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SCATTERED DATA POINTS BEST INTERPLATION AS A PROBLEM OF THE BEST RECOVERY IN THE SENSE OF SARD

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Abstract

The problem of the best recovery in the sense of Sard of a linear functional Lf on the basis of information $T(f) = \{L_j f, \mathrm{j} = 1, 2, \ldots, \mathrm{N}\}$ is studied. It is shown that in the class of bivariate functions with restricted (n, m) -derivative, known on the (n, m)-grid lines, the problem of the best recovery of a linear functional leads to the best approximation of $L(K_n K_m)$ in the space $S = \mathrm{span}\{\mathrm{L_j(K_n \bar{K}_m), j} = 1, 2, \ldots, \mathrm{N}\}$, where $K_n(x, t) = K(x, t) - L_n^x(K(., t); x)$ is the difference between the truncated power kernel $K(x, t) = (x-t)_+^{n-1}/(n-1)!$ and its Lagrange interpolation formula. In particular, the best recovery of a bivariate function is considered, if scattered data points and blending grid are given. An algorithm is designed and realized using the software product MATLAB.

1 Introduction

Let H be a Hilbert space. Suppose that L and $L_1, \ldots, L_N$ are linear functionals defined on H. The problem is to find the best method $S_* f$ of recovery of the functional L on the basis of the information $Tf = (L_1 f, L_2 f, \ldots, L_N f)$.

Given a method Sf, the error is

$$E(S) = \sup_{f \in B} |Lf - Sf|,$$

where B usually is the unit ball in H. The method S_* is the best method of recovery of Lf if its error is minimal, i.e.

$$E(T) = \inf_S \sup_{f \in B} |Lf - Sf| = \sup_{f \in B} |Lf - S_* f|. \tag{1}$$

In accordance with Smolyak's Lemma [6], a linear best method of recovery of a linear functional exists under certain restrictions:

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Lemma Smolyak: *If the linear functionals* $L, L_j, \mathrm{j} = 1, 2, \ldots, \mathrm{N}$ *are defined on a convex and centrally symmetric body* B *in a linear space and*

$$\sup\{Lf : f : L_j f = 0, \mathrm{j} = 1, 2, \ldots, \mathrm{N}\} < \infty,$$

then there exists a best linear method for recovery of L*, i.e.there exist numbers* $A_1, A_2, \ldots, A_N$ *such that*

$$E(T) = \inf_{C_j} \sup_{f \in B} |Lf - \sum C_j L_j f| = \sup_{f \in B} |Lf - \sum A_j L_j f|. \tag{2}$$

The linear methods which are exact for a class of functions, are usually studied [5], [7]. This gives an opportunity for Peano's theorem application. An explicit representation and an algorithm for the best recovery of univariate function can be found in [11].

Results for bivariate functions in some special cases can be found in [1], [2], [3] and [4]. We will not set here any restrictions on the linear approximation methods , but some additional information about the function we will need.

2 Best recovery of a linear functional in the sense of Sard

The Sobolev's class of functions is defined as usually:

$$W_p^n[a, A] = \{f \in C^{n-1}[a, A] : f^{(n-1)} \text{abs.cont.}, \|\mathrm{f}^{(\mathrm{n})}\|_{\mathrm{p}} < \infty\}.$$

We denote the truncated power kernels resp:

$$K(x, t) := \frac{(x-t)_+^{n-1}}{(n-1)!}, \quad K(y, \tau) := \frac{(y-\tau)_+^{m-1}}{(m-1)!}.$$

Set the Lagrange interplation formula $L_n^x(f; x) = \sum_{i=1}^n f(x_i) l_{ni}(x), \quad l_{ni}(x) = \prod_{r=1, r\neq i}^n \frac{x - x_r}{x_i - x_r}, \quad l_{ni}(x_j) = \delta_{ij}$. We shall denote the difference between the truncated power kernel and its Lagrange formula by K_n , resp. $\bar{K}_m$:

$$K_n(x, t) := K(x, t) - L_n^x(K(., t); x), \quad \bar{K}_m(y, \tau) := K(y, \tau) - L_m^y(K(., \tau); y).$$

We shall omit the bar in the symbols $\bar{K}_m$ and $\bar{l}_{jm}$, since there is no chance of confusion.

