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A REPRESENTATION FORMULA FOR CURVES IN $\mathbb{C}^3$ WITH PRESET INFINITESIMAL ARC LENGTH

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Abstract

We consider an algebraic representation formula for meromorphic curves in $\mathbb{C}^3$ with preset infinitesimal arc length, i. e., a differential operator $\mathbf{M}$ assigning to triples (f, h, d) of meromorphic functions meromorphic curves $\Phi = (\varphi_1, \varphi_2, \varphi_3)^\top$ such that d is the infinitesimal arclength of Φ, in this way obtaining the complete solution of the differential equation $\varphi_1'^2 + \varphi_2'^2 + \varphi_3'^2 = d^2$ in terms of derivatives of f, h, d only and without integrations. Computer algebra systems are an excellent tool to handle formulas of this type. We give simple *Mathematica* code and apply it to work out some examples, graphics as well as algebraic expressions of complex curves with special properties. For the case $d = 0$ of null curves, we give some graphical examples of minimal surfaces constructed in this way, showing deformations and symmetries. We give an expression for the curvature κ of Φ in terms of the Schwarzian derivative of f and for the case $d = 1$ a simple differential relation for f and h equivalent to the condition $\kappa = 1$

1 Introduction

The **isotropic cone** $\mathcal{I} \subset \mathbb{C}^3$ consists of all vectors $z = (z_1, z_2, z_3)^\top \in \mathbb{C}^3$, $z \neq 0$, such that $z_1^2 + z_2^2 + z_3^2 = 0$. A **null curve** $\Phi(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))^\top$ in $\mathbb{C}^3$ is understood as a curve whose tangent at each point is a line on $\mathcal{I}$, i. e., ${\varphi_1'}^2 + {\varphi_2'}^2 + {\varphi_3'}^2 = 0$. We will always assume that Φ is **full**, i. e., that Φ', Φ'', Φ''' are linearly independent.

Consider the open dense subset

$$\mathcal{I}_0 = \left\{ z = (z_1, z_2, z_3)^\top \in \mathcal{I} \quad \middle| \quad z_1 - \mathbf{i}\, z_2 \neq 0 \right\} \quad \left(\mathbf{i} = \sqrt{-1} \right).$$

A parametrization of $\mathcal{I}_0$ is given by the bijective map

$$\mathbf{W} : (f, \omega) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\}) \longrightarrow \frac{\omega}{2} \begin{pmatrix} 1 - f^2 \\ \mathbf{i}\left(1 + f^2\right) \\ 2f \end{pmatrix}, \tag{1.1}$$

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whose inverse is

$$\mathbf{W}^{-1} : (z_1, z_2, z_3)^\top \in \mathcal{I}_0 \longrightarrow \left(\frac{z_3}{z_1 - \mathbf{i}\, z_2}, z_1 - \mathbf{i}\, z_2 \right). \tag{1.2}$$

Replacing f, ω by meromorphic functions $f(z), \omega(z)$, integration gives the Weierstraß representation formula

$$\mathbf{WEI}_{\omega,f}(z) = \int \mathbf{W}(f(z), \omega(z))\mathbf{d}z \tag{1.3}$$

of null curves in $\mathbb{C}^3$ and, up to translations any parametrized null curve Φ satisfying $\varphi_1' \neq \mathbf{i}\,\varphi_2'$ is represented in a unique way by (1.3). We call the corresponding functions f and ω the Weierstraß data of Φ.

An algebraic representation formula for null curves, i. e., a formula containing only derivatives of the input data, was obtained by N. Hitchin, in [3] and later on by M. Kokubu, M.Umehara, K. Yamada, K.,in [8]. The basic idea of their construction is recalled here in section 4. Considering deformations of null curves, we derived the same representation formula in [5] and [6]. and proved a number of its properties, such as bijectivity and a group equivariance.In section 2 we indicate, how this representation formula is decoded in an computer algebra system like *Mathematica* we show how this equivariance is related to symmetries of the corresponding minimal surfaces by graphical examples in section 3

2 Deformations of null curves

In this section we recall results of [6].

Definition 2.1 *The* **natural parameter** *of a null curve* $\Phi(z)$ *in* $\mathbb{C}^3$ *is defined by* $p'(z) = \sqrt[4]{\langle \Phi''(z), \Phi''(z) \rangle}$*. The* **curvature** κ_Φ^2 *of a null curve* $\Phi(z)$ *is defined by*

$$\kappa_\Phi^2 = \sqrt{\left\langle \frac{\mathbf{d}^3\Phi}{\mathbf{d}p^3}, \frac{\mathbf{d}^3\Phi}{\mathbf{d}p^3} \right\rangle} \tag{2.1}$$

The functions p and κ are a complete system of invariants of null curves. In terms of the Weierstraß data (ω, f) the natural parameter is given by $p'(z) = \sqrt{f'(z)\,\omega(z)}$. Putting $\omega(z) = 1/f'(z)$, then $\mathbf{WEI}_f^* = \mathbf{WEI}_{1/f',f}$ is a null curve in natural parametrization, whose curvature is given by Schwarzian derivative $\mathcal{S}(f)$ of f:

$$\kappa^2_{\mathbf{WEI}_f^*}(z) = \mathcal{S}(f)(z) = \frac{3\, f''(z)^2 - 2\, f'(z)\, f^{(3)}(z)}{f'(z)^2}. \tag{2.2}$$

The original curve Φ can be reconstructed from p and κ by solving a linear system similar to the classical Frenet equations in $\mathbb{R}^3$. (See for instance [6] for details.)

Theorem 2.1 *Assume that Φ is a* **full** *null curve in natural parametrization (i. e., $p'(z) = \text{const} = 1$), $h(z)$ an arbitrary meromorphic function and $\kappa_\Phi(z)$ the curvature of Φ. Then, the following linear combination of Φ', Φ'', Φ'''*

$$\Delta = \left(h\,\kappa_\Phi^2 + h''\right)\Phi' - h'\Phi'' + h\Phi^{(3)}. \tag{2.3}$$

is a new null curve as well as the sum $\Phi(z) + \Delta(z)$.

