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IDENTITIES FOR CURVATURE TENSORS IN GENERALIZED FINSLER SPACE

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Abstract

In the some previous works we have obtained several curvature tensors in the generalized Finsler space GF_N (the space with non-symmetric basic tensor and non-symmetric connection in Rund's sence).

In this work we study identities for the mentioned tensors (the antisymmetry with respect of two indices, the cyclic symmetry, the symmetry with respect of pairs of indices).

1 Introduction

The generalized Finsler space (GF_N) is a differentiable manifold with non-symmetric basic tensor
$g_{ij}(x^1,\dots,x^N,\dot{x}^1,\dots,\dot{x}^N)\equiv g_{ij}(x,\dot{x})$, where

$$g_{ij}(x,\dot{x})\neq g_{ji}(x,\dot{x}),\quad (g=det(g_{ij})\neq 0,\ \dot{x}=dx/dt). \tag{1.1}$$

Based on (1.1), one defines the symmetric respectively anti-symmetric part of g_{ij}

$$g_{\underline{ij}}=\frac{1}{2}(g_{ij}+g_{ji}),\qquad g_{\underset{\vee}{ij}}=\frac{1}{2}(g_{ij}-g_{ji}), \tag{1.2}$$

where, following [8], is

$$a)\ g_{\underline{ij}}(x,\dot{x})=\frac{1}{2}\frac{\partial^2F^2(x,\dot{x})}{\partial\dot{x}^i\partial\dot{x}^j},\qquad b)\ \frac{\partial g_{\underset{\vee}{ij}}}{\partial\dot{x}^k}=0, \tag{1.3}$$

where $F(x,\dot{x})$ is a metric function in GF_N, having the properties known from the theory of usual Finsler space (F_N) (see e.g. [7]).

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The lowering and the raising of indices one defines by the tensors $g_{\underline{ij}}$ and h^{ij} respectively, where h^{ij} is defined as follows

$$g_{\underline{ij}}\, h^{jk} = \delta_i^k, \quad (\underline{g} = det(g_{\underline{ij}}) \neq 0). \tag{1.4}$$

We can define generalized Cristoffel symbols of the 1^{st} and the 2^{nd} kind:

$$\begin{aligned} \gamma_{i.jk} &= \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}) \neq \gamma_{i.kj}, \\ \gamma^i_{jk} = h^{ip}\gamma_{p.jk} &= \frac{1}{2}h^{ip}(g_{jp,k} - g_{jk,p} + g_{pk,j}) \neq \gamma^i_{kj}, \end{aligned} \tag{1.5}$$

where, e.g., $g_{ji,k} = \partial g_{ji}/\partial x^k$.

Then we have

$$\gamma^p_{jk} g_{\underline{ip}} = \gamma_{s.jk} h^{ps} g_{\underline{ip}} = \gamma_{s.jk}\delta^s_i = \gamma_{i.jk}. \tag{1.6}$$

Introducing a tensor C_{ijk} like as at F_N, we have

$$C_{ijk}(x,\dot{x}) \overset{def}{=} \frac{1}{2} g_{ij,\dot{x}^k} \underset{(1.3b)}{=} \frac{1}{2} g_{\underline{ij},\dot{x}^k} = \frac{1}{4} F^2_{\dot{x}^i\dot{x}^j\dot{x}^k}, \tag{1.7}$$

where " $\underset{(1.3b)}{=}$ " signifies "equal based on (1.3b)". We see that C_{ijk} is symmetric in relation to each pair of indices. Also, we have

$$C^i_{jk} \overset{def}{=} h^{ip} C_{pjk} \underset{(1.7)}{=} h^{ip} C_{jpk} = h^{ip} C_{jkp}. \tag{1.8}$$

With help of coefficients

$$P^i_{jk} = \gamma^i_{jk} - C^i_{jp}\gamma^p_{sk}\dot{x}^s \neq P^i_{kj} \tag{1.9}$$

one obtains coefficients of non-symmetric affine connections in the Rund's sence [7], [9]:

$$P^{*i}_{jk} = \gamma^i_{jk} - h^{iq}(C_{jqp}P^p_{ks} + C_{kqp}P^p_{js} - C_{jkp}P^p_{qs})\dot{x}^s \underset{(1.6)}{\neq} P^{*i}_{kj}, \tag{1.10}$$

$$P^*_{i.jk} = P^{*r}_{jk} g_{\underline{ir}} = \gamma_{i.jk} - (C_{ijp}P^p_{ks} + C_{ikp}P^p_{js} - C_{jkp}P^p_{is})\dot{x}^s \neq P^*_{i.kj}. \tag{1.11}$$

