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ON UNIQUE COMMON FIXED POINT THEOREMS FOR THREE AND FOUR SELF MAPPINGS

H. Bouhadjera

Abstract

In this contribution, two unique common fixed point theorems for three and four self mappings are given which improve and extend the results of Aage and Salunke [1] and others.

1 Introduction

In his paper [9], Sessa introduced the concept of weakly commuting mappings and obtained some common fixed point theorems. In 1986, Jungck [3] extend commuting and weakly commuting mappings by giving the concept of compatible mappings. After that, the same author with Murthy and Cho [4] defined another extension of weakly commuting mappings called the concept of compatible mappings of type (A). Later on, Pathak and Khan [8] gave the notion of compatible mappings of type (B) which extends the concept of compatible mappings of type (A). In this way, the first author with Cho, Kang and Madharia [7] gave another extension of the concept of compatible mappings of type (A) by introducing the notion of compatible mappings of type (C). Another type of compatibility called compatibility of type (P) was introduced in [6]. Recently, Jungck and Rhoades [5] gave an extension of all concepts of commutativity, weak commutativity and compatibility by introducing the notion of weakly compatible mappings. More recently, Al-Thagafi and Shahzad [2] defined the concept of occasionally weakly compatible mappings (shortly (owc)) as an extension of the concept of weakly compatible mappings.

So on this way we have proved some common fixed point theorems for three and four owc mappings satisfying a contractive condition.

1.1 Definition *self mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly commuting pair if, for all $x \in \mathcal{X}$*

$$d(fgx, gfx) \leq d(fx, gx).$$

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1.2 Definition *self mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be*
(1) *compatible if,*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

(2) *compatible of type (A) if,*

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0,$$

(3) *compatible of type (B) if,*

$$\begin{aligned} \lim_{n\to\infty} d(fgx_n, g^2x_n) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right] \\ &\text{and} \\ \lim_{n\to\infty} d(gfx_n, f^2x_n) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right], \end{aligned}$$

(4) *compatible of type (C) if,*

$$\begin{aligned} \lim_{n\to\infty} d(fgx_n, g^2x_n) &\leq \frac{1}{3}\Big[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) \\ &\quad + \lim_{n\to\infty} d(ft, g^2x_n)\Big] \\ &\text{and} \\ \lim_{n\to\infty} d(gfx_n, f^2x_n) &\leq \frac{1}{3}\Big[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) \\ &\quad + \lim_{n\to\infty} d(gt, f^2x_n)\Big], \end{aligned}$$

(5) *compatible of type (P) if,*

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

1.3 Definition *self mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly compatible if they commute at their coincidence points.*

1.4 Definition *Two self mappings f and g of a set $\mathcal{X}$ are owc iff, there is a point t in $\mathcal{X}$ which is a coincidence point of f and g at which f and g commute.*

In their paper [1] Aage and Salunke proved the following results:

1.5 Theorem *Suppose f, g and h be three self mappings of a complete metric space $(\mathcal{X}, d)$ into itself satisfying the conditions:*
(i) $f(\mathcal{X}) \cup g(\mathcal{X}) \subset h(\mathcal{X})$.

(ii) $d(fx, gy) \leq \alpha d(hx, hy) + \beta[d(fx, hx) + d(gy, hy)] + \gamma[d(hx, gy) + d(hy, fx)]$,

for all x, $y \in \mathcal{X}$ *and* α, β *and* γ *are non-negative reals such that* $\alpha + 2\beta + 2\gamma < 1$.
(iii) One of f, g *and* h *is continuous.*
(iv) (f, h) *and* (g, h) *are compatible of type* (A).
Then f, g *and* h *have a unique common fixed point.*

1.6 Theorem *Suppose* f, g, h *and* k *are four self mappings of a complete metric space* $(\mathcal{X}, d)$ *into itself satisfying the conditions:*
(i) $f(\mathcal{X}) \subset k(\mathcal{X})$, $g(\mathcal{X}) \subset h(\mathcal{X})$.

$$(ii)\ d(fx, gy) \leq \alpha d(hx, ky) + \beta[d(fx, hx) + d(gy, ky)] + \gamma[d(hx, gy) + d(ky, fx)],$$

for all x, $y \in \mathcal{X}$ *and* α, β *and* γ *be non-negative reals such that* $\alpha + 2\beta + 2\gamma < 1$.
(iii) One of f, g, h *and* k *is continuous.*
(iv) (f, h) *and* (g, k) *are compatible of type* (A).
Then f, g, h *and* k *have a unique common fixed point.*

Note that compatible mappings of type (A) are occasionally weakly compatible but, the converse is not true in general. The following example shows this fact.