A proof of this as well as of some generalizations will be given in section 5 below.
Consider Δ for $h = \varepsilon\, h_0$, where h_0 is a fixed function and ε a complex parameter. Since Δ is linear in h, it approaches 0 as $\varepsilon \longrightarrow 0$ and $\Phi + \Delta$.approaches Φ. Therefore,we give the following definition:

Definition 2.2 *We call the operator* $\mathbf{VAR} : (\Phi, h) \longrightarrow \mathbf{VAR}_{\Phi,h} := \Delta$ *the* **variation** *and* $\mathbf{DEF}_{\Phi,h} = \Phi + \mathbf{VAR}_{\Phi,h}$ *the* **deformation** *of Φ by the function h*

If Φ is not given in natural parametrization, a similar but more involved formula for Δ is obtained by the help of the chain rule involving the derivatives of $p(z)$ up to order 3. Computation of algebraic expressions of these operators and invariants can be accomplished by means of computer algebra. The following *Mathematica* programs encode the Weierstraß -formulas and their derivatives as `wei` and `weip` respectively, the natural parameter as `natparprime`, the curvature κ as `curv`, and the variation $\mathbf{VAR}_{\Phi,h}$ of a null curve in arbitrary parametrization as `var`,

```
weip[om_,f_][z_]:=Simplify[om[z]{1-f[z]^2,I(1+f[z]^2),2f[z]}/2]
weip[f_][z_]:=weip[1/D[f[#],#]&,f][z]
wei[om_,f_][z_]:=Integrate[weip[om,f][zz],zz]/.zz->z
wei[f_][z_]:=Integrate[weip[f][zz],zz]/.zz->z

natparprime[phi_][z_]:=(-phi''[z].phi''[z]//Simplify)^(1/4)//PowerExpand;
    Sqrt[Sqrt[phi''[z].phi''[z] // Simplify]] // PowerExpand;

curv[phi_][z_]:=Module[{pp1=D[phi[zz],zz]//Simplify,pp3,
   cp1=natparprime[phi][zz],cp2,cp3,tt},pp3=D[pp1,zz,zz]//Simplify;
   cp2=D[cp1,zz]//Simplify;cp3=D[cp2,zz]//Simplify;
   tt=(pp3.pp3+9(cp2*cp1)^2-2*cp1^3*cp3)*cp1^-6;
   (tt/.zz->z)//Simplify//Sqrt]


variation[phi_][h_][z_]:=Module[{
   cpz=natparprime[phi][zz],phiz=D[phi[zz],zz]//Simplify,
   mc=curv[phi][zz],chainrulematrix,curveinp,curveinz,
   cpzz,cpzzz,phizz,phizzz,tt,test,phip,phipp,phippp,hp,hpp,hz,hzz},
   test=phiz.phiz//Simplify;
   If[test==0,
      (phizz=D[phiz,zz]//Simplify;phizzz=D[phizz,zz]//Simplify;
      cpzz=D[cpz,zz]//Simplify;cpzzz=D[cpzz,zz]//Simplify;
      chainrulematrix={{cpz,0,0},{cpzz,cpz^2,0},{cpzzz,
         3*cpz*cpzz,cpz^3}}//Simplify;
      curveinz={phiz,phizz,phizzz};
      curveinp=Inverse[chainrulematrix].curveinz//Simplify;
      phip=curveinp[[1]];phipp=curveinp[[2]];phippp=curveinp[[3]];
      hz=D[h[zz],zz]//Simplify;hzz=D[hz,zz]//Simplify;
      hp=hz/cpz//Simplify;hpp=-hz cpzz cpz^-3 +hzz cpz^-2//Simplify;
```

```
    tt=(-h[zz] mc^2+hpp) phip-hp phipp+h[zz] phippp;
    (tt/.zz->z)//Simplify),
       Print["Not a minimal curve."],
       Print["Something wrong."]]]

deformation[phi_][h_][z_]:=phi[z]+var[phi][h][z]
```

Expressing the curve Φ by its Weierstraß data f, ω, we obtain from $\mathbf{VAR}_{\Phi,h}$ an algebraic representation formulas free of integrations. For $\Phi(z) = \mathbf{WEI}_{\omega,f}$, the assignment $(\omega, f, h) \longrightarrow \mathbf{VAR}_{\Phi,h}$ is a differential operator in terms of ω, f, h, linear in h.

In the case $\Phi(z) = \mathbf{WEI}^*_f$ of the Weierstraß curve in natural parametrization we obtain a differential operator

$$(f, h) \longrightarrow \mathbf{VAR}_{f,h}(z) := \mathbf{VAR}_{\mathbf{WEI}_f,h}(z) \tag{2.4}$$

in terms of f, h, denoted by the same symbol, with the following explicit form:

$$\mathbf{VAR}_{f,h} = \frac{-1}{2f'^3}\begin{pmatrix} \mathbf{i}\{f'\left((h'f''+f'h'')f^2-2f'^2h'f-h'f''-f'h''\right)+h \\ \quad\left(2f'^4-2ff''f'^2+\left(f^2-1\right)f^{(3)}f'-\left(f^2-1\right)f''^2\right)\} \\ f'\left((h'f''+f'h'')f^2-2f'^2h'f+h'f''+f'h''\right)-h \\ \quad\left(2f'^4-2ff''f'^2+\left(f^2+1\right)f^{(3)}f'-\left(f^2+1\right)f''^2\right) \\ 2\mathbf{i}\left(h'f'^3-(fh''-hf'')f'^2-\right. \\ \quad\left.f\left(h'f''+hf^{(3)}\right)f'+fhf''^2\right) \end{pmatrix}. \tag{2.5}$$

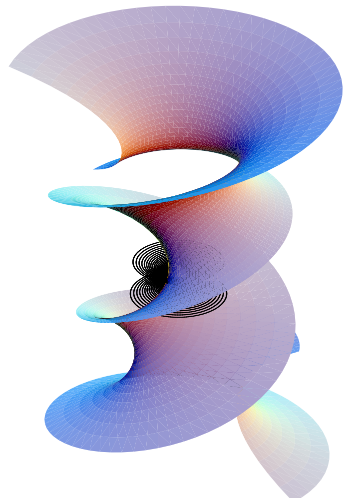
The infinitesimal natural parameter of $\mathbf{VAR}_{f,h}$ is a differential operator, linear in h, whose coefficients can be expressed in terms of the Schwarzian derivative $\mathcal{S}(f)$, as can be easily verified with the Mathematica terms of above:

$$\begin{cases} {p'_{f,h}}^2 & = \mathbf{i}\left(h''' + \dfrac{2f'f^{(3)}-3f''^2}{f'(z)^2}h' + \dfrac{3f''^3-4f'f''f^{(3)}+f'^2f^{(4)}}{f'^3}h\right) \\ & = \mathbf{i}\left(h''' - \mathcal{S}(f)\,h' - \dfrac{1}{2}\mathcal{S}(f)'\,h\right). \end{cases} \tag{2.6}$$

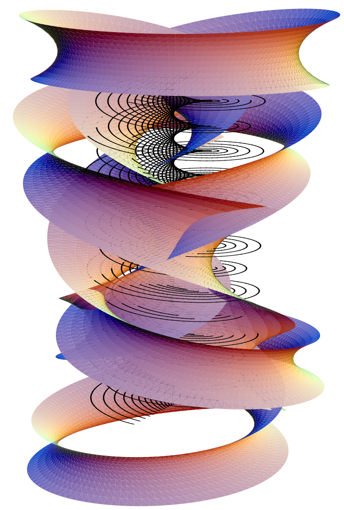
Similarly, the curvature of $\mathbf{VAR}_{f,h}$ is a linear differential operator of order 5 in h whose coefficients depend only on $\mathcal{S}(f)$.