In GF_N we denote double anti-symmetric and double symmetric part for connection P^* respectively:

$$a)\ T^{*i}_{jk}(x,\dot{x}) = P^{*i}_{[jk]} = P^{*i}_{jk} - P^{*i}_{kj}, \quad b)\ P^{*i}_{(jk)} = P^{*i}_{jk} + P^{*i}_{kj}, \tag{1.12}$$

where T^{*i}_{jk} is the torsion tensor.

We define four kinds of covariant derivative of a tensor in the space GF_N . For example, for a tensor $a^i_j(x,\xi)$ is

$$a^i_{\underset{\substack{1\\2\\3\\4}}{j|m}}(x,\xi) = a^i_{j,m} + a^i_{j,\dot{p}}\xi^p_{,m} + P^{*i}_{\substack{pm\\mp\\pm\\mp}} a^p_j - P^{*p}_{\substack{jm\\mj\\mj\\jm}} a^i_p. \tag{1.13}$$

where $\xi(x)$ is an arbitrary tangent vector in the tangent space $T_N(x)$, and $a^i_{j,\dot{p}} = \partial a^i_j/\partial \dot{x}^p$.

Theorem 1.1. *For the tensor* $g_{\underline{ij}}(x,\dot{x})$ *based on four kinds of derivatives (1.13) is valid [3, 4]*

$$g_{\underline{ij}\underset{\theta}{|}m}(x,\xi) = 2C_{ijp}(\xi^p_{,m} + P^{*p}_{ms}\dot{x}^s), \quad \theta = 1,2,3,4, \tag{1.14}$$

$$g_{\underline{ij}\underset{\theta}{|}m}(x,\dot{x}) = 2C_{ijp}\dot{x}^p_{\underset{2}{|}m} = 2C_{ijp}\dot{x}^p_{\underset{4}{|}m}, \quad \theta = 1,2,3,4, \tag{1.15}$$

and also it is easy to prove:

Theorem 1.2. *For the Chronecker symbol is in force*

$$\delta^i_{j\underset{\theta}{|}m} = 0, \quad \theta = 1,2,3,4. \tag{1.16}$$

Theorem 1.3. *For* h^{ij} *is in force [3, 4]*

$$h^{ij}_{\underset{\theta}{|}m} = -h^{ip}h^{jq}g_{\underline{pq}\underset{\theta}{|}m}, \quad \theta = 1,2,3,4. \tag{1.17}$$

In the work [3] we have obtained 10 Ricci type identities in a general case for the tensor $a^{r_1\dots r_u}_{t_1\dots t_v}(x,\xi)$. Then based on (1.13) and using the first and the second kind of covariant derivative we study the differences

$$a^{r_1\dots r_u}_{t_1\dots t_v\underset{\substack{1\\2\\1\\2\\1\\1\\1\\2\\2\\1}}{|}m\underset{\substack{1\\2\\2\\1\\1\\1\\1\\2\\2\\2}}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{\substack{1\\2\\1\\2\\2\\1\\2\\1\\2\\2}}{|}n\underset{\substack{1\\2\\2\\1\\2\\2\\1\\2\\1\\1}}{|}m}, \tag{1.18}$$

where, $a^{r_1\dots r_u}_{t_1\dots t_v\underset{\alpha}{|}m\underset{\beta}{|}n} = (a^{r_1\dots r_u}_{t_1\dots t_v\underset{\alpha}{|}m})_{\underset{\beta}{|}n}$, $\alpha,\beta \in \{1,2\}$. In this manner we get three curvature tensors:

$$\underset{1}{\widetilde{K}}{}^i{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}, \tag{1.19}$$

$$\underset{2}{\widetilde{K}}{}^i{}_{jmn} = P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{mp} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}, \tag{1.20}$$

$$\underset{3}{\widetilde{K}}{}^i{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{nm}(P^{*i}_{pj} - P^{*i}_{jp}) + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}, \tag{1.21}$$

