Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.yu/filomat

Filomat **23:1** (2009), 69–83

VECTOR INEQUALITIES FOR POWERS OF SOME OPERATORS IN HILBERT SPACES

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Abstract

Vector inequalities for powers of some operators in Hilbert spaces with applications for operator norm, numerical radius, commutators and self-commutators are given.

1 Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [13, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [13, p. 8]:

$$w(T)=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [13, p. 9]:

$$w(T)\le\|T\|\le 2w(T), \tag{1.2}$$

for any $T\in B(H)$

For more results on numerical radii, see [14], Chapter 11.

For other results and historical comments on the above see [13, p. 39–41]. For recent inequalities involving the numerical radius, see [2]-[10], [15], [19]-[21] and [22].

The Schwarz inequality for positive operators asserts that if T is a positive operator in $B(H)$, then

$$|\langle Tx,y\rangle|^2\le\langle Tx,x\rangle\langle Ty,y\rangle\ \text{ for all } x,y\in H. \tag{1.3}$$

2000 *Mathematics Subject Classifications.* 47A12 (47A30, 47A63, 47B15.
Key words and Phrases. Hilbert space, Bounded linear operators, Vector inequalities, Operator norm and numerical radius, Commutator, Self-commutator.
Received: September 9, 2008
Communicated by Vladimir Rakočević

For an arbitrary operator T in $B(H)$ the following "mixed Schwarz" inequality has been established by Kato in [18] (see also [12] and [14, p. 265]):

$$|\langle Tx,y\rangle|^2 \le \langle (T^*T)^\alpha x,x\rangle \left\langle (TT^*)^{1-\alpha} y,y\right\rangle \text{ for all } x,y\in H \tag{1.4}$$

and for $\alpha\in[0,1]$.

An important consequence of Kato's inequality (1.4) is the famous Heinz inequality (see [1], [16], [17], [18]) which says that if T,A and B are operators in $B(H)$ such that A and B are positive and $\|Tx\|\le\|Ax\|$ and $\|T^*y\|\le\|By\|$ for all x,y in H then

$$|\langle Tx,y\rangle| \le \|A^\alpha x\| \left\|B^{1-\alpha}y\right\|$$

for all $x,y\in H$ and for $\alpha\in[0,1]$.

In this paper we establish some vector inequalities for powers of various operators in Hilbert spaces. Applications for norm and numerical radius inequalities are provided. Particular cases for commutators and self-commutators are also given.

2 Vector Inequalities for Two Operators

The first results concerning powers of two operators is incorporated in:

Theorem 1. *For any $A,B\in B(H)$ and $r\ge1$ we have the vector inequality:*

$$|\langle Ax,By\rangle|^r \le \frac{1}{2}\left[\langle (A^*A)^r x,x\rangle + \langle (B^*B)^r y,y\rangle\right], \tag{2.1}$$

where $x,y\in H$, $\|x\|=\|y\|=1$.

In particular, we have the norm inequality

$$\|B^*A\|^r \le \frac{1}{2}\left(\|(A^*A)^r\| + \|(B^*B)^r\|\right) \tag{2.2}$$

and the numerical radius inequality

$$w^r(B^*A) \le \frac{1}{2}\|(A^*A)^r + (B^*B)^r\|, \tag{2.3}$$

respectively.

The constant $\frac{1}{2}$ is best possible in all inequalities (2.1), (2.2) and (2.3).

Proof. By the Schwarz inequality in the Hilbert space $(H;\langle .,.\rangle)$ we have:

$$\begin{aligned}|\langle B^*Ax,y\rangle| &= |\langle Ax,By\rangle| \le \|Ax\|\cdot\|By\| \\ &= \langle A^*Ax,x\rangle^{1/2}\cdot\langle B^*By,y\rangle^{1/2}, \qquad x,y\in H.\end{aligned} \tag{2.4}$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t)=t^r$, $r\ge1$, we have successively,