We mention an unexpected coincidence of the second expression in (2.6) with an explicit formula for the coadjoint action of the Virasoro algebra on its regular dual space (see Exercise 1.6.2 in [9]).

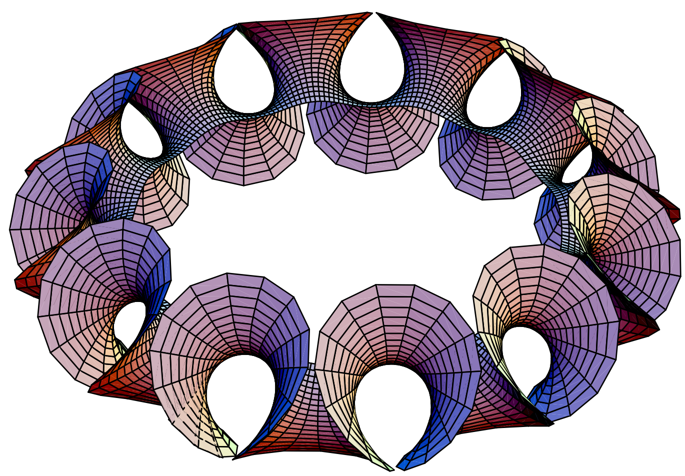
The real part $\mathbf{x}(\mathbf{u},\mathbf{v}) = \Re(\mathbf{\Phi}(\mathbf{u}+\mathbf{i}\,\mathbf{v}))$ of any null curve $\Phi(z)$ in $\mathbb{C}^3$ is a minimal surface in $\mathbb{R}^3$ in conformal parametrization. We give two examples of deformations of null curves and show plots of the corresponding minimal surfaces. At first, a deformation of a catenoid and its associated minimal surface, a helicoid, are visualized. The parametrized null curve of the catenoid is given by $\mathbf{VAR}_{f,h}$, where $f(z) = e^{7z}$ and the deforming function is chosen as $h(z) = \dfrac{1}{17}\left(e^{2z} + \sin(z/7)\right)$. The original catenoid and helicoid are indicated by circles and helical lines respectively.

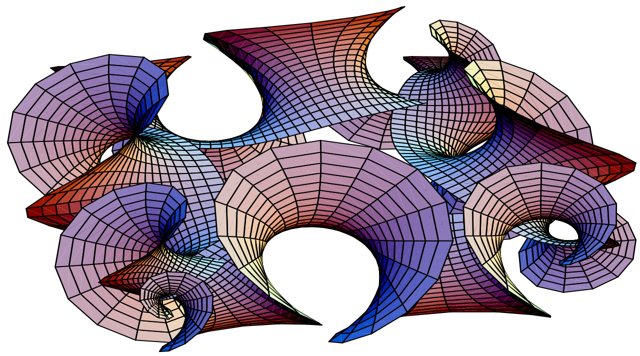


Deformation of a catenoid



Deformation of a helicoid





The other example shows the minimal surface corresponding to the null curve $\Phi_n(z)$ with Weierstraß data $\omega(z) = z^{2-1/n}$ and $f(z) = \frac{\mathbf{i}\, n\, z^{1/n-1}}{2\,n-1}$ and a deformation by the function $h(z) = a\, z^2$, $a \in \mathbb{C}$ for the values $n = 6$ and $a = 0.3$.

3 Symmetries of $(f, h) \longrightarrow \mathbf{VAR}_{f,h}$

We mention some nice properties of $\mathbf{VAR}_{f,h}$. The algebraic Weierstraß map (1.1) is equivariant with respect to the group homomorphism $\mu : \mathrm{Sl}(2, \mathbb{C}) \to \mathrm{SO}(3, \mathbb{C})$ sending a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{C})$ to the following matrix of $\mathrm{SO}(3, \mathbb{C})$:

$$\mu(M) = \frac{1}{2} \begin{pmatrix} a^2 - b^2 - c^2 + d^2 & \mathbf{i}\left(a^2 + b^2 - c^2 - d^2\right) & 2\,(c\,d - a\,b) \\ \mathbf{i}\left(-a^2 + b^2 - c^2 + d^2\right) & a^2 + b^2 + c^2 + d^2 & 2\,\mathbf{i}\,(a\,b + c\,d) \\ 2\,(b\,d - a\,c) & -2\,\mathbf{i}\,(a\,c + b\,d) & 2\,(b\,c + a\,d) \end{pmatrix}. \tag{3.1}$$

More precisely, this homomorphism is the one obtained pulling back the natural action of $\mathrm{SO}(3, \mathbb{C})$ on the isotropic cone $\mathcal{I}$ to the Weierstraß data f, ω. If M acts on a pair (ω, f) by linear fractional transformations $f \longrightarrow f_1 = (a\,f + b)/(c\,f + d)$ then

$$\mu(M)\,\mathbf{W}(\omega, f) = \mathbf{W}(\omega_1, f_1), \text{ where } \omega_1 = (c\,f + d)^2\,\omega, \quad f_1 = \frac{a\,f + b}{c\,f + d}. \tag{3.2}$$

The Weierstraß formula in natural parametrization has a similar property of equivariance: If $\Phi = \mathbf{WEI}^*_f$, and $\Phi_1 = \mathbf{WEI}^*_{f_1}$ then $\Phi_1 = \mu(M)\,\Phi$. This poperty is passed on to all derivatives of $\mathbf{WEI}^*_f$ and the curvature of $\mathbf{WEI}^*_f$ remains unchanged. Therefore, it is passed on to $\mathbf{VAR}_{f,h}$ too:

Theorem 3.1 *$(f,h) \to \mathbf{VAR}_{f,h}$ is equivariant with respect to the group homomorphism $\mu : Sl(2,\mathbb{C}) \to SO(3,\mathbb{C})$ given by (3.1), where now M acts on a pair (f,h) by linear fractional transformation of the first argument f, leaving the second unchanged.*

In addition to this, $(f,h) \longrightarrow \mathbf{VAR}_{f,h}$ is bijective and its inverse is given by elementary operations.

Theorem 3.2 *The range of $(f,h) \to \mathbf{VAR}_{f,h}$ is the set of all null curves $\Phi(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))^\top$ satisfying the condition $\varphi_1' - \mathbf{i}\,\varphi_2' \neq 0$ and a pair of functions (f,h) such that $\Phi = \mathbf{VAR}_{f,h}$ is given by*

$$f(z) = \frac{\varphi_3'(z)}{\varphi_1'(z) - \mathbf{i}\,\varphi_2'(z)}, \quad h(z) = \left\langle \mathbf{WEI}_f^*, \Phi(z) \right\rangle. \tag{3.3}$$

Moreover, If $\Phi_1 = \Phi + (a,b,c)^\top$ is a translate of Φ by a vector $(a,b,c)^\top \in \mathbb{C}^3$, then the corresponding data f_1, h_1 for Φ_1 are $f_1 = f$ and

$$h_1(z) = h(z) + \frac{-b + \mathbf{i}\,a + 2\,\mathbf{i}\,c\,f(z) + (-b - \mathbf{i}\,a)\,f^2(z)}{2\,f'(z)}. \tag{3.4}$$

Finally, there is the natural behavior of $\mathbf{VAR}_{f,h}$ with respect to changes of variables.