In the work [4] we use the third and the fourth kind of covariant derivative (1.13), and in that manner we get ten new Ricci type identities. In these identities appear the same quantities like in the (1.18), but in different distribution. Only in the last case appears a new curvature tensor $\underset{4}{\widetilde{K}}$:

$$a^j_{i\underset{3}{|}m\underset{4}{|}n} - a^j_{i\underset{4}{|}n\underset{3}{|}m} = \underset{4}{\widetilde{K}}{}^i{}_{pmn}a^p_j + \underset{3}{\widetilde{K}}{}^p{}_{jnm}a^i_p\ ,$$

where

$$\underset{4}{\widetilde{K}}{}^i{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{nj,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{mn}(P^{*i}_{pj} - P^{*i}_{jp}) + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}. \tag{1.22}$$

In the work [5] we obtained combined Ricci type identities using (1.18). For example:

$$\begin{aligned} &a^i_{j\underset{1}{|}m\underset{2}{|}n} - a^i_{j\underset{1}{|}n\underset{2}{|}m} + a^i_{j\underset{2}{|}m\underset{1}{|}n} - a^i_{j\underset{2}{|}n\underset{1}{|}m} \\ &\qquad = (\underset{1}{\widetilde{A}} + \underset{3}{\widetilde{A}})^i_{pmn}a^p_j - (\underset{2}{\widetilde{A}} + \underset{4}{\widetilde{A}})^p_{jmn}a^i_p + T^{*p}_{mn}(a^i_{j\underset{1}{|}p} - a^i_{j\underset{2}{|}p}), \end{aligned} \tag{1.23}$$

where we note

$$(\underset{1}{\widetilde{A}} + \underset{3}{\widetilde{A}})^i_{jmn} = \underset{1}{\widetilde{A}}{}^i{}_{jmn} + \underset{3}{\widetilde{A}}{}^i{}_{jmn},$$

$$\begin{aligned} \underset{1}{\widetilde{A}}{}^i{}_{jmn} &= P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{mp} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}, \\ \underset{3}{\widetilde{A}}{}^i{}_{jmn} &= P^{*i}_{mj,n} - P^{*i}_{nj,m} + P^{*p}_{mj}P^{*i}_{pn} - P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{mj,\dot{s}}\xi^s_{,n} - P^{*p}_{nj,\dot{s}}\xi^s_{,m}, \end{aligned}$$

and similarly in other cases. In this manner we get 8 new curvature tensors that we call "derived curvature tensors". Further, in the work [6] we consider independent between curvature tensors (1.19-1.22) and these 8 new derived curvature tensors.

We proved the following theorem [6]:

Theorem 1.4. *From 12 curvature tensors in the space GF_N with non-symmetric connection P^{*i}_{jk} there are 5 independent ones, while the rest we can express as linear combinations of these 5 tensors (e.g. $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, $\underset{3}{\widetilde{K}}$, $\underset{4}{\widetilde{K}}$, $\underset{5}{\widetilde{K}} \equiv \underset{2}{\widetilde{K}}{}^*$), and the curvature tensor $\widetilde{K}$ of the associated symmetric connection $P^{*i}_{(jk)}$, where*

$$\begin{aligned} \underset{5}{\widetilde{K}}{}^i{}_{jmn} \equiv \underset{2}{\widetilde{K}}{}^{*i}{}_{jmn} = &\frac{1}{2}(P^{*i}_{\underline{jm},n} - P^{*i}_{\underline{jn},m} + P^{*p}_{jm}P^{*i}_{pn} + P^{*p}_{mj}P^{*i}_{np} - P^{*p}_{jn}P^{*i}_{mp} \\ &- P^{*p}_{nj}P^{*i}_{pm} + P^{*p}_{\underline{jm},\dot{s}}\xi^s_{,n} - P^{*p}_{\underline{jn},\dot{s}}\xi^s_{,m}) \end{aligned} \tag{1.24}$$

$$\underset{1}{\widetilde{K}}{}^* = 2\widetilde{K} - \frac{1}{2}(\underset{1}{\widetilde{K}} + \underset{2}{\widetilde{K}}), \quad \underset{3}{\widetilde{K}}{}^* = 2\widetilde{K} - \underset{5}{\widetilde{K}}, \quad \underset{4}{\widetilde{K}}{}^* = 2\widetilde{K} - \underset{1}{\widetilde{K}}, \tag{1.25}$$