$$\begin{aligned}\langle A^*Ax,x\rangle^{1/2}\cdot\langle B^*By,y\rangle^{1/2} &\le \frac{\langle A^*Ax,x\rangle + \langle B^*By,y\rangle}{2} \\ &\le \left(\frac{\langle A^*Ax,x\rangle^r + \langle B^*By,y\rangle^r}{2}\right)^{\frac{1}{r}}\end{aligned} \tag{2.5}$$

for any $x, y \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $z \in H$ with $\|z\| = 1$ we have the inequality (see for instance [20])

$$\langle Pz, z\rangle^r \leq \langle P^r z, z\rangle . \tag{2.6}$$

Applying this property to the positive operators A^*A and B^*B, we deduce that

$$\left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*By, y\rangle^r}{2}\right)^{\frac{1}{r}} \leq \left(\frac{\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r y, y\rangle}{2}\right)^{\frac{1}{r}} \tag{2.7}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, on making use of the inequalities (2.4), (2.5) and (2.7), we get the inequality:

$$|\langle (B^*A)\, x, y\rangle|^r \leq \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r y, y\rangle\right] \tag{2.8}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, which proves (2.1).

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.8) and since the operators $(A^*A)^r$ and $(B^*B)^r$ are self-adjoint, we deduce the desired inequality (2.2).

Now, if we take $y = x$ in (2.1), then we get

$$|\langle (B^*A)\, x, x\rangle|^r \leq \frac{1}{2}\left[\langle [(A^*A)^r + (B^*B)^r]\, x, x\rangle\right] \tag{2.9}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.9) we get (2.3).

The sharpness of the constant follows by taking $r = 1$ and $B = A$ in all inequalities (2.1), (2.2) and (2.3). The details are omitted. $\square$

Corollary 1. *For any $A \in B(H)$ and $r \geq 1$ we have the vector inequalities:*

$$|\langle Ax, y\rangle|^r \leq \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + 1\right], \tag{2.10}$$

and

$$\left|\langle A^2 x, y\rangle\right|^r \leq \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + \langle (AA^*)^r y, y\rangle\right], \tag{2.11}$$

where $x, y \in H$, $\|x\| = \|y\| = 1$.

In particular, we have the norm inequalities

$$\|A\|^r \leq \frac{1}{2}\left(\|(A^*A)^r\| + 1\right) \tag{2.12}$$

and

$$\left\|A^2\right\|^r \leq \frac{1}{2}\left(\|(A^*A)^r\| + \|(AA^*)^r\|\right), \tag{2.13}$$

respectively.

We also have the numerical radius inequalities

$$w^r(A) \le \frac{1}{2}\|(A^*A)^r + I\| \tag{2.14}$$

and

$$w^r\left(A^2\right) \le \frac{1}{2}\|(A^*A)^r + (AA^*)^r\|, \tag{2.15}$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A, B \in B(H)$, any $\alpha \in (0,1)$ and $r \ge 1$, we have the vector inequality:*

$$|\langle Ax, By\rangle|^{2r} \le \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + (1-\alpha)\left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y\right\rangle \tag{2.16}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have the norm inequality

$$\|B^*A\|^{2r} \le \alpha \left\|(A^*A)^{\frac{r}{\alpha}}\right\| + (1-\alpha)\left\|(B^*B)^{\frac{r}{1-\alpha}}\right\| \tag{2.17}$$

and the numerical radius inequality

$$w^{2r}(B^*A) \le \left\|\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\right\|, \tag{2.18}$$

respectively.

Proof. By Schwarz's inequality, we have:

$$\begin{aligned} |\langle (B^*A)x, y\rangle|^2 &\le \langle (A^*A)x, x\rangle \cdot \langle (B^*B)y, y\rangle \\ &= \left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x\right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} y, y\right\rangle, \end{aligned} \tag{2.19}$$

for any $x, y \in H$.