1.7 Example *Let* $\mathcal{X} = [0, \infty[$ *with the usual metric. Define* f, $g : \mathcal{X} \to \mathcal{X}$ *by:*

$$fx = \begin{cases} 0 & \text{if } x \in [0,1[\\ x^3 & \text{if } x \in [1,\infty[, \end{cases} \quad gx = \begin{cases} 2x & \text{if } x \in [0,1[\\ \dfrac{1}{x^2} & \text{if } x \in [1,\infty[. \end{cases}$$

We have $f(1) = 1 = g(1)$ *and* $fg(1) = 1 = gf(1)$.
Now, consider the sequence $x_n = 1 + \dfrac{1}{n}$ *for* $n \in \{1, 2, \ldots\}$. *We have* $fx_n = x_n^3 \to 1$ *and* $gx_n = \dfrac{1}{x_n^2} \to 1$ *as* $n \to \infty$. *But,*

$$d(fgx_n, ggx_n) \to 2 \neq 0.$$

Therefore, f *and* g *are occasionally weakly compatible but not compatible of type* (A).

For our main results we need the following definition:

1.8 Definition *Let* $\mathcal{X}$ *be a set. A symmetric on* $\mathcal{X}$ *is a mapping* $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *such that*

$$d(x, y) = 0 \text{ iff } x = y, \text{ and } d(x, y) = d(y, x) \text{ for } x, y \text{ in } \mathcal{X}.$$

2 Main Results

2.1 A unique common fixed point for three mappings

2.1 Theorem *Let* $\mathcal{X}$ *be a set with a symmetric* d. *Suppose* f, g *and* h *are three self mappings of* $(\mathcal{X}, d)$ *satisfying the conditions:*

$$(1)\ \int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{\alpha d(hx,hy)+\beta[d(fx,hx)+d(gy,hy)]+\gamma[d(hx,gy)+d(hy,fx)]} \varphi(t)dt,$$

for all x, $y \in \mathcal{X}$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable non-negative such that $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon > 0$ and α, β, γ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$,
(2) pair of mappings (f, h) or (g, h) is owc.
Then f, g and h have a unique common fixed point.

Proof
Suppose that f and h are owc, then, there is an element $u \in \mathcal{X}$ such that $fu = hu$ and $fhu = hfu$.
First, we prove that $fu = gu$. Indeed, by using inequality (1), we get

$$\begin{aligned}\int_0^{d(fu,gu)} \varphi(t)dt &\leq \int_0^{\alpha d(hu,hu)+\beta[d(fu,hu)+d(gu,hu)]+\gamma[d(hu,gu)+d(hu,fu)]} \varphi(t)dt \\ &= \int_0^{(\beta+\gamma)d(fu,gu)} \varphi(t)dt < \int_0^{d(fu,gu)} \varphi(t)dt\end{aligned}$$

which is a contradiction, hence, $gu = fu = hu$.
Again, suppose that $ffu \neq fu$. The use of condition (1) gives

$$\int_0^{d(ffu,gu)} \varphi(t)dt \leq \int_0^{\alpha d(hfu,hu)+\beta[d(ffu,hfu)+d(gu,hu)]+\gamma[d(hfu,gu)+d(hu,ffu)]} \varphi(t)dt;$$

i.e.,

$$\int_0^{d(ffu,fu)} \varphi(t)dt \leq \int_0^{(\alpha+2\gamma)d(ffu,fu)} \varphi(t)dt < \int_0^{d(ffu,fu)} \varphi(t)dt,$$

this contradiction implies that $ffu = fu = hfu$.
Now, suppose that $gfu \neq fu$. By inequality (1) we have

$$\int_0^{d(fu,gfu)} \varphi(t)dt \leq \int_0^{\alpha d(hu,hfu)+\beta[d(fu,hu)+d(gfu,hfu)]+\gamma[d(hu,gfu)+d(hfu,fu)]} \varphi(t)dt;$$