$$\underset{5}{\widetilde{K}}{}^* = 4\widetilde{K} - \underset{1}{\widetilde{K}} - 2\underset{5}{\widetilde{K}}, \quad \underset{6}{\widetilde{K}}{}^* = -\underset{1}{\widetilde{K}} + 2\underset{5}{\widetilde{K}}, \quad \underset{7}{\widetilde{K}}{}^* = -\underset{2}{\widetilde{K}} + 2\underset{5}{\widetilde{K}}, \quad \underset{8}{\widetilde{K}}{}^* = 4\widetilde{K} - \underset{1}{\widetilde{K}} - 2\underset{5}{\widetilde{K}}. \tag{1.26}$$

2 The mixed curvature tensors of the space GF_N

In the space F_N with symmetric affine connection for δ-differentiation we have one curvature tensor

$$\widetilde{K}{}^i{}_{jmn} = P^{*i}_{jm,n} - P^{*i}_{jn,m} + P^{*p}_{jm}P^{*i}_{pn} - P^{*p}_{jn}P^{*i}_{pm} + P^{*p}_{jm,\dot{s}}\xi^s_{,n} - P^{*p}_{jn,\dot{s}}\xi^s_{,m}, \tag{2.1}$$

As it is known, the curvature tensor of the space F_N with symmetric affine connection possesses the next property [7]:

$$\widetilde{K}^i{}_{jmn} = -\widetilde{K}^i{}_{jnm}, \tag{2.2}$$

Introduce the denotation

$$\underset{jmn}{\mathfrak{S}} \widetilde{K}^i{}_{jmn} = \widetilde{K}^i{}_{jmn} + \widetilde{K}^i{}_{mnj} + \widetilde{K}^i{}_{njm} \tag{2.3}$$

and similarly in other cases. It is easy to prove

$$\underset{jmn}{\mathfrak{S}} \widetilde{K}^i{}_{jmn} = 0. \tag{2.4}$$

As the tensors $\underset{1}{\widetilde{K}}, \underset{2}{\widetilde{K}}, \underset{3}{\widetilde{K}}, \underset{4}{\widetilde{K}}, \underset{1}{\widetilde{K}}^*, \dots, \underset{8}{\widetilde{K}}^*$, are generalizations of the tensor $\widetilde{K}$ and reduce to this one in the case of symmetric connection we have to investigate the properties (2.2, 2.4) for these tensors.
From the expressions in the §1 one gets

$$\underset{1}{\widetilde{K}}{}^i{}_{jmn} = -\underset{1}{\widetilde{K}}{}^i{}_{jnm}, \ \underset{jmn}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^i{}_{jmn} = \underset{jmn}{\mathfrak{S}} (T^{*i}_{jm,n} + T^{*p}_{jm} P^{*i}_{pn} + T^{*i}_{jm,\dot{s}} \xi^s_{,n}), \tag{2.5}$$

$$\underset{2}{\widetilde{K}}{}^i{}_{jmn} = -\underset{2}{\widetilde{K}}{}^i{}_{jnm}, \ \underset{jmn}{\mathfrak{S}} \underset{2}{\widetilde{K}}{}^i{}_{jmn} = \underset{jmn}{\mathfrak{S}} (T^{*i}_{mj,n} + T^{*p}_{mj} P^{*i}_{np} + T^{*i}_{mj,\dot{s}} \xi^s_{,n}), \tag{2.6}$$

$$\underset{3}{\widetilde{K}}{}^i{}_{jmn} \neq \underset{3}{\widetilde{K}}{}^i{}_{jnm}, \ \underset{jmn}{\mathfrak{S}} \underset{3}{\widetilde{K}}{}^i{}_{jmn} = \underset{jmn}{\mathfrak{S}} T^{*p}_{jm} T^{*i}_{np}, \tag{2.7}$$

$$\underset{4}{\widetilde{K}}{}^i{}_{jmn} \neq \underset{4}{\widetilde{K}}{}^i{}_{jnm}, \ \underset{jmn}{\mathfrak{S}} \underset{4}{\widetilde{K}}{}^i{}_{jmn} = 0. \tag{2.8}$$

