \infty,$$

which contradict (3.2). □

Now we show that, for some values of δ, there are functions with a.e. divergent Cesàro means.

Theorem 3.2. *Let given numbers α, β, p, and δ be such that $\alpha > -1/2$; $\beta > -1$;*

$$1 < p < \frac{4(\alpha+1)}{2\alpha+3};$$
$$0 \leq \delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}.$$

There is an $f \in L^p(d\mu)$, supported in $[0,1]$, whose Cesàro means $\sigma_N^\delta f(x)$ is divergent almost everywhere on $[-1,1]$.

Proof. From Egorov's theorem and [9, Lemma 1.1] (see also [15, Theorem 3.1.22]) it follows that if the series (1.1) is Cesàro summable of order δ on a set E of positive measure in $[-1,1]$ then there is a subset $E_1 \subset E$ of positive measure where

$$|n^{-\delta} c_n(f) p_n^{(\alpha,\beta,M,N)}(x)| \leq c$$

uniformly for $x \in E_1$. Hence, from Proposition 2.2, we have

$$|n^{-\delta} c_n(f) \left(\cos(k\theta+\gamma) + O(n^{-1})\right)| \leq c$$

uniformly for $\cos\theta \in E_1$. Using again the Cantor-Lebesgue Theorem we obtain

$$|\frac{c_n(f)}{n^\delta}| \leq c, \qquad \forall n \geq 1. \tag{3.4}$$

Suppose that

$$\delta < \frac{2\alpha+2}{p} - \frac{2\alpha+3}{2}.$$

For q conjugate of p

$$\delta < \alpha + \frac{1}{2} - \frac{2\alpha+2}{q}.$$

From the argument given in the [9, Subsection 1.4] and Proposition 2.4, for the linear functional $c_n'(f)$, it follows that there is an $f \in L^p(d\mu)$, supported on $[0,1]$, such that

$$\frac{c_n'(f)}{n^\delta} \to \infty, \qquad as\ n \to \infty.$$

So, from (3.3) and [1, (3),(4)], it follows that

$$\frac{c_n(f)}{n^\delta} \to \infty, \qquad as\ n \to \infty.$$

Combining the above results with (3.4) it follows that, for this f, the $\sigma_N^\delta f(x)$ diverges almost everywhere. □

Remark 3.1. *Using formulae in [2], which relates the Riesz and Cesàro means of order $\delta \geq 0$, we conclude that the Theorem 3.2 also holds for the Riesz means.*

4 Necessary conditions for the norm convergence

Let $S_n f$ be the n-th partial sum of the expansion (1.1)

$$S_n f(x) = \sum_{k=0}^{n} \hat{f}(k) p_k^{(\alpha,\beta,M,N)}(x)$$

If $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$, then (see [3], [5], and [7] in a more general framework)

$$||S_n f||_{L^p(d\mu)} \leq C||f||_{L^p(d\mu)} \qquad \forall n \geq 0, \ \forall f \in L^p(d\mu)$$

if and only if p belongs to the open interval (p_0, q_0).

Now we will give a new proof of the following theorem.

Theorem 4.1. *Let $\alpha \geq \beta \geq -1/2$ and $\alpha > -1/2$. If there exists a constant $c > 0$ such that*

$$\|S_n f\|_{L^p(d\mu)} \leq c\|f\|_{L^p(d\mu)} \tag{4.1}$$

for every $f \in S_p$ and $n \geq 0$, then $p \in (p_0, q_0)$

Proof. For the proof, we apply the same argument as in [10] (see also [12]). Assume that (4.1) holds true. Then

$$\|\hat{f}(n) p_n^{(\alpha,\beta,M,N)}(x)\|_{L^p(d\mu)} \leq 2c\|f\|_{L^p(d\mu)}.$$

Therefore

$$\|p_n^{(\alpha,\beta,M,N)}(x)\|_{L^p(d\mu)} \|p_n^{(\alpha,\beta,M,N)}(x)\|_{L^q(d\mu)} < \infty,$$

where p is the conjugate of q. By Corollary 2.2, it follows that the last inequality holds if and only if $p \in (p_0, q_0)$.

The proof of Theorem 4.1 is complete. □

Remark 4.1. *Using the symmetry properties [6, (2.5)], we get the same results as above with α replaced by β.*

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