that is,

$$\int_0^{d(fu,gfu)} \varphi(t)dt \leq \int_0^{(\beta+\gamma)d(fu,gfu)} \varphi(t)dt < \int_0^{d(fu,gfu)} \varphi(t)dt,$$

the above contradiction implies that $gfu = fu$. Put $fu = gu = hu = t$, so, t is a common fixed point of mappings f, g and h.
Now, let t and z be two distinct common fixed points of mappings f, g and h; i.e., $ft = gt = ht = t$ and $fz = gz = hz = z$. As $t \neq z$, then, $d(t, z) > 0$. From condition (1) we have

$$\begin{aligned}\int_0^{d(t,z)} \varphi(t)dt &= \int_0^{d(ft,gz)} \varphi(t)dt \\ &\leq \int_0^{\alpha d(ht,hz)+\beta[d(ft,ht)+d(gz,hz)]+\gamma[d(ht,gz)+d(hz,ft)]} \varphi(t)dt \\ &= \int_0^{(\alpha+2\gamma)d(t,z)} \varphi(t)dt < \int_0^{d(t,z)} \varphi(t)dt\end{aligned}$$

hence $d(t,z)=0$ and $z=t$. Thus the common fixed point is unique. ■

If we put $\varphi(t)=1$ in the above theorem, we get the following result.

2.2 Corollary *(Theorem 2.1 of [1] improved) Let $\mathcal{X}$ be a set with a symmetric d and let f, g and h be three self mappings of $(\mathcal{X},d)$ such that:*

(1) $d(fx,gy)\le \alpha d(hx,hy)+\beta[d(fx,hx)+d(gy,hy)]+\gamma[d(hx,gy)+d(hy,fx)]$

for all x, $y\in\mathcal{X}$ and α, β, γ are non-negative reals such that $\alpha+2\beta+2\gamma<1$,
(2) f and h or g and h are owc.
Then f, g and h have a unique common fixed point.

2.2 A unique common fixed point for four mappings

Now, we give our second main result.

2.3 Theorem *Let $\mathcal{X}$ be a set endowed with a symmetric d. Suppose f, g, h and k are four self mappings of $(\mathcal{X},d)$ satisfying the following conditions:*

$$(1)\ \int_0^{d(fx,gy)}\varphi(t)dt\le\int_0^{\alpha d(hx,ky)+\beta[d(fx,hx)+d(gy,ky)]+\gamma[d(hx,gy)+d(ky,fx)]}\varphi(t)dt,$$

for all x, $y\in\mathcal{X}$, where $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable non-negative such that $\int_0^{\epsilon}\varphi(t)dt>0$ for each $\epsilon>0$ and α, β, γ be non-negative reals such that $\alpha+2\beta+2\gamma<1$,
(2) pairs of mappings (f,h) and (g,k) are owc.
Then f, g, h and k have a unique common fixed point.

Proof
Since pairs of mappings (f,h) and (g,k) are owc, then, there exist two points u and v in $\mathcal{X}$ such that $fu=hu$ and $fhu=hfu$, $gv=kv$ and $gkv=kgv$.
First, we prove that $fu=gv$. Indeed, from inequality (1) we have

$$\begin{aligned}\int_0^{d(fu,gv)}\varphi(t)dt &\le \int_0^{\alpha d(hu,kv)+\beta[d(fu,hu)+d(gv,kv)]+\gamma[d(hu,gv)+d(kv,fu)]}\varphi(t)dt\\ &= \int_0^{(\alpha+2\gamma)d(fu,gv)}\varphi(t)dt<\int_0^{d(fu,gv)}\varphi(t)dt\end{aligned}$$

a contradiction. Hence, $d(fu,gv)=0$ and $fu=hu=gv=kv$.
Now, suppose that $ffu=fhu=hfu\ne fu$. Then, we have

$$\int_0^{d(ffu,gv)}\varphi(t)dt\le\int_0^{\alpha d(hfu,kv)+\beta[d(ffu,hfu)+d(gv,kv)]+\gamma[d(hfu,gv)+d(kv,ffu)]}\varphi(t)dt;$$

that is,

$$\int_0^{d(ffu,fu)}\varphi(t)dt\le\int_0^{(\alpha+2\gamma)d(ffu,fu)}\varphi(t)dt,$$