Theorem 3.3 *Under a change $z \to t(z)$ of the parameter, $\mathbf{VAR}_{f,h}$ transforms in the following way:*

$$\mathbf{VAR}_{f,h}(t(z)) = \mathbf{VAR}_{\widetilde{f},\widetilde{h},}(z), \text{ where } \widetilde{f}(z) = f(t(z)), \quad \widetilde{h}(z) = \frac{h(t(z))}{t'(z)}. \tag{3.5}$$

Therefore, if $\mathbf{f}$ and $\mathbf{h}$ are a meromorphic function and a meromorphic vector field on a Riemann surface then, defining $(\mathbf{f},\mathbf{h}) \to \mathbf{VAR}_{\mathbf{f},\mathbf{h}}$ as in (2.4) in terms of a local coordinate z, the result is independent of the choice of z.

For easy proofs of theorems 3.1, 3.2, 3.3 a computer algebra system such as *Mathematica* can be used. The following is a simple Mathematica-code for $\mathbf{VAR}_{f,h}$ based on the more involved expression `variation` of $\mathbf{VAR}_{\Phi,h}$ given above.

```
var[f_][h_][z_]=variation[wei[f][#] &][h][z]//Simplify
```

The equivariance of theorem 3.1 can be used to construct symmetric minimal surfaces. We end this section with graphical examples of suuch minimal surfaces.

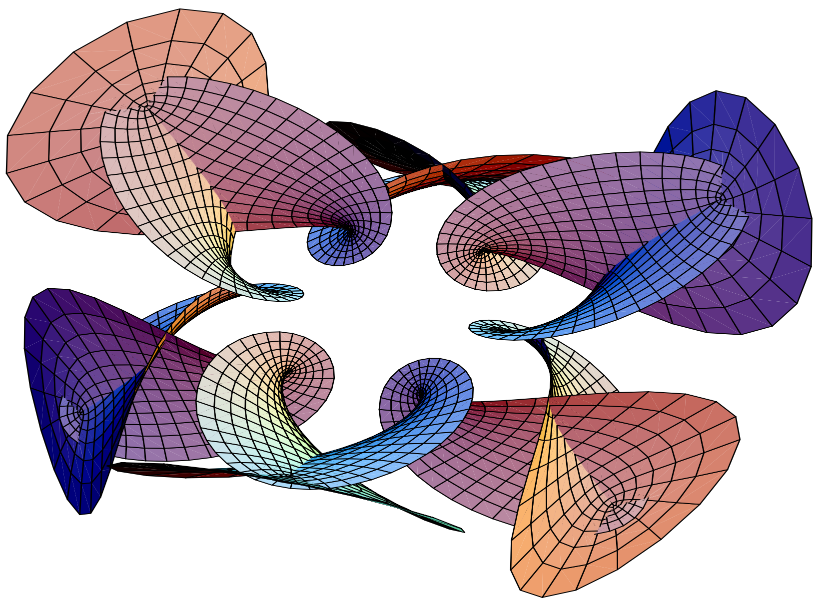
The subgroup $G_{\text{Quat}} \subset \text{Sl}(2,\mathbb{C})$, consisting of all matrices $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ with $z, w \in \mathbb{C}$, $|z|^2 + |w|^2 = 1$ is mapped under the group homomorphism μ of (3.1) onto the real orthogonal group SO(3)) $\subset$ SO$(3,\mathbb{C})$. For a real number $r \in \mathbb{R}$ define the following 1-parametric subgroups of SO(3):

$$\begin{cases} \mathbf{D}_{xy}(r) &= \text{rotation around the } z\text{-axis by the angle } r \\ \mathbf{D}_{xz}(r) &= \text{rotation around the } y\text{-axis by the angle } r \\ \mathbf{D}_{yz}(r) &= \text{rotation around the } x\text{-axis by the angle } r \end{cases}.$$

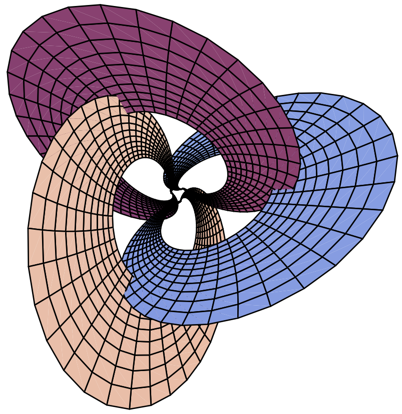
Under μ they correspond to the following 1-parametric subgroups: of $G_{\text{Quat}} \subset \text{Sl}(2, \mathbb{C})$:

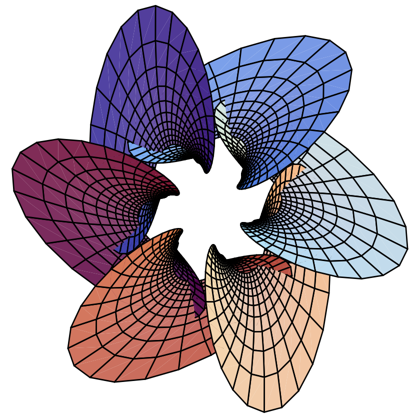
$$\begin{cases} \mathbf{M}_{xy}(r) = \begin{pmatrix} e^{-\mathbf{i}\, r/2} & 0 \\ 0 & e^{\mathbf{i}\, r/2} \end{pmatrix}, \quad \mathbf{M}_{xz}(r) = \begin{pmatrix} \cos\left(\frac{r}{2}\right) & -\sin\left(\frac{r}{2}\right) \\ \sin\left(\frac{r}{2}\right) & \cos\left(\frac{r}{2}\right) \end{pmatrix}, \\ \mathbf{M}_{yz}(r) = \begin{pmatrix} \cos\left(\frac{r}{2}\right) & -\mathbf{i}\sin\left(\frac{r}{2}\right) \\ -\mathbf{i}\sin\left(\frac{r}{2}\right) & \cos\left(\frac{r}{2}\right) \end{pmatrix} \end{cases} \tag{3.6}$$

Example 1. The function $f(z) = e^{z/3}\,\sinh(z)$ is translation invariant under $\mathbf{M}_{xy}\,(2\,\pi/3)$ and the function $h(z) = \sinh(z)$ is $2\,\mathbf{i}\,\pi$-periodic. This implies that the corresponding minimal surfaces is $2\,\pi/3$-periodic.