Theorem 1: *If* $f(x, y) \in W_2^{n,m}[a, A] \times [b, B]$ *and* $x_1, \ldots, x_n$ *are in* $(a, A), y_1, \ldots y_m$ *are in* (b, B)*, then for every* $(x, y) \in [a, A] \times [b, B]$ *the following equality holds:*

$$\begin{aligned} f(x, y) = &\sum_{k=1}^n f(x_k, y) l_{nk}(x) + \sum_{j=1}^m f(x, y_j) l_{mj}(y) - \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) l_{ni}(x) l_{mj}(y) \\ &+ \int_a^A \int_b^B K_n(x, t) K_m(y, \tau) f^{(n,m)}(t, \tau) dt d\tau. \end{aligned} \tag{3}$$

We consider the Hilbert space H with the scalar product

$$(f,g) := \int_a^A \int_b^B f(x,y)g(x,y)dxdy$$

and L_2-norm

$$\|f\| = \left\{ \int_a^A \int_b^B f^2(x,y)dxdy \right\}^{\frac{1}{2}}.$$

The class of functions B is defined as follows:

$$B = \{f(x,y) \in W_2^{n,m}[a,A] \times [b,B] : \|f^{(n,m)}\| \leq 1\}. \tag{4}$$

It is clear that B is a convex and centrally symmetric body and the Smolyak's lemma holds.

We suppose, that the function $f(x,y) \in B$ is known on the grid lines

$$x = x_k, k = 1,2,\ldots,n, \quad y = y_j, j = 1,2,\ldots,m. \tag{5}$$

The proofs of the following Theorem 2. and Theorem 3. can be found in [12].

Theorem 2: *The problem of finding the best linear method* $Lf \sim \sum A_j L_j$ *is equivalent to the problem of the best approximation of* $L(K_n(x,t)K_m(y,\tau))$ *in the space* $\mathrm{span}\{\mathrm{L_j}(\mathrm{K_n K_m}), \mathrm{j} = 1,2,\ldots,\mathrm{N}\}$ *in the sense that the coefficients* $A_j, j = 1,2,\ldots,N$ *of the best linear method can be obtained after the best approximation of* $L(K_n(x,t)K_m(y,\tau))$ *in the space* $\mathrm{span}\{\mathrm{L_j}(\mathrm{K_n K_m}), \mathrm{j} = 1,2,\ldots,\mathrm{N}\}$.

Further we use Gramm-Schmidt orthogonalization:

$$Q_1 K := L_1 K, \ldots\ldots, Q_i K = \sum_{j=1}^{i} f_{ij} L_j K. \tag{6}$$

We shall consider here linear methods for approximation without any restrictions, but some additonal information we need-the function's traces must be known:

Theorem 3: *Let* L *be a linear functional such that* $L(K_n K_m)$ *is integrable on* $[a,A] \times [b,B]$. *If* $f \in B$ *is known on the grid lines (5) , then the best linear method for recovery of* Lf *on the basis of information* $T(f) = (L_1 f, L_2 f, \ldots, L_N f)$ *is*

$$\begin{aligned} Lf \sim & Lb_f + \sum_{j=1}^{N} A_j L_j (f - b_f) \\ & = Lb_f + \sum_{j=1}^{N} \sum_{i=j}^{N} \frac{(L(K_n K_m), Q_i(K_n K_m))}{(Q_i(K_n K_m), Q_i(K_n K_m))} f_{ij} L_j (f - b_f), \end{aligned} \tag{7}$$

where $\{Q_i(K_n K_m), i = 1,2,\ldots,N\}$ *is an orthogonal system in the* $\mathrm{span}\{\mathrm{L_i}(\mathrm{K_n K_m})\}$, $\{f_{ij}\}$ *are coefficients and* K_n, K_m *are the modified kernels.*

3 Best recovery of a bivariate function

Let a point $(x, y) \in [a, A] \times [b, B]$ be fixed. We consider the special case $Lf := f(x, y), \quad L_j f := f(a_j, b_j), j = 1, 2, \ldots, N$. As we assume above, the function f is known on the grid lines (5). The formula (7) holds.