It is well known that (see for instance [20]) if P is a positive operator and $q \in (0,1]$ then for any $u \in H$, $\|u\| = 1$, we have

$$\langle P^q u, u\rangle \le \langle Pu, u\rangle^q. \tag{2.20}$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ $(\alpha \in (0,1))$, we have

$$\begin{aligned} &\left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x\right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} y, y\right\rangle \\ &\qquad \le \left\langle (A^*A)^{\frac{1}{\alpha}} x, x\right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\right\rangle^{1-\alpha}, \end{aligned} \tag{2.21}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha) b$, $\alpha \in (0,1)$, $a, b \geq 0$, we get

$$\left\langle (A^*A)^{\frac{1}{\alpha}} x, x\right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\right\rangle^{1-\alpha} \leq \alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\right\rangle \quad (2.22)$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

$$\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}}, \qquad \alpha \in (0,1), \ a, b \geq 0,$$

we deduce that

$$\begin{aligned} &\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\right\rangle \qquad (2.23) \\ &\leq \left[\alpha \left\langle (A^*A)^{\frac{1}{\alpha}} x, x\right\rangle^{r} + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}} y, y\right\rangle^{r}\right]^{\frac{1}{r}} \\ &\leq \left[\alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y\right\rangle\right]^{\frac{1}{r}}, \end{aligned}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, where, for the last inequality we used the inequality (2.6) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.19), (2.21), (2.22) and (2.23), we get

$$|\langle (B^*A) x, y\rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y\right\rangle \quad (2.24)$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, and the inequality (2.16) is proved.

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.24) produces the desired inequality (2.17).

The numerical radius inequality follows from (2.24) written for $y = x$. The details are omitted. □

The following particular instances are of interest:

Corollary 2. *For any $A \in B(H)$ and $\alpha \in (0,1)$, $r \geq 1$, we have the vector inequalities*

$$|\langle Ax, y\rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + 1 - \alpha, \quad (2.25)$$

$$\left|\left\langle A^2x, y\right\rangle\right|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + (1-\alpha) \left\langle (AA^*)^{\frac{r}{1-\alpha}} y, y\right\rangle \quad (2.26)$$

and

$$|\langle Ax, Ay\rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + (1-\alpha) \left\langle (A^*A)^{\frac{r}{1-\alpha}} y, y\right\rangle, \quad (2.27)$$

respectively, where $x, y \in H$, $\|x\| = \|y\| = 1$.

We have the norm inequalities

$$\|A\|^{2r} \le \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + 1 - \alpha \tag{2.28}$$

and

$$\left\|A^2\right\|^{2r} \le \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + (1-\alpha) \left\| (AA^*)^{\frac{r}{1-\alpha}} \right\|, \tag{2.29}$$

respectively.

We have the numerical radius inequalities

$$w^{2r}(A) \le \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) I \right\| \tag{2.30}$$

and

$$w^{2r}\left(A^2\right) \le \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(AA^*)^{\frac{r}{1-\alpha}} \right\|, \tag{2.31}$$

respectively.

Moreover, we have the norm inequality

$$\|A\|^{4r} \le \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(A^*A)^{\frac{r}{1-\alpha}} \right\|. \tag{2.32}$$

3 Vector Inequalities for the Sum of Two Products

The following result concerning four operators may be stated:

Theorem 3. *For any $A, B, C, D \in B(H)$ and $r, s \ge 1$ we have:*

$$\begin{aligned} &\left| \left\langle \left[\frac{B^*A + D^*C}{2} \right] x, y \right\rangle \right|^2 \\ &\quad \le \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$\left\| \frac{B^*A + D^*C}{2} \right\|^2 \le \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(B^*B)^s + (D^*D)^s}{2} \right\|^{\frac{1}{s}}. \tag{3.2}$$

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle ., . \rangle)$ we have:

$$\begin{aligned} &|\langle (B^*A + D^*C) x, y \rangle|^2 \\ &= |\langle B^*Ax, y \rangle + \langle D^*Cx, y \rangle|^2 \\ &\le [|\langle B^*Ax, y \rangle| + |\langle D^*Cx, y \rangle|]^2 \\ &\le \left[\langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*By, y \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y \rangle^{\frac{1}{2}} \right]^2, \end{aligned} \tag{3.3}$$

for any $x, y \in H$.