Example 2. For functions $f(z) = \dfrac{1+e^{2\,z}}{z^{2\,z/3}}$ and $f(z) = \dfrac{1+e^{2\,z}}{z^{z/3}}$ and a constant function $h(z) = -1+\mathbf{i}$ we obtain in a similar way the following surfaces:



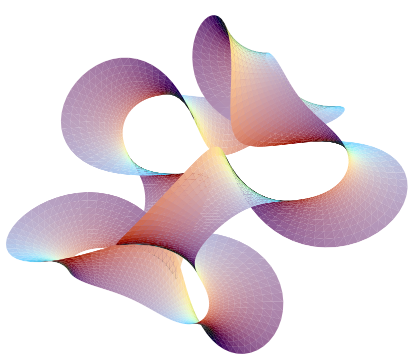


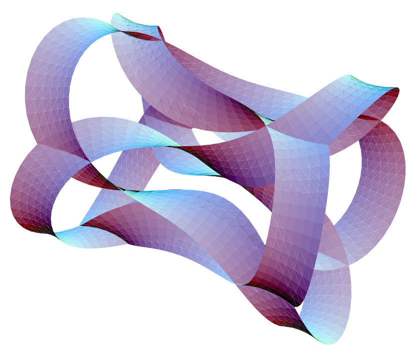
$f(z) = \dfrac{1+e^{2\,z}}{z^{2\,z/3}}$ $f(z) = \dfrac{1+e^{2\,z}}{z^{z/3}}$

Example 3. Examples of surfaces symmetric with respect to a rotation around the x-axis are obtained from the function $f(z) = \tanh(z)$. Under the action of $\mathbf{M}_{yz}(r) \in \mathrm{Sl}(2,\mathbb{C})$ $f(z)$ is transformed into

$$f_1(z) = \frac{\cos(r/2)\,\tanh(z) - i\,\sin(r/2)}{-i\,\sin(r/2)\,\tanh(z) + \cos(r/2)} = \tanh\left(z - \frac{\mathbf{i}\,r}{2}\right)$$

i. e., the action of $\mathbf{M}_{yz}(r)$ coincides with translation by $-\mathbf{i}\,(r/2)$. An $\mathbf{i}\,r/2$-periodic function is $h(z) = \exp(4\,z\,\pi/r)$. Therefore $\mathbf{VAR}_{f,h}$ is invariant under rotations around the x-axis with (real) angle r. Below we show parts of the surfaces obtained for $r = 5\,\pi/2$ and $r = 7\,\pi/2$.





$r = 5\,\pi/2$ $r = 7\,\pi/2$

4 Defining VAR by a natural operator

In [5] we gave an other construction of an integration free representation formula in $\mathbb{C}^3$. Let Σ be a Riemann surface and denote by $\mathcal{A}_\Sigma$ the set of non-constant meromorphic functions, by $\mathcal{X}_\Sigma$ the space of meromorphic vector fields, and $\mathcal{L}_\Sigma$ the space of meromorphic 1-forms. Moreover, let $(\mathcal{A}_\Sigma \times \mathcal{L}_\Sigma)^*_n \subset \mathcal{A}_\Sigma \times \mathcal{L}_\Sigma$ be the set of pairs (f, ω) such that the products $f^k \omega$ are exact for $k = 1, \ldots, n-1$. A sequence $\mathbf{M}_n : \mathcal{A}_\Sigma \times \mathcal{X}_\Sigma \to \mathcal{L}_\Sigma$ of nonlinear differential operators such that $(\mathbf{f}, \mathbf{M}_n(\mathbf{f}, \mathbf{h})) \in (\mathcal{A}_\Sigma \times \mathcal{L}_\Sigma)^*_n$ for all $(\mathbf{f}, \mathbf{h}) \in \mathcal{A}_\Sigma \times \mathcal{X}_\Sigma$, $n \geq 3$, is defined by the following recursive procedure:

$$\mathbf{M}_0(\mathbf{f}, \mathbf{h}) = \langle \mathbf{df}, \mathbf{h} \rangle \, \mathbf{df} \quad \text{and} \quad \mathbf{M}_n(\mathbf{f}, \mathbf{h}) = \mathbf{d}\left(\frac{\mathbf{M}_{n-1}(\mathbf{f}, \mathbf{h})}{\mathbf{df}} \right). \tag{4.1}$$

In the local setting, i. e., for meromorphic functions f, h on $\mathbb{C}$ explicit expressions for $\mathbf{M}_n(f, h)$ can be computed by hand. In the case $n = 3$, the result for $\mathbf{M}_3(f, h)$ differs only by the factor $f'(z)$ from the natural parameter $p'_{f,h}$ of $\mathbf{VAR}_{f,h}$ as given in (2.6).

Proposition 4.1 *The natural parameter $p'_{f,h}$ of $\mathbf{VAR}_{f,h}$ is related to $\mathbf{M}_3$ as follows: $p'_{f,h} = \mathbf{M}_3(f, h) f'(z)$.*

Meromorphic functions $\mathbf{g}_{k,n}$ such that $\mathbf{dg}_{k,n} = \mathbf{f}^k \, \mathbf{M}_n(\mathbf{f}, \mathbf{h})$ can be given explicitely, again in terms of $\mathbf{M}_i(\mathbf{f}, \mathbf{h}), i = 0, \ldots, n-1$. Namely,

$$\mathbf{g}_{k,n} = \sigma_{k,n}(\mathbf{f}, \mathbf{h}) = \sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{j!} \mathbf{f}^j \, \frac{\mathbf{M}_{j+n-k-1}(\mathbf{f}, \mathbf{h})}{\mathbf{df}}, \quad \mathbf{g}_{0,n} = \frac{\mathbf{M}_{n-1}(\mathbf{f}, \mathbf{h})}{\mathbf{df}}. \tag{4.2}$$

Now, if in the Weierstraß formula (1.3) the form ω is given as $\omega = \mathbf{M}_n(\mathbf{f}, \mathbf{h})$ the corresponding integrals are expressed explicitely by the operators $\sigma_{k,n}(\mathbf{f}, \mathbf{h})$ and in the case $n = 3$, up to a factor $\mathbf{i}$, the result agrees with (2.5):

Theorem 4.1 *For any meromorphic function $\mathbf{f} \in \mathcal{A}_\Sigma$ and any meromorphic vector field $\mathbf{h} \in \mathcal{X}_\Sigma$ holds*

$$\mathbf{i}\, \mathbf{WEI}_{\omega, \mathbf{f}}(z) = \mathbf{VAR}_{\mathbf{f}, \mathbf{h}}(z) \ \textit{where}\ \omega = \mathbf{M}_3(\mathbf{f}, \mathbf{h}).$$

Therefore, replacing in (1.3) $\omega, f\,\omega, f^2\,\omega$ by $\sigma_{0,3}(f, h), \sigma_{1,3}(f, h), \sigma_{2,3}(f, h)$ respectively, the recursion (4.2) leads to the following explicit form of $\mathbf{VAR}_{f,h}(z)$:

$$\mathbf{VAR}_{f,h} = \frac{\mathbf{i}}{f'} \left\{ \mathbf{M}_{0,3} \begin{pmatrix} 1 \\ \mathbf{i} \\ 0 \end{pmatrix} + \mathbf{M}_{1,3} \begin{pmatrix} f \\ -\mathbf{i}\, f \\ -1 \end{pmatrix} + \frac{\mathbf{M}_{2,3}}{2} \begin{pmatrix} 1 - f^2 \\ \mathbf{i}\left(1 + f^2\right) \\ 2\, f \end{pmatrix} \right\}. \tag{4.3}$$

The recursion procedure (4.1) and the resulting 'free Weierstraß formula' (4.3) occurred in a number of other papers, for instance [3], [8], [10], [11].