We see that only $\underset{4}{\widetilde{K}}{}^i{}_{jmn}$ possesses the cyclic symmetry of the form (2.4). Crossing to the derived curvature tensors, we remark that one can use the relations between curvature tensors (1.19-1.22) and derived curvature tensors given by (1.24-1.26). By virtue of (1.24-1.26, 2.2) we get

$$\underset{1}{\widetilde{K}}{}^{*i}_{jmn} = -\underset{1}{\widetilde{K}}{}^{*i}_{jnm}, \quad \underset{jmn}{\mathfrak{S}} \underset{1}{\widetilde{K}}{}^{*i}_{jmn} = \underset{jmn}{\mathfrak{S}} T^{*p}_{mj} T^{*i}_{pn} \tag{2.9}$$

From

$$\underset{2}{\widetilde{K}}{}^{*i}_{jmn} = \widetilde{K}^i{}_{jmn} + \frac{1}{4} T^{*p}_{jm} T^{*i}_{pn} + \frac{1}{4} T^{*p}_{jn} T^{*i}_{pm} \tag{2.10}$$

we have

$$\underset{2}{\widetilde{K}}{}^{*i}_{jmn} \neq -\underset{2}{\widetilde{K}}{}^{*i}_{jnm}, \quad \underset{jmn}{\mathfrak{S}} \underset{2}{\widetilde{K}}{}^{*i}_{jmn} = 0. \tag{2.11}$$

Using the previous symmetry properties and (1.25, 2.10) we get

$$\underset{3}{\widetilde{K}}{}^{*i}_{jmn} \neq -\underset{3}{\widetilde{K}}{}^{*i}_{jnm}, \quad \underset{jmn}{\mathfrak{S}} \underset{3}{\widetilde{K}}{}^{*i}_{jmn} = 0. \tag{2.12}$$

From (1.25, 2.1) is

$$\underset{4}{\widetilde{K}}{}^{*i}_{jmn} = -\underset{4}{\widetilde{K}}{}^{*i}_{jnm}, \quad \underset{jmn}{\mathfrak{S}} \underset{4}{\widetilde{K}}{}^{*i}_{jmn} = 0. \tag{2.13}$$

Using the relations (1.24-1.26) and the properties of the tensors $\underset{1}{\widetilde{K}}, \underset{1}{\widetilde{K}}, \ldots \underset{4}{\widetilde{K}}, \underset{2}{\widetilde{K}}^*$ it is easy to prove that the tensors $\underset{5}{\widetilde{K}}^*, \ldots \underset{8}{\widetilde{K}}^*$ do not posses the symmetry properties of the type (2.2,2.4). Based on exposed, we obtain the following.

Theorem 2.1. *Among obtained 12 mixed curvature tensors in the space GF_N the property (2.2) possess*
$\underset{1}{\widetilde{K}}, \underset{2}{\widetilde{K}}, \underset{1}{\widetilde{K}}^*, \underset{4}{\widetilde{K}}^*$ *and the property (2.4) the tensors* $\underset{4}{\widetilde{K}}, \underset{2}{\widetilde{K}}^*, \underset{3}{\widetilde{K}}^*, \underset{4}{\widetilde{K}}^*$, *while in the remain cases one obtains complicated relations, generalizing (2.2, 2.4).*

3 The covariant curvature tensors of the space GF_N and their properties

In GF_N one defines covariant curvature tensors

$$\underset{p}{\widetilde{K}}_{ijmn} = g_{\underline{is}}\underset{p}{\widetilde{K}}{}^s{}_{jmn}, \quad \underset{q}{\widetilde{K}}{}^*_{ijmn} = g_{\underline{is}}\underset{q}{\widetilde{K}}{}^{*s}_{jmn} \quad (p = \overline{1,4};\ q = \overline{1,8}) \tag{3.1}$$

As it is known, in a Finsler space (for which is $g_{ij} = g_{ji}$) for covariant curvature tensor the following relations hold [7, 11]

$$\widetilde{K}_{ijmn} + \widetilde{K}_{ijnm} = 0,\ \widetilde{K}_{ijmn} + \widetilde{K}_{jimn} = g_{ij;[nm]}, \tag{3.2}$$

where (;) denotes covariant differentiation based on the symmetric part of the connection P^*. Also, we have

$$\underset{jmn}{\mathfrak{S}}\,\widetilde{K}_{ijmn} = 0. \tag{3.3}$$

Since the space GF_N is a generalization of the usual Finsler space we have to examine relations of the type (3.2, 3.3) for covariant curvature tensors of the GF_N.