Now, on utilising the elementary inequality:

$$(ab + cd)^2 \le \left(a^2 + c^2\right)\left(b^2 + d^2\right), \qquad a, b, c, d \in \mathbb{R},$$

we then conclude that:

$$\begin{aligned}\langle A^*Ax, x\rangle^{\frac{1}{2}} \cdot \langle B^*By, y\rangle^{\frac{1}{2}} + \langle C^*Cx, x\rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y\rangle^{\frac{1}{2}} \\ \le (\langle A^*Ax, x\rangle + \langle C^*Cx, x\rangle) \cdot (\langle B^*By, y\rangle + \langle D^*Dy, y\rangle),\end{aligned} \tag{3.4}$$

for any $x, y \in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r, s \ge 1$ that

$$\begin{aligned}(\langle A^*Ax, x\rangle + \langle C^*Cx, x\rangle) \cdot (\langle B^*By, y\rangle + \langle D^*Dy, y\rangle) \\ \le 4 \cdot \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2}\right] y, y\right\rangle^{\frac{1}{s}}\end{aligned} \tag{3.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Consequently, by (3.3) – (3.5) we have:

$$\begin{aligned}\left|\left\langle \left[\frac{B^*A + D^*C}{2}\right] x, y\right\rangle\right|^2 \\ \le \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2}\right] y, y\right\rangle^{\frac{1}{s}}\end{aligned} \tag{3.6}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, which provides the desired result (3.1).

Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ in (3.6) we deduce the desired inequality (3.2). □

Remark 1. *If we make $y = x$ in (3.6) and take the supremum over $\|x\| = 1$, then we get the inequality*

$$w^2\left(\frac{B^*A + D^*C}{2}\right) \le \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(B^*B)^s + (D^*D)^s}{2}\right\|^{\frac{1}{s}},$$

which is not as good as (3.2) since we always have

$$w^2\left(\frac{B^*A + D^*C}{2}\right) \le \left\|\frac{B^*A + D^*C}{2}\right\|^2.$$

Remark 2. *If $s = r$, then the inequality (3.1) becomes :*

$$\begin{aligned}\left|\left\langle \left[\frac{B^*A + D^*C}{2}\right] x, y\right\rangle\right|^{2r} \\ \le \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2}\right] x, x\right\rangle \cdot \left\langle \left[\frac{(B^*B)^r + (D^*D)^r}{2}\right] y, y\right\rangle\end{aligned} \tag{3.7}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *while (3.2) is equivalent with*

$$\left\| \frac{B^*A + D^*C}{2} \right\|^{2r} \le \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(B^*B)^r + (D^*D)^r}{2} \right\|. \tag{3.8}$$

Corollary 3. *For any* $A, C \in B(H)$ *we have:*

$$\left| \left\langle \left(\frac{A + C}{2} \right) x, y \right\rangle \right|^{2r} \le \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \tag{3.9}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$*. In particular, we have the norm inequality*

$$\left\| \frac{A + C}{2} \right\|^{2r} \le \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|, \tag{3.10}$$

where $r \ge 1$*.*

The result is obvious by choosing $B = D = I$ in Theorem 3.

Corollary 4. *For any* $A, C \in B(H)$ *we have:*

$$\left| \left\langle \left(\frac{A^2 + C^2}{2} \right) x, y \right\rangle \right|^2 \le \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(AA^*)^s + (CC^*)^s}{2} \right] y, y \right\rangle^{\frac{1}{s}} \tag{3.11}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$*. Also, we have the norm inequality*

$$\left\| \frac{A^2 + C^2}{2} \right\|^2 \le \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^{\frac{1}{r}} \cdot \left\| \frac{(AA^*)^s + (CC^*)^s}{2} \right\|^{\frac{1}{s}} \tag{3.12}$$

for all $r, s \ge 1$*.*

If $s = r$*, then we have, in particular,*

$$\left| \left\langle \left(\frac{A^2 + C^2}{2} \right) x, y \right\rangle \right|^{2r} \le \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2} \right] x, x \right\rangle \cdot \left\langle \left[\frac{(AA^*)^r + (CC^*)^r}{2} \right] y, y \right\rangle \tag{3.13}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and the norm inequality*

$$\left\| \frac{A^2 + C^2}{2} \right\|^{2r} \le \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\| \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\| \tag{3.14}$$

for $r \ge 1$*.*

The result is obvious by choosing $B = A^*$ and $D = C^*$ in Theorem 3.