For $k = 1$ and $k = 2$ we have

$$\begin{cases} \sigma_{1,n}(\mathbf{f},\mathbf{h}) = f\,\dfrac{\mathbf{M}_{n-1}(\mathbf{f},\mathbf{h})}{\mathbf{df}} - \dfrac{\mathbf{M}_{n-2}(\mathbf{f},\mathbf{h})}{\mathbf{df}} \text{ and} \\ \sigma_{2,n}(\mathbf{f},\mathbf{h}) = \mathbf{f}^2\,\dfrac{\mathbf{M}_{n-1}(\mathbf{f},\mathbf{h})}{\mathbf{df}} - 2\,\mathbf{f}\,\dfrac{\mathbf{M}_{n-2}(\mathbf{f},\mathbf{h})}{\mathbf{df}} + 2\,\dfrac{\mathbf{M}_{n-3}(\mathbf{f},\mathbf{h})}{\mathbf{df}} \end{cases} \tag{4.4}$$

Explicit local expressions of $\sigma_{k,3}(f,h)$ for $k = 0, 1, 2$ are

$$\begin{cases} \sigma_{0,3}(f,h) &= \dfrac{1}{f'(z)^3}\Big(f'\,h'\,f'' - h\,f''^2 + f'^2\,h'' + h\,f'\,f^{(3)}\Big), \\ \sigma_{1,3}(f,h) &= \dfrac{1}{f'(z)^3}\Big(f\,f'^2\,h'' + f\,h\,f'\,f^{(3)} - f'^3\,h' \\ & \quad -h\,f'^2\,f'' + f\,f'\,h'\,f'' - f\,h\,f''^2\Big), \\ \sigma_{2,3}(f,h) &= \dfrac{1}{f'(z)^3}\Big(2\,h\,f'^4 - 2\,f\,f'^3\,h' - 2\,f\,h\,f'^2\,f'' \\ & \quad +f^2\,f'\,h'\,f'' - f^2\,h\,f''^2 + f^2\,f'^2\,h'' + f^2\,h\,f'\,f^{(3)}\Big). \end{cases} \tag{4.5}$$

5 A free representation formula for curves in $\mathbb{C}^3$ with preset arc length

Let $z \in \mathbb{C} \longrightarrow \Phi(z) \in \mathbb{C}^3$ be a full null curve, and let z be the *natural parameter* on Φ, i. e., $\langle \Phi''(z), \Phi''(z)\rangle = 1$. Furthermore let $\kappa(z)$ be the *minimal curvature* of Φ i. e., $\left\langle \Phi^{(3)}(z), \Phi^{(3)}(z)\right\rangle = \kappa^2(z)$

Successive differentiation of these equations leads to expressions for all scalar products $\left\langle \Phi^{(i)}, \Phi^{(j)}\right\rangle$. We display them here for $i, j \leq 4$ in the following table.

$$\begin{cases} \langle \Phi', \Phi'\rangle = 0 & \langle \Phi', \Phi''\rangle = 0 & \left\langle \Phi', \Phi^{(3)}\right\rangle = -1 & \left\langle \Phi', \Phi^{(4)}\right\rangle = 0 \\ & \langle \Phi'', \Phi''\rangle = 1 & \left\langle \Phi'', \Phi^{(3)}\right\rangle = 0 & \left\langle \Phi'', \Phi^{(4)}\right\rangle = -\kappa^2 \\ & & \left\langle \Phi^{(3)}, \Phi^{(3)}\right\rangle = \kappa^2 & \left\langle \Phi^{(3)}, \Phi^{(4)}\right\rangle = \kappa^3\,\kappa' \\ & & & \left\langle \Phi^{(4)}, \Phi^{(4)}\right\rangle = ? \end{cases} \tag{5.1}$$

¿From this table we infer that

$$\Phi^{(4)} = -\kappa^3\,\kappa'\,\Phi' - \kappa^2\,\Phi'', \tag{5.2}$$

namely, considering an ansatz for $\Phi^{(4)}$ as a linear combination $\Phi^{(4)} = a\,\Phi' + b\,\Phi'' + c\,\Phi^{(3)}$ and multiplying it with Φ', Φ'', $\Phi^{(3)}$, (5.1) gives $a = -\kappa^3\kappa'$, $b = -\kappa^2$ and $c = 0$ and we obtain (5.2). Moreover, equation (5.2) permits to compute $\left\langle \Phi^{(4)}, \Phi^{(4)}\right\rangle = \kappa^4$, completing in this way the array (5.1).

The 'natural' Weierstraß-representation formula is obtained by putting $\omega = 1/f'$ in (1.3):

$$\Phi(z) = \mathbf{WEI}^*_f(z) = \frac{1}{2}\int_{z_0}^{z} \frac{1}{f'} \begin{pmatrix} 1 - f^2 \\ \mathbf{i}(1 - f^2) \\ 2\,f \end{pmatrix} \mathbf{d}\zeta. \tag{5.3}$$

Let us introduce the vector

$$\mathbf{K}_f(z) = \begin{pmatrix} -i\, f(z) \\ -f(z) \\ i \end{pmatrix}. \tag{5.4}$$

Then we have

$$\big\langle \mathbf{K}_f(z), \mathbf{WEI}_{\omega,f}{}'(z)\big\rangle = 0, \quad \big\langle \mathbf{K}_f(z), \mathbf{WEI}^{*}_f{}'(z)\big\rangle = 0, \tag{5.5}$$

and

$$\begin{cases} \mathbf{WEI}''_{\omega,f}(z) & = & \dfrac{\omega'(z)}{\omega(z)}\mathbf{WEI}'_{\omega,f}(z) & - & \mathbf{i}\, f(z)\, g'(z)\, \mathbf{K}_g(z) \\ \mathbf{WEI}^{*}_f{}''(z) & = & -\dfrac{f''(z)}{f'(z)}\mathbf{WEI}^{*}_f{}'(z) & + & \mathbf{K}_f(z). \end{cases} \tag{5.6}$$

The osculating spaces $\mathcal{S}$ of the curves $\mathbf{WEI}_{\omega,f}(z)$ and $\mathbf{WEI}^{*}_f(z)$ are the linear subspaces of $\mathbb{C}^3$ spanned by the pairs of vectors $\big(\mathbf{WEI}_{\omega,f}{}'(z), \mathbf{WEI}_{\omega,f}{}''(z)\big)$ and $\big(\mathbf{WEI}^{*}_f{}'(z), \mathbf{WEI}^{*}_f{}''(z)\big)$ respectively. Therefore, the equations (5.6) show that the pairs $\big(\mathbf{K}_g(z), \mathbf{WEI}_{\omega,f}{}'(z)\big)$ and $\big(\mathbf{K}_f(z), \mathbf{WEI}^{*}_f{}'(z)\big)$ form orthogonal bases of these osculating spaces.