We take points $x_1, \ldots, x_n$ in (a, A) and points $y_1, \ldots, y_m$ in (b, B) and we calculate the terms in formula (7):

$$K_n(x, t) = \frac{(x-t)_+^{n-1}}{(n-1)!} - \sum_{i=1}^{n} \frac{(x_i - t)_+^{n-1}}{(n-1)!} l_{ni}(x).$$

$$K_m(y, \tau) = \frac{(y-\tau)_+^{m-1}}{(m-1)!} - \sum_{j=1}^{m} \frac{(y_j - \tau)_+^{m-1}}{(m-1)!} l_{mj}(y)$$

$$L_i(K_n K_m) = K_n(a_i, t) K_m(b_i, \tau)$$

$$(L(K_n K_m), L_i(K_n K_m)) = (K_n(x, t) K_m(y, \tau), K_n(a_i, t) K_m(b_i, \tau)) =$$

$$= \int_a^A [K(x, t) - \sum_{k=1}^{n} K(x_k, t) l_{nk}(x)][K(a_i, t) - \sum_{r=1}^{n} K(x_r, t) l_{nr}(a_i)] dt.$$

$$. \int_b^B [K(y, \tau) - \sum_{j=1}^{m} K(y_j, \tau) l_{mj}(y)][K(b_i, \tau) - \sum_{s=1}^{m} K(y_s, \tau) l_{ms}(b_i)] d\tau. \tag{8}$$

Set

$$I_n(x, y) = I_n(y, x) := \int_a^A (x-t)_+^{n-1} (y-t)_+^{n-1} dt. \tag{9}$$

It is easy to see that the first integral in equation (8) is equal to $\frac{1}{(n-1)!^2} S(x, a_i)$, where

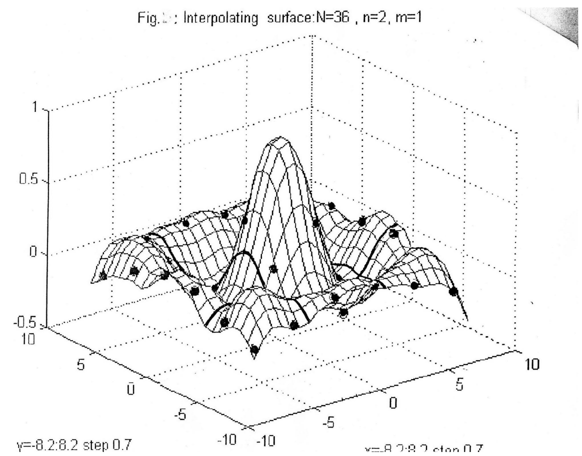
$$\begin{aligned} S(x, a_i) : = I_n(x, a_i) - \sum_{r=1}^{n} l_{nr}(a_i) I_n(x, x_r) - \sum_{k=1}^{n} l_{nk}(x) I_n(x_k, a_i) \\ + \sum_{k=1}^{n} \sum_{r=1}^{n} l_{nk}(x) l_{nr}(a_i) I_n(x_k, x_r). \end{aligned} \tag{10}$$

By analogy we obtain the second integral in [8]. The equation (8) is equivalent to

$$(K_n K_m, L_i(K_n K_m)) = \frac{1}{(n-1)!^2 (m-1)!^2} S(x, a_i) S(y, b_i).$$

The calculation of the coefficients f_{ij} depends on the terms

$$\begin{aligned} (L_i(K_n K_m), L_j(K_n K_m)) &= (K_n(a_i, t) K_m(b_i, \tau), K_n(a_j, t) K_m(b_j, \tau)) \\ &= \frac{1}{(n-1)!^2 (m-1)!^2} S(a_i, a_j) S(b_i, b_j). \end{aligned} \tag{11}$$