hence $d(ffu, fu) = 0$ i.e. $ffu = fu$, since $\alpha + 2\gamma < 1$. Thus $ffu = hfu = fu$. Similarly $gfu = kfu = fu$. Put $fu = t$, therefore t is a common fixed point of mappings f, g, h and k.
Let t and z be two different common fixed points of mappings f, g, h and k. i.e. $ft = gt = ht = kt = t$ and $fz = gz = hz = kz = z$. By condition (1)

$$\begin{aligned}\int_0^{d(t,z)} \varphi(t)dt &= \int_0^{d(ft,gz)} \varphi(t)dt \\ &\leq \int_0^{\alpha d(ht,kz)+\beta[d(ft,ht)+d(gz,kz)]+\gamma[d(ht,gz)+d(kz,ft)]} \varphi(t)dt \\ &= \int_0^{(\alpha+2\gamma)d(t,z)} \varphi(t)dt < \int_0^{d(t,z)} \varphi(t)dt,\end{aligned}$$

since $\alpha + 2\gamma < 1$, we have $z = t$. Hence the proof. ∎

Putting $\varphi(t) = 1$ in the above theorem, we obtain the next result.

2.4 Corollary *(Theorem 2.2 of [1] improved) Let d be a symmetric of a set $\mathcal{X}$. Suppose f, g, h and k are four self mappings of $(\mathcal{X}, d)$ satisfying the conditions:*

(1) $d(fx, gy) \leq \alpha d(hx, ky) + \beta[d(fx, hx) + d(gy, ky)] + \gamma[d(hx, gy) + d(ky, fx)]$

for all x, $y \in \mathcal{X}$ and α, β, γ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$,
(2) (f, h) and (g, k) are owc.
Then f, g, h and k have a unique common fixed point.

Before giving an example which illustrates and shows the generality of our results against the other results, we present the following corollary which whose proof is as easy as the proof of the theorem.

2.5 Corollary *Endowe the set $\mathcal{X}$ with a symmetric d. Suppose f, g, h and k are four self mappings of $(\mathcal{X}, d)$ satisfying the following conditions:*

$$(1) \quad \int_0^{d(fx,gy)} \varphi(t)dt \leq \int_0^{\alpha d(hx,ky)+\beta[d(hx,gy)+d(ky,fx)]} \varphi(t)dt,$$

for all x, $y \in \mathcal{X}$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable non-negative such that $\int_0^\epsilon \varphi(t)dt > 0$ for each $\epsilon > 0$ and α, β be non-negative reals such that $\alpha + 2\beta < 1$,
(2) f and h as well as g and k are owc.
Then f, g, h and k have a unique common fixed point.

2.6 Remark *It is clear that, our results improve and extend those of Aage and Salunke, the references therein and other results because we are removed the inclusions between the images of the mappings, and we are weaken several conditions on the space $(\mathcal{X}, d)$, the contractive condition and all the mappings by using the integral* *type. And as every contractive or strict contractive condition of integral type automatically includes a corresponding contractive or strict contractive condition, not involving integrals, by siting $\varphi(t) = 1$ over $\mathbb{R}^+$. So, results of [1] and the references therein become special cases of our results.*

The following example support our results.

2.7 Example *Let $\mathcal{X} = [0, \infty[$ with the symmetric $d(x, y) = (x - y)^2$. Define*

$$fx = gx = \begin{cases} 0 & \text{if } x \in [0,1[\\ 1 & \text{if } x \in [1,\infty[, \end{cases} \quad hx = \begin{cases} 3 & \text{if } x \in [0,1[\\ \dfrac{1}{x} & \text{if } x \in [1,\infty[, \end{cases}$$

and

$$kx = \begin{cases} 9 & \text{if } x \in [0,1[\\ \dfrac{1}{\sqrt{x}} & \text{if } x \in [1,\infty[. \end{cases}$$

First, it is clear to see that d is not a metric and mappings f, g, h and k are discontinuous at $x = 1$. Also f and h as well as g and k are occasionally weakly compatible. Take $\varphi(x) = 3x^2$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{5}$, $\gamma = \frac{1}{6}$, we have

(1) *For x, $y \in [0, 1[$, we have $fx = gy = 0$, $hx = 3$, $ky = 9$ and*

$$\begin{aligned} \int_0^{d(fx,gy)} \varphi(t)dt &= \int_0^0 3t^2 dt = 0 \\ &\leq \int_0^{\alpha d(hx,ky)+\beta[d(fx,hx)+d(gy,ky)]+\gamma[d(hx,gy)+d(ky,fx)]} 3t^2 dt \\ &= (42)^3. \end{aligned}$$