We are going now to construct a generalization $(f,h,d) \in \mathcal{A}\times\mathcal{A}\times\mathcal{A} \longrightarrow \Delta_{f,h,d}$ of the operator $\mathbf{VAR}_{f,h}$ representing curves of arbitrary arc length d in $\mathbb{C}^3$ and such that $\mathbf{VAR}_{f,h} = \Delta_{f,h,0}$. Let $\mathcal{C}$ be the set of all parametrized meromorphic curves $\Psi : \mathbb{C} \longrightarrow \mathcal{A}$. Decompose $\mathcal{C}$ into mutually disjoint classes $\mathcal{C}_d$ according to the infinitesimal complex arc length $d(z)$, i. e., for fixed $d \in \mathcal{A}$ denote by $\mathcal{C}_d$ the set of all curves $\Psi \in \mathcal{C}$ with $\langle \Psi', \Psi'\rangle = d^2(z)$. The class $\mathcal{C}_0$ is just the set of all meromorphic minimal curves.

Let $\Phi(z)$ be a minimal curve in natural parametrization, i. e., put $\Phi(z) = \mathbf{WEI}^{*}_f(z)$. Assuming that $\Phi(z)$ is nonplanar, the vectors $\Phi'(z)$, $\Phi''(z)$ and $\Phi'''(z)$ form a basis of $\mathbb{C}^3$ for each z. Therefore, any other meromorphic curve $\Delta : \mathbb{C} \longrightarrow \mathbb{C}^3$ can be represented as a linear combination

$$\Delta(z) = v_1(z)\, \Phi'(z) + v_2(z)\, \Phi''(z) + v_3(z)\, \Phi'''(z), \tag{5.7}$$

for certain meromorphic functions $v_1, v_2, v_3 \in \mathcal{A}$.

Denote $\Psi(z) = \Phi(z) + \Delta(z)$. We establish conditions to be imposed on v_1, v_2, v_3 that in order that $\Psi(z)$ and $\Delta(z)$ have the same infinitesimal complex arc length, ı.e, adding $\Phi(z)$ to $\Delta(z)$ preserves the class of $\Delta(z)$.

We find that $\Psi'(z)$ and $\Delta'(z)$ have the same infinitesimal length if and only if

$$v_2(z) = -v_3'(z). \tag{5.8}$$

Indeed, the equations (5.1) give

$$\left\{ \begin{array}{lll} |\Psi'|^2 - |\Delta'|^2 & = & |\Phi'|^2 + 2\,\langle \Phi', \Delta' \rangle + |\Delta'|^2 - |\Delta'|^2 = 2\,\langle \Phi', \Delta' \rangle \\ & = & 2\,\left\langle \Phi', v_1 \Phi'' + v_2 \Phi''' + v_3 \Phi^{(4)} + v_1' \Phi' + v_2' \Phi'' + v_3' \Phi''' \right\rangle \\ & = & -2\, v_2 - 2\, v_3'. \end{array} \right. \tag{5.9}$$

Putting $v_2(z) = -v_3'(z)$, we get $\Delta = v_1\,\Phi' - v_3'\,\Phi'' + v_3\,\Phi'''$ and from (5.2) we infer that the derivative of Δ is the following linear combination of Φ' and Φ'' only:

$$\left\{ \begin{array}{ll} \Delta' & = v_1'\,\Phi' + v_1\,\Phi'' - v_3''\,\Phi'' + v_3\,\left(\kappa_\Phi \kappa_\Phi' \Phi' + \kappa_\Phi^2 \Phi'' \right) \\ & = \left(v_1' + v_3\,\kappa_\Phi\,\kappa_\Phi' \right) \Phi' + \left(v_1 - v_3'' + v_3\,\kappa_\Phi^2 \right) \Phi''. \end{array} \right. \tag{5.10}$$

The conditions $\langle \Phi', \Phi' \rangle = \langle \Phi', \Phi'' \rangle = 0$ and $\langle \Phi'', \Phi'' \rangle = 1$ imply

$$\langle \Delta', \Delta' \rangle = \left(v_1 - v_3'' + v_3\,\kappa_\Phi^2 \right)^2. \tag{5.11}$$

Consequently, both Δ and Ψ belong to $\mathcal{C}_d$ if and only if $v_2 + v_3' = 0$ and $\left(v_1 - v_3'' + v_3\,\kappa_\Phi^2 \right)^2 = d^2(z)$.

Theorem 5.1 *The mapping*

$$\Delta : (\Phi, h, d) \in \mathcal{C}_0 \times \mathcal{A} \times \mathcal{A} \longrightarrow \Delta_{\Phi,h,d} = \left(h'' - h\,\kappa_\Phi^2 + d \right) \Phi' - h'\,\Phi'' + h\,\Phi''' \tag{5.12}$$

is a surjective mapping of $\mathcal{C}_0 \times \mathcal{A} \times \mathcal{A}$ *onto* $\mathcal{D}$ *mapping* $\mathcal{C}_0 \times \mathcal{A} \times \{d\}$ *onto* $\mathcal{D}_d$*. A left inverse operator to* Δ *is given as follows: Given* $\Delta = \Delta_{\Phi,h,d}$ *determine at first* $d(z)$ *with*

$$d^2(z) = \langle \Delta', \Delta' \rangle \tag{5.13}$$

next compute Φ *by putting*

$$\Delta'(z) = \begin{pmatrix} \delta_1(z) \\ \delta_2(z) \\ \delta_3(z) \end{pmatrix}, \quad f(z) = \frac{\delta_3(z) + d(z)}{\delta_1(z) - i\,\delta_2(z)} \text{ and } \Phi(z) = \mathbf{WEI}_f^*(z), \tag{5.14}$$

and finally, h is the scalar product

$$h(z) = \langle \Phi'(z), \Delta(z) \rangle. \tag{5.15}$$

Proof:

By the construction it is clear that $\Delta_{\Phi,h,d} \in \mathcal{C}_d$ and we have only to show that the mapping $\Delta \longrightarrow (\Phi, h, d)$ defined by (5.13), (5.14) and (5.15) is a right inverse to Δ. Now, equation (5.13) is a consequence of the construction of Δ. Equation (5.15) follows direclty from (5.12) and (5.1).