From (6) it follows that $(K_nK_m, Q_i(K_nK_m)) = \sum_{k=1}^{i} f_{ik}(K_nK_m, L_k(K_nK_m))$ and

$$(Q_i(K_nK_m), Q_i(K_nK_m)) = \frac{1}{(n-1)!^2(m-1)!^2} \sum_{r=1}^{i} \sum_{s=1}^{i} f_{ir} f_{is} S(a_r, a_s) S(b_r, b_s). \quad (12)$$

Finally, replacing these terms in formula (7), we obtain

$$\begin{aligned} f(x,y) &\sim b_f(x,y) \\ &+ \sum_{j=1}^{N} \sum_{i=j}^{N} f_{ij} \frac{\sum_{k=1}^{i} f_{ik} S(x, a_k) S(y, b_k)}{\sum_{r=1}^{i} \sum_{s=1}^{i} f_{is} f_{ir} S(a_r, a_s) S(b_r, b_s)} (f(a_j, b_j) - b_f(a_j, b_j)), \end{aligned} \quad (13)$$

where $S(x, a_k)$ is given by (10). We find, that the integral in (9) is equal to

$$I_n(x,y) = \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{2n-k-1} (\min(x,y) - a)^{2n-k-1} |x-y|^k. \quad (14)$$

From this representation it follows that the integral $I_n(x, x_r)$ in (10) is a polynomial of degree $2n-1$ for $x < x_r$ and of degree $n-1$ for $x > x_r$.

The error of the best recovery in an arbitrary point (x, y) is obtained in the proof of Theorem 2 [12]. After finding the coefficients A_j we replace them in

$$E(T) = \|K_nK_m - \sum A_j L_j(K_nK_m)\|.$$

The formula (11) is exact for blending functions $f \equiv b_f$, belonging to the class

$$B^{(n,m)} := \{f(x,y) \in W_2^{n,m}[a,A] \times [b,B] : f^{(n,m)} \equiv 0\}. \quad (15)$$

A program has been realized for interpolation of scattered data, based on formulae (11) and (10), (12), using the software product MATLAB v.4. See on the fig.

the interpolating surface for the so called Mexican hat, if $N = 36$ points, $n = 2$, $m = 1$.

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References

[1] Nielson G., *Bivariate Spline Functions and the Approximation of Linear Functionals,* Numer.Math., **21** (1973), 138–160.

[2] Mansfield L. E., *Optimal Approximations and Error Bounds in Spaces of Bivariate Functions,* J. Approx. theory, **5** (1972), 77–96.

[3] Mansfield L. E., *On the Optimal Approximation of Linear Functionals in Spaces of Bivariate Functions,* SIAM J. Numer. Anal., **8** (1971), 115–126.

[4] Ritter D., *Two Dimensional Spline Functions and best Approximation of Linear Functionals,* J. Approx. Theory, **3** (1970), 352–368.

[5] Sard A., *Linear approximation* , Math. Surveys 9, Providence, Rhode Island, 1963.

[6] Smolyak S. A., *Ph. D. Thesis: On the Optimal Recovery of Functions and Functionals of them,* Moscow State University, 1965.

[7] Bojanov B. D., *Optimal recovery of differentiable functions,* Math. USSR Sbornik, v. **69** (1991), 357-377.

[8] Bojanov B. D., Hakopian H. A., Sahakian A. A., *Spline Functions and Multivariate Interpolations*, Kluwer Academic Publishers, Dordrecht, 1993.

[9] Schoenberg I. J., *On the interpolation by spline functions and their minimal properties*, in Proc. Conference on Approximatiom Theory, Oberwolfach, ISNM 5 (1964), 109–129.

[10] Laurent P. J., *Approximation et optimisation*, Hermann, Paris, 1972.

[11] Dicheva N. K., *On the best recovery of a linear functional and its applications*, Proceed of BEM XXI, eds. C. A. Brebbia and H. Power, WIT Press, Southampton, Boston (1999), 739-747.

[12] Dicheva N. K., *On the best recovery of a linear functional in a certain class of bivariate functions,* Numer. Funct. Anal. and Optim., **21(7&8)**, 2000, 829-843.

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