For curvature tensor of the first kind, we have:

$$\begin{aligned}\underset{1}{\widetilde{K}}_{ijmn} = g_{\underline{is}}\underset{1}{\widetilde{K}}{}^s{}_{jmn} =& g_{\underline{is}}P^{*s}_{jm,n} - g_{\underline{is}}P^{*s}_{jn,m} + P^{*p}_{jm}P^*_{i.pn} - P^{*p}_{jn}P^*_{i.pm} \\ &+ g_{\underline{is}}P^{*s}_{jm,\dot{q}}\xi^q_{,n} - g_{\underline{is}}P^{*s}_{jn,\dot{q}}\xi^q_{,m}\end{aligned} \tag{3.4}$$

As

$$P^*_{i.jm,n} = (g_{\underline{is}}P^{*s}_{jm})_{,n} = g_{\underline{is},n}P^{*s}_{jm} + g_{\underline{is}}P^{*s}_{jm,n}, \tag{3.5}$$

we have

$$\begin{aligned}\underset{1}{\widetilde{K}}_{ijmn} =& P^*_{i.jm,n} - P^*_{i.jn,m} + g_{\underline{rs}}P^{*r}_{im}P^{*s}_{jn} - g_{\underline{rs}}P^{*r}_{in}P^{*s}_{jm} \\ &+ P^*_{i.jm,\dot{q}}\xi^q_{,n} - P^*_{i.jn,\dot{q}}\xi^q_{,m} - P^{*p}_{jm}g_{\underline{ip}\underset{1}{|}n} + P^{*p}_{jn}g_{\underline{ip}\underset{1}{|}m}\end{aligned} \tag{3.6}$$

By direct calculation we have

$$g_{\underline{ij}\underset{1}{|}mn} - g_{\underline{ij}\underset{1}{|}nm} = -g_{\underline{pj}}\underset{1}{\widetilde{K}}{}^p{}_{imn} - g_{\underline{ip}}\underset{1}{\widetilde{K}}{}^p{}_{jmn} - T^{*p}_{mn}g_{\underline{ij}\underset{1}{|}p} \tag{3.7}$$

i.e.

$$\underset{1}{\widetilde{K}}_{ijmn} + \underset{1}{\widetilde{K}}_{jimn} = g_{\underline{ij}\underset{1}{|}[nm]} + T^{*p}_{nm} g_{\underline{ij}\underset{1}{|}p}. \tag{3.8}$$

From (2.5) and (3.4) we conclude

$$\underset{1}{\widetilde{K}}_{ijmn} = -\underset{1}{\widetilde{K}}_{ijnm}, \quad \underset{1}{\widetilde{K}}_{ijmn} + \underset{1}{\widetilde{K}}_{jimn} = g_{\underline{ij}\underset{1}{|}[nm]} - T^{*p}_{mn} g_{\underline{ij}\underset{1}{|}p}. \tag{3.9}$$

From (2.5), we obtain the relation

$$\begin{aligned}\underset{jmn}{\mathfrak{S}} \underset{1}{\widetilde{K}}_{ijmn} =& \underset{jmn}{\mathfrak{S}} (g_{\underline{is}} T^{*s}_{jm,n} + g_{\underline{is}} T^{*p}_{jm} P^{*s}_{pn} + g_{\underline{is}} T^{*s}_{jm,\dot{q}} \xi^q_{,n}) \\ =& \underset{jmn}{\mathfrak{S}} (P^*_{i.[jm],n} - P^{*s}_{[jm]} g_{\underline{is}\underset{1}{|}n} - g_{\underline{rs}} P^{*s}_{[jm]} P^{*r}_{in} + P_{i.[jm],\dot{q}} \xi^q_{,n}).\end{aligned} \tag{3.10}$$