In order to prove (5.14) we differentiate (5.12) and obtain

$$\Delta_{\Phi,h,d}' == \left(q^2 + \mathbf{i}\,d' \right) \Phi' + \mathbf{i}\,d\,\Phi'' \text{ where } q^2 = h''' - h'\,\kappa_\Phi - h\,\kappa_\Phi\,\kappa_\Phi'. \tag{5.16}$$

If $\Phi = \mathbf{WEI}^*_f$, one infers from (5.4) and (5.6) that

$$\Delta'_{\Phi,h,d}(z) = r(z)\,\Phi'(z) + \mathbf{i}\,d(z)\,\mathbf{K}_f(z) = r(z)\,\Phi'(z) + d(z)\begin{pmatrix} f(z) \\ -\mathbf{i}\,f(z) \\ -1 \end{pmatrix} \tag{5.17}$$

where $r(z) = q^2(z) + \mathbf{i}\,d(z)\left(1 - \dfrac{f''(z)}{f'(z)}\right)$, but the special form of this function will have no meaning for our considerations. Namely, looking at the vector components of equation (5.17) one observes that $\delta_1 - \mathbf{i}\,\delta_2 = r\ (\varphi_1 - \mathbf{i}\,\varphi_2)$ and $\delta_3 = r\,\varphi_3 - d$, where we have put $\Phi'(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))$. Therefore, by the inversion formula (1.2) of the Weierstraß representation

$$\frac{\delta_3 + d}{\delta_1 - \mathbf{i}\,\delta_2} = \frac{\varphi_3}{\varphi_1 - \mathbf{i}\,\varphi_2} = f(z).\ \blacksquare \tag{5.18}$$

Note that we have:

Proposition 5.1 $\Delta_{\Phi,h,d}$ *is an affine map in* h *with associated linear map* $\Delta_{\Phi,h,0}$.

The proof follows immediately from (5.12). Propositions (5.1)and (5.1) give a full description of the inverse image of a meromorphic curve under Δ, for as one can show, in the case $\Phi = \mathbf{WEI}^*_f$ the operator $\Delta_{\Phi,h,0}$ has kernel

$$\mathbf{Ker}_{\mathbf{VAR}f,\cdot} = \left\{\frac{a + b\,f(z) + c\,f^2(z)}{f'(z)};\ a, b, c \in \mathbb{C}\right\} \tag{5.19}$$

6 A representation formula for curves of curvature 1 in $\mathbb{C}^3$

In this section we consider for curves $\alpha(z)$ in $\mathbb{C}^3$ the ordinary curvature

$$\kappa = \sqrt{\frac{\alpha' \times \alpha''}{\langle \alpha', \alpha' \rangle^3}}. \tag{6.1}$$

At first we give the Mathematica code for the operator (5.12) and its inverses (5.13) and (5.14).

```
delta[f_,h_,d_][z_]:=var[f][h][z]+d[z]weip[f][z]//Simplify;
deltainvers1[phi_][z_]:=Module[{u=-D[phi[z],z]//Simplify,dd},
   dd=PowerExpand[Sqrt[Simplify[u.u]]];(u[[3]]+dd)/(u.{1,-I,0})//Simplify];
deltainvers2[phi_][z_]:=Module[{ff=deltainvers1[phi][z]},
   -phi[z].{I(1-ff^2),-1-ff^2,2*I*ff}/(2*D[ff,z])//Simplify]
deltainvers3[phi_][z_]:=Module[{u=D[phi[z],z]//Simplify},
   PowerExpand[Sqrt[Simplify[u.u]]]]
deltainvers[phi_][z_]:=Module[{ff=deltainvers1[phi][z],
   u=D[phi[z],z]//Simplify},
   {ff,phi[z].{I(1-ff^2),-1-ff^2,2*I*ff}/(2*D[ff,z])//Simplify,
   PowerExpand[Sqrt[Simplify[u.u]]]}]
```

Considering for instance the circle $\alpha(z) = (\cos z, \sin z, 0)$, these programs give the data $f(z) = \mathbf{i}\exp(\mathbf{i}\,z), h(z) = \mathbf{i}, d(z) = 1$. An arbitrary curve $\alpha(z)$ of infinitesimal arc length 1 is represented as follows:

```
alpha[z_] = delta[f, h, 1 &][z] // Simplify
```

The response is an expression differing only by the summand $d\,\mathbf{WEI}^*_f$ from the expression displayed $\mathbf{VAR}_{f,h}$ displayed above.

Translating now the formula (6.1) in Mathematica code, namely

```
kappa[alpha_][t_]:=Module[{ap=D[alpha[t],t],app=D[alpha[t],t,t],normal},
   normal=Simplify[Factor[Cross[ap,app]]];
   Sqrt[Simplify[normal.normal/(ap.ap)^3]]]
curvature[f_,h_][z_]=kappa[alpha][z];
```

the above term returns an involved expression for the curvature κ_α of α. However, it is seen, that it depends only on two invariants, namely the Schwarzian derivative $\mathcal{S}(f)$ of f and the product $\mathbf{M}_3(f,h)\,f'$,

$$\left\{\begin{array}{rcl} \kappa_\alpha^2 & = & 2\,\mathbf{i}\,(f'\,\mathbf{M}_3'(f,h) + \mathbf{M}_3(f,h)\,f'') - \mathcal{S}(f)^2 - \mathbf{M}_3(f,h)^2\,f'^2 \\ & = & 2\,\mathbf{i}\,\dfrac{\mathbf{d}\,(f'\,\mathbf{M}_3(f,h))}{\mathbf{d}z} - \left(f'\,\mathbf{M}_3(f,h)\right)^2 - \mathcal{S}(f)^2, \end{array}\right. \tag{6.2}$$

Therefore we obtain:

Proposition 6.1 *A generic meromorphic curve $\alpha(z)$ with infinitesimal arc length 1 and preset curvature κ is given by $\alpha(z) = \Delta_{f,h,1}(z)$, where the functions f and h are subject to the condition*

$$\kappa = 2\,\mathbf{i}\,\frac{\mathbf{d}\mathbf{N}(f,h)}{\mathbf{d}z} - \mathbf{N}^2(f,h) - \mathcal{S}(f)^2, \ \textit{where}\ \mathbf{N}(f,h) = \mathbf{M}_3(f,h)\,f'. \tag{6.3}$$

A nontrivial solution is obtained as follows: Put $f(z) = \dfrac{a + b\,z}{c + d\,z}$. Then $\mathcal{S}(f)(z) = 0$ and `curvature[f,h][z]` returns $\kappa = \sqrt{\mathbf{i}\,\left(h^{(3)}\right)^2 + (1+\mathbf{i})\,\sqrt{2}\,h^{(4)}}$. The differential equation $\kappa = 1$ has the solution

$$h(z) = \frac{c}{6}\,\left(c_2 + c_3\,z + c_4\,z^2 + z^3 + 2\,\mathbf{Li}_3\,\left(-e^{z+2\,c\,c_1}\right)\right),$$

where $c = \dfrac{1-\mathbf{i}}{\sqrt{2}}$, c_1, c_2, c_3, c_4 are constants and

$$\mathbf{Li}_3(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^3} = \int \left(\frac{1}{x}\int \frac{\log(1-x)}{x}\mathbf{d}x\right)\mathbf{d}x.$$

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