Another particular result of interest is the following one:

Corollary 5. *For any $A, B \in B(H)$ we have:*

$$\left|\left\langle \left[\frac{B^*A + A^*B}{2}\right] x, y\right\rangle\right|^2 \leq \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(A^*A)^s + (B^*B)^s}{2}\right] y, y\right\rangle^{\frac{1}{s}} \tag{3.15}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$\left\|\frac{B^*A + A^*B}{2}\right\|^2 \leq \left\|\frac{(A^*A)^r + (B^*B)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(A^*A)^s + (B^*B)^s}{2}\right\|^{\frac{1}{s}} \tag{3.16}$$

for any $r, s \geq 1$.

In particular we have

$$\left|\left\langle \left[\frac{B^*A + A^*B}{2}\right] x, y\right\rangle\right|^{2r} \leq \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2}\right] x, x\right\rangle \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2}\right] y, y\right\rangle \tag{3.17}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\left\|\frac{B^*A + A^*B}{2}\right\|^r \leq \left\|\frac{(A^*A)^r + (B^*B)^r}{2}\right\| \tag{3.18}$$

where $r \geq 1$.

The proof is obvious by choosing $D = A$ and $C = B$ in Theorem 3.

Another particular case that might be of interest is the following one.

Corollary 6. *For any $A, D \in B(H)$ we have:*

$$\left|\left\langle \left(\frac{A + D}{2}\right) x, y\right\rangle\right|^2 \leq \left\langle \left[\frac{(A^*A)^r + I}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(DD^*)^s + I}{2}\right] y, y\right\rangle^{\frac{1}{s}} \tag{3.19}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\left\|\frac{A + D}{2}\right\|^2 \leq \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(DD^*)^s + I}{2}\right\|^{\frac{1}{s}}, \tag{3.20}$$

where $r, s \geq 1$.

In particular we have

$$|\langle Ax, y\rangle|^2 \le \left\langle \left[\frac{(A^*A)^r + I}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(AA^*)^s + I}{2}\right] y, y\right\rangle^{\frac{1}{s}} \tag{3.21}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and the norm inequality*

$$\|A\|^2 \le \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(AA^*)^s + I}{2}\right\|^{\frac{1}{s}}. \tag{3.22}$$

Moreover, for any $r \ge 1$ *we have*

$$|\langle Ax, y\rangle|^{2r} \le \left\langle \left[\frac{(A^*A)^r + I}{2}\right] x, x\right\rangle \cdot \left\langle \left[\frac{(AA^*)^r + I}{2}\right] y, y\right\rangle \tag{3.23}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\|A\|^{2r} \le \left\|\frac{(A^*A)^r + I}{2}\right\| \cdot \left\|\frac{(AA^*)^r + I}{2}\right\|. \tag{3.24}$$

The proof of (3.19) is obvious by the Theorem 3 on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D. The details are omitted.

Remark 3. *If* $T \in B(H)$ *and* $T = A + iC$, *i.e.,* A *and* C *are its Cartesian decomposition, then we get from (3.9)*

$$|\langle Tx, y\rangle|^{2r} \le 2^{2r-1} \langle [(A^*A)^r + (C^*C)^r] x, x\rangle \tag{3.25}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$. *In particular, we have the norm inequality*

$$\|T\|^{2r} \le 2^{2r-1} \|(A^*A)^r + (C^*C)^r\|, \tag{3.26}$$

where $r \ge 1$.