As in the case of the tensor $\underset{1}{\widetilde{K}}_{ijmn}$ one concludes that

$$\begin{aligned}\underset{2}{\widetilde{K}}_{ijmn} =& P^*_{i.mj,n} - P^*_{i.nj,m} + g_{\underline{rs}} P^{*r}_{mi} P^{*s}_{nj} - g_{\underline{rs}} P^{*r}_{ni} P^{*s}_{mj} \\ &+ P^*_{i.mj,\dot{q}} \xi^q_{,n} - P^*_{i.nj,\dot{q}} \xi^q_{,m} - P^{*p}_{mj} g_{\underline{ip}\underset{2}{|}n} + P^{*p}_{nj} g_{\underline{ip}\underset{2}{|}m}\end{aligned} \tag{3.11}$$

$$\underset{2}{\widetilde{K}}_{ijmn} = -\underset{2}{\widetilde{K}}_{ijnm}, \quad \underset{2}{\widetilde{K}}_{ijmn} + \underset{2}{\widetilde{K}}_{jimn} = g_{\underline{ij}\underset{2}{|}[nm]} - T^{*p}_{mn} g_{\underline{ij}\underset{2}{|}p}. \tag{3.12}$$

$$\begin{aligned}\underset{jmn}{\mathfrak{S}} \underset{2}{\widetilde{K}}_{ijmn} =& \underset{jmn}{\mathfrak{S}} (g_{\underline{is}} T^{*i}_{mj,n} + g_{\underline{is}} T^{*p}_{mj} P^{*i}_{np} + g_{\underline{is}} T^{*i}_{mj,\dot{s}} \xi^s_{,n}) \\ =& \underset{jmn}{\mathfrak{S}} (P^*_{i.[mj],n} - P^{*s}_{[mj]} g_{\underline{is}\underset{2}{|}n} - g_{\underline{rs}} P^{*s}_{[mj]} P^{*r}_{in} + P_{i.[mj],\dot{q}} \xi^q_{,n}).\end{aligned} \tag{3.13}$$

i.e. for $\underset{1}{\widetilde{K}}$ and $\underset{2}{\widetilde{K}}$ a relation of the form (3.3) does not hold.

In accordance with (3.1, 1.21) is

$$\begin{aligned}\underset{3}{\widetilde{K}}_{ijmn} =& P^*_{i.jm,n} - P^*_{i.nj,m} + g_{\underline{rs}} P^{*s}_{im} P^{*p}_{nj} - g_{\underline{rs}} P^{*r}_{ni} P^{*s}_{jm} \\ &+ P^*_{i.jm,\dot{q}} \xi^q_{,n} - P^*_{i.nj,\dot{q}} \xi^q_{,m} - P^{*p}_{jm} g_{\underline{ip}\underset{3}{|}n} + P^{*p}_{nj} g_{\underline{ip}\underset{3}{|}m}\end{aligned} \tag{3.14}$$

wherefrom one concludes

$$\underset{3}{\widetilde{K}}_{ijmn} \neq -\underset{3}{\widetilde{K}}_{ijnm}, \quad \underset{3}{\widetilde{K}}_{ijmn} + \underset{3}{\widetilde{K}}_{jimn} = g_{\underline{ij}\underset{2}{|}n\underset{1}{|}m} - g_{\underline{ij}\underset{1}{|}m\underset{2}{|}n}, \tag{3.15}$$

$$\underset{jmn}{\mathfrak{S}} \underset{3}{\widetilde{K}}_{ijmn} = \underset{jmn}{\mathfrak{S}} g_{\underline{is}} T^{*p}_{mj} T^{*s}_{np} = \underset{jmn}{\mathfrak{S}} T^{*p}_{mj} T^*_{i.np} \tag{3.16}$$

For the tensor $\underset{4}{\widetilde{K}}_{ijmn}$ is

$$\underset{4}{\widetilde{K}}_{ijmn} \neq -\underset{4}{\widetilde{K}}_{ijnm}, \quad \underset{4}{\widetilde{K}}_{ijmn} + \underset{4}{\widetilde{K}}_{jimn} = h^{ij}_{\underset{3}{|}m\underset{4}{|}n} - h^{ij}_{\underset{4}{|}n\underset{3}{|}m}. \tag{3.17}$$

From (3.1) we have

$$\underset{jmn}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}_{ijmn} = \underset{jmn}{\mathfrak{S}}(g_{\underline{is}}\underset{4}{\widetilde{K}}{}^{s}{}_{jmn}) = g_{\underline{is}}\underset{jmn}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{s}{}_{jmn}$$

and based on (2.8), it follows that

$$\underset{jmn}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}_{ijmn} = 0. \tag{3.18}$$