(2) *For x, $y \in [1, \infty[$, we have $fx = gy = 1$, $hx = \frac{1}{x}$, $ky = \frac{1}{\sqrt{y}}$ and*

$$\begin{aligned} \int_0^{d(fx,gy)} \varphi(t)dt &= \int_0^0 3t^2 dt = 0 \\ &\leq \int_0^{\alpha d(hx,ky)+\beta[d(fx,hx)+d(gy,ky)]+\gamma[d(hx,gy)+d(ky,fx)]} 3t^2 dt \\ &= \left[\frac{1}{4}\left(\frac{1}{x} - \frac{1}{\sqrt{y}}\right)^2 + \frac{11}{30}\left[\left(1 - \frac{1}{x}\right)^2 + \left(1 - \frac{1}{\sqrt{y}}\right)^2\right]\right]^3. \end{aligned}$$

(3) *For* $x \in [0,1[$, $y \in [1,\infty[$, *we have* $fx = 0$, $gy = 1$, $hx = 3$, $ky = \frac{1}{\sqrt{y}}$ *and*

$$\begin{aligned}
&\int_0^{d(fx,gy)} \varphi(t)dt = \int_0^1 3t^2 dt = 1 \\
\leq & \int_0^{\alpha d(hx,ky)+\beta[d(fx,hx)+d(gy,ky)]+\gamma[d(hx,gy)+d(ky,fx)]} 3t^2 dt \\
= & \left[\frac{1}{4}\left(3-\frac{1}{\sqrt{y}}\right)^2 + \frac{1}{5}\left(9+\left(1-\frac{1}{\sqrt{y}}\right)^2\right) + \frac{1}{6}\left(4+\frac{1}{y}\right)\right]^3 \\
= & \left[\frac{59}{12} + \frac{37}{60y} - \frac{19}{10\sqrt{y}}\right]^3.
\end{aligned}$$

(4) *Finally, for* $x \in [1,\infty[$, $y \in [0,1[$, *we have* $fx = 1$, $gy = 0$, $hx = \frac{1}{x}$, $ky = 9$ *and*

$$\begin{aligned}
&\int_0^{d(fx,gy)} \varphi(t)dt = \int_0^1 3t^2 dt = 1 \\
\leq & \int_0^{\alpha d(hx,ky)+\beta[d(fx,hx)+d(gy,ky)]+\gamma[d(hx,gy)+d(ky,fx)]} 3t^2 dt \\
= & \left[\frac{1}{4}\left(9-\frac{1}{x}\right)^2 + \frac{1}{5}\left(81+\left(1-\frac{1}{x}\right)^2\right) + \frac{1}{6}\left(64+\frac{1}{x^2}\right)\right]^3 \\
= & \left[\frac{1}{4}\left(9-\frac{1}{x}\right)^2 + \frac{1}{5}\left(1-\frac{1}{x}\right)^2 + \frac{1}{6x^2} + \frac{403}{15}\right]^3.
\end{aligned}$$

So, all the hypotheses of Theorem 2.3 are satisfied and 1 is the unique common fixed point of mappings f, g, h *and* k.

Now, consider the sequence $x_n = 1 + \frac{1}{n}$ *for* $n = 1, 2, \ldots$. *We have* $fx_n = 1 = gx_n$, $hx_n = \frac{1}{x_n} \to 1$ *and* $kx_n = \frac{1}{\sqrt{x_n}} \to 1$ *as* $n \to \infty$. *Also, we have*

$$\begin{aligned}
d(fhx_n, hhx_n) &\to 9 \neq 0, \\
d(gkx_n, kkx_n) &\to 81 \neq 0;
\end{aligned}$$

that is, neither f *and* h *nor* g *and* k *are compatible of type* (A). *Again, we have* $f(\mathcal{X}) = \{0,1\} \nsubseteq k(\mathcal{X}) =]0,1] \cup \{9\}$ *and* $g(\mathcal{X}) = \{0,1\} \nsubseteq h(\mathcal{X}) =]0,1] \cup \{3\}$.

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Laboratoire de Mathématiques Appliquées, Université Badji Mokhtar, B. P. 12, 23000, Annaba, Algérie
E-mail: b_hakima2000@yahoo.fr