Now, if we use the inequality (3.19) for T, A *and* B, *then we get:*

$$|\langle Tx, y\rangle|^2 \le 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(A^*A)^r + I] x, x\rangle^{\frac{1}{r}} \cdot \langle [(CC^*)^s + I] y, y\rangle^{\frac{1}{s}} \tag{3.27}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and the norm inequality*

$$\|T\|^2 \le 2^{2-\frac{1}{r}-\frac{1}{s}} \|(A^*A)^r + I\|^{\frac{1}{r}} \cdot \|(CC^*)^s + I\|^{\frac{1}{s}}, \tag{3.28}$$

where $r, s \ge 1$. *In particular, we have*

$$|\langle Tx, y\rangle|^{2r} \le 2^{2r-2} \langle [(A^*A)^r + I] x, x\rangle \cdot \langle [(CC^*)^r + I] y, y\rangle \tag{3.29}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and the norm inequality*

$$\|T\|^{2r} \le 2^{2r-2} \|(A^*A)^r + I\| \cdot \|(CC^*)^r + I\|, \tag{3.30}$$

for any $r \ge 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot,\cdot)$, where, as in [2],

$$w_e(T,U) := \sup_{\|x\|=1}\left(|\langle Tx,x\rangle|^2 + |\langle Ux,x\rangle|^2\right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. *For any $A,B,C,D \in B(H)$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have the vector inequality:*

$$\begin{aligned}&|\langle Ax,By\rangle|^2 + |\langle Cx,Dy\rangle|^2\\ &\qquad\le \langle[(A^*A)^p + (C^*C)^p]x,x\rangle^{1/p}\cdot\langle[(B^*B)^q + (D^*D)^q]y,y\rangle^{1/q}\end{aligned} \tag{3.31}$$

for each $x,y\in H$ with $\|x\|=\|y\|=1$.

In particular, we have the inequality for the Euclidean radius:

$$w_e^2(B^*A,D^*C) \le \|(A^*A)^p + (C^*C)^p\|^{1/p}\cdot\|(B^*B)^q + (D^*D)^q\|^{1/q}. \tag{3.32}$$

Proof. On utilising the elementary inequality

$$ac+bd \le (a^p+b^p)^{1/p}\cdot(c^q+d^q)^{1/q},\ a,b,c,d\ge 0 \text{ and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,$$

then for any $x,y\in H$, $\|x\|=\|y\|=1$ we have the inequalities:

$$\begin{aligned}&|\langle B^*Ax,y\rangle|^2 + |\langle D^*Cx,y\rangle|^2\\ &\le \langle A^*Ax,x\rangle\cdot\langle B^*By,y\rangle + \langle C^*Cx,x\rangle\cdot\langle D^*Dy,y\rangle\\ &\le (\langle A^*Ax,x\rangle^p + \langle C^*Cx,x\rangle^p)^{1/p}\cdot(\langle B^*By,y\rangle^q + \langle D^*Dy,y\rangle^q)^{1/q}\\ &\le (\langle (A^*A)^p x,x\rangle + \langle (C^*C)^p x,x\rangle)^{1/p}\cdot(\langle (B^*B)^q y,y\rangle + \langle (D^*D)^q y,y\rangle)^{1/q}\\ &= \langle[(A^*A)^p + (C^*C)^p]x,x\rangle^{1/p}\cdot\langle[(B^*B)^q + (D^*D)^q]y,y\rangle^{1/q}.\end{aligned}$$

For the second inequality, let us make the choice $y=x$ to get

$$\begin{aligned}&|\langle B^*Ax,x\rangle|^2 + |\langle D^*Cx,x\rangle|^2\\ \le\ &\langle[(A^*A)^p + (C^*C)^p]x,x\rangle^{1/p}\cdot\langle[(B^*B)^q + (D^*D)^q]x,x\rangle^{1/q},\end{aligned}$$

for any $x\in H$, $\|x\|=1$. Taking the supremum over $x\in H$, $\|x\|=1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.32). □

The following particular case is of interest.