Further, let us examine the symmetry properties of the tensors $\underset{q}{\widetilde{K}}{}^{*}_{ijmn}$ $(q = \overline{1,8})$ From (1.25) is

$$\underset{1}{\widetilde{K}}{}^{*}_{ijmn} = g_{\underline{is}}\underset{1}{\widetilde{K}}{}^{*s}{}_{jmn} = 2\widetilde{K}{}_{ijmn} - \frac{1}{2}(\underset{1}{\widetilde{K}}{}_{ijmn} + \underset{2}{\widetilde{K}}{}_{ijmn}) \tag{3.19}$$

and using (3.2, 3.9, 3.12) we conclude

$$\underset{1}{\widetilde{K}}{}^{*}_{ijmn} = -\underset{1}{\widetilde{K}}{}^{*}_{jimn}, \tag{3.20}$$

The tensor $\underset{1}{\widetilde{K}}{}^{*}_{ijmn}$ does not possess any cyclic symmetry. Using (2.10) we obtain

$$\underset{2}{\widetilde{K}}{}^{*}_{ijmn} = \widetilde{K}{}^{*}_{ijmn} + \frac{1}{4}h^{ps}(T^{*}_{s.jm}T^{*}_{i.pn} + T^{*}_{s.jn}T^{*}_{i.pm}), \tag{3.21}$$

wherefrom we see that the tensor $\underset{2}{\widetilde{K}}{}_{ijmn}$ is antisymmetric with respect neither on m, n nor on i, j.

From (2.11) we have

$$\underset{jmn}{\mathfrak{S}}\underset{2}{\widetilde{K}}{}^{*}_{ijmn} = 0. \tag{3.22}$$

Using (1.25) and properties of the tensors $\widetilde{K}$, $\underset{2}{\widetilde{K}}{}^{*} \equiv \underset{5}{\widetilde{K}}$ we conclude that

$$\underset{jmn}{\mathfrak{S}}\underset{3}{\widetilde{K}}{}^{*}_{ijmn} = 0, \tag{3.23}$$

while the tensor $\underset{3}{\widetilde{K}}{}^{*}_{ijmn}$ does not possess other symmetry properties.

From (1.25) is

$$\underset{4}{\widetilde{K}}{}^{*}_{ijmn} = g_{\underline{is}}\underset{4}{\widetilde{K}}{}^{*s}{}_{jmn} = 2\widetilde{K}{}_{ijmn} - \underset{1}{\widetilde{K}}{}_{ijmn}, \tag{3.24}$$

and taking into consideration (3.2, 3.9) and properties of the tensors $\widetilde{K}$, $\underset{1}{\widetilde{K}}$ one concludes

$$\underset{4}{\widetilde{K}}{}^{*}_{ijmn} = -\underset{4}{\widetilde{K}}{}^{*}_{ijnm}, \qquad \underset{jmn}{\mathfrak{S}}\underset{4}{\widetilde{K}}{}^{*}_{ijmn} = 0. \tag{3.25}$$

From (1.24-1.26) and corresponding properties of the tensors $\widetilde{K}$, $\underset{1}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}$, $\underset{2}{\widetilde{K}}{}^{*}$, it follows that the tensors $\underset{5}{\widetilde{K}}{}^{*}, \ldots, \underset{8}{\widetilde{K}}{}^{*}$ do not possess any symmetric properties. Consequently, we have proved

Theorem 3.1. *For Covariant curvature tensors in GF_N among the properties of the the type (3.2, 3.3) are valid: For $\underset{1}{\widetilde{K}}$ the property (3.9, 3.10), for $\underset{2}{\widetilde{K}}$ the properties (3.12, 3.13), for $\underset{3}{\widetilde{K}}$ the property (3.16), for $\underset{4}{\widetilde{K}}$ the property (3.18), for $\underset{1}{\widetilde{K}}{}^*$ the property (3.20), for $\underset{2}{\widetilde{K}}{}^*$ the property (3.22),for $\underset{3}{\widetilde{K}}{}^*$ the property (3.23), for $\underset{4}{\widetilde{K}}{}^*$ the properties (3.25). In the remain cases are obtained complicated relations generalizing (3.2, 3.3).*

Remark. For $g_{ij}(x, \dot{x}) = g_{ji}(x, \dot{x})$ we obtain usual Finsler space F_N. If $g_{ij}(x) \neq g_{ji}(x)$ one obtains a generalized Riemannian space GR_N [2]. For $g_{ij}(x) = g_{ji}(x)$ GF_N reduces to the Riemannian space R_N.

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