Corollary 7. *For any $A,C\in B(H)$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have:*

$$|\langle Ax,y\rangle|^2 + |\langle Cx,y\rangle|^2 \le 2^{1/q}\langle[(A^*A)^p + (C^*C)^p]x,x\rangle^{1/p} \tag{3.33}$$

for each $x,y\in H$, with $\|x\|=\|y\|=1$. In particular,

$$w_e^2(A,C) \le 2^{1/q}\|(A^*A)^p + (C^*C)^p\|^{1/p}.$$

The proof follows from (3.31) and (3.32) for $B = D = I$.

Corollary 8. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$|\langle Ax, y\rangle|^2 + |\langle Dx, y\rangle|^2 \le \langle [(A^*A)^p + I]\, x, x\rangle^{1/p} \cdot \langle [(DD^*)^q + I]\, y, y\rangle^{1/q} \tag{3.34}$$

for each $x, y \in H$, with $\|x\| = \|y\| = 1$. In particular,

$$w_e^2(A, D) \le \|(A^*A)^p + I\|^{1/p} \cdot \|(DD^*)^q + I\|^{1/q}.$$

4 Inequalities for the Commutator

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$\|TU - UT\| \le 2\|T\|\|U\|. \tag{4.1}$$

In [11], the following result has been obtained as well

$$\|TU - UT\| \le 2\min\{\|T\|, \|U\|\}\min\{\|T - U\|, \|T + U\|\}. \tag{4.2}$$

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator:

Proposition 1. *For any $T, U \in B(H)$ and $r, s \ge 1$ we have the vector inequality*

$$|\langle (TU - UT)\, x, y\rangle|^2 \le 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(U^*U)^r + (T^*T)^r]\, x, x\rangle^{\frac{1}{r}} \cdot \langle [(UU^*)^s + (TT^*)^s]\, y, y\rangle^{\frac{1}{s}}, \tag{4.3}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$\|TU - UT\|^2 \le 2^{2-\frac{1}{r}-\frac{1}{s}} \|(U^*U)^r + (T^*T)^r\|^{\frac{1}{r}} \cdot \|(UU^*)^s + (TT^*)^s\|^{\frac{1}{s}}. \tag{4.4}$$

In particular, we have

$$|\langle (TU - UT)\, x, y\rangle|^{2r} \le 2^{2r-2} \langle [(U^*U)^r + (T^*T)^r]\, x, x\rangle \cdot \langle [(UU^*)^r + (TT^*)^r]\, y, y\rangle \tag{4.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\|TU - UT\|^{2r} \le 2^{2r-2} \|(U^*U)^r + (T^*T)^r\| \cdot \|(UU^*)^r + (TT^*)^r\|, \tag{4.6}$$

for any $r \ge 1$.

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. □

Now, for $U = T^*$ we can state the following corollary.

Corollary 9. *For any $T \in B(H)$ we have the vector inequality for the self commutator:*

$$\begin{aligned} &|\langle (TT^* - T^*T)\, x, y\rangle|^2 \\ &\qquad \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(TT^*)^r + (T^*T)^r]\, x, x\rangle^{\frac{1}{r}} \cdot \langle [(TT^*)^s + (T^*T)^s]\, y, y\rangle^{\frac{1}{s}} \end{aligned} \tag{4.7}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$\|TT^* - T^*T\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(TT^*)^r + (T^*T)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (T^*T)^s\|^{\frac{1}{s}}. \tag{4.8}$$

In particular we have

$$\begin{aligned} &|\langle (TT^* - T^*T)\, x, y\rangle|^{2r} \\ &\qquad \leq 2^{2r-2} \langle [(TT^*)^r + (T^*T)^r]\, x, x\rangle \cdot \langle [(TT^*)^r + (T^*T)^r]\, y, y\rangle \end{aligned} \tag{4.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\|TT^* - T^*T\|^r \leq 2^{r-1} \|(TT^*)^r + (T^*T)^r\|, \tag{4.10}$$

for any $r \geq 1$.

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