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FUGLEDE AND ELEMENTARY OPERATORS ON BANACH SPACE

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Abstract

We generalize the notion of Fuglede-Putnam's property to general $*-$Banach algebra in the sense of Fuglede operator and study the elementary operator of length ≤ 2 in the context of this property

1 Introduction

Suppose $\mathcal{A}$ is a complex linear algebra, with identity 1: then an involution $*: \mathcal{M} \to \mathcal{M}$ on a linear subspace $\mathcal{M} \subseteq \mathcal{A}$ is a mapping which is conjugate linear and self inverting: for each $x, y \in \mathcal{M}$ and each $\alpha, \beta \in \mathbb{C}$

$$(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*; \ (x^*)^* = x. \tag{1.1}$$

We shall describe $x \in \mathcal{A}$ as hermitian, whenever

$$x \in \mathcal{M} \text{ and } x^* = x. \tag{1.2}$$

It is easily checked that

$$H + iH = \mathcal{M}; \ H \cap iH = \{0\}. \tag{1.3}$$

The canonical example, when $\mathcal{A}$ is a Banach algebra, comes from the numerical range: $x \in \mathcal{A}$ is said to be hermitian provided

$$\mathcal{V}_{\mathcal{A}}(x) = \{\varphi(a) : \varphi \in \text{state}(\mathcal{A})\} \subseteq \mathbb{R}; \tag{1.4}$$

here state($\mathcal{A}$) consists of the linear functionals $\varphi \in \mathcal{A}^*$ for which $\|\varphi\| = 1 = \varphi(1)$. It is well known ([4] Lemma 5.2) that

$$x \in \mathcal{A} \ hermitian \Longleftrightarrow \forall t \in \mathbb{R} : \left\|e^{itx}\right\| = 1. \tag{1.5}$$

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It is also known ([4] Lemma 5.7) that if $H = H_{\mathcal{A}}$ denotes the hermitian elements of $\mathcal{A}$ in the sense of (1.4) then the second part of (1.3) holds: thus if we define the space $\mathcal{M} = H + iH$ as in the first part of (1.3) we can define an involution $* : \mathcal{M} \to \mathcal{M}$ by setting

$$(h + ik)^* = h - ik \ (h, k \in H). \tag{1.6}$$

If $* : \mathcal{M} \to \mathcal{M} \subseteq A$ is an involution we define $a \in \mathcal{A}$ to be normal iff

$$a \in \mathcal{M} \text{ and } a^*a = aa^* \in \mathcal{A} : \tag{1.7}$$

note that is not necessary that $a^*a \in \mathcal{M}$. Equivalently, with respect to (1.6),

$$a = h + ik \text{ with } h, k \in H \text{ and } hk = kh. \tag{1.8}$$

Let $\mathcal{A} = B(\mathcal{H})$ be the algebra of all bounded operators acting on a complex separable Hilbert space $\mathcal{H}$ and $A, B \in B(\mathcal{H})$, we say that the pair (A, B) satisfies the Fuglede-Putnam's property if $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$ where $\delta_{A,B}$ denotes the generalized derivation defined on $B(\mathcal{H})$ by $\delta_{A,B}(X) = AX - XB$.

Many mathematicians have extended this property for several classes of operators. For detailed study for this property, the reader is referred to [2, 3, 6, 9, 15].

In this note we wish to discuss the "Fuglede-Putnam property" in algebras $\mathcal{A} = B(\mathcal{X})$ for Banach spaces $\mathcal{X}$, in particular for "elementary operators".

We will use the following further notations, the range of an operator $T \in B(\mathcal{X})$ is denoted by ran T and the commutator $AB - BA$ is denoted by $[A, B]$. The set of complex numbers is denoted by $\mathbb{C}$.

2 Fuglede Operators

Suppose $* : \mathcal{M} \to \mathcal{M} \subseteq \mathcal{A}$ is an involution in the sense (1.1) and suppose in particular that $\mathcal{A} = B(\mathcal{X})$ for a Banach space $\mathcal{X}$: then

Definition 2.1 *We define $T \in \mathcal{M} \subseteq B(\mathcal{X})$ to be*
Fuglede iff

$$ker\ T \subseteq ker\ T^*; \tag{2.1}$$

reduced iff

$$ker\ T \subseteq ker\ TT^*; \tag{2.2}$$

natural iff

$$ker\ TT^* = ker\ T^*. \tag{2.3}$$

These definitions come from [10], following an idea of Shulman and Turowska [13]; in [10, Definition 6] the condition (2.3) was described by saying that T^* was " ultra weakly *-orthogonal". We remark that if $\mathcal{X}$ is a Hilbert space then every operator $T \in \mathcal{A}$ satisfies (2.3). An equivalent version of (2.3) is that ker $T \cap$ran $T^* = \{0\}$. The simplest relationships between the concepts of Definition 2.1 are

Theorem 2.2 *If $T \in \mathcal{A} = B(\mathcal{X})$ for a Banach space $\mathcal{X}$ then*

$$T \text{ natural and reduced } \Longrightarrow T \text{ Fuglede} \Longrightarrow T \text{ reduced.} \tag{2.4}$$

Also

$$T \text{ normal } \Longrightarrow T \text{ reduced.} \tag{2.5}$$

Proof. If T is natural and reduced then ker $T \subseteq$ ker TT^* giving (2.1). If T is normal then ker $T^* \subseteq$ ker $TT^* =$ ker T^*T, giving (2.2). ■

Note that the normality need not in general, imply the Fuglede property.

3 Elementary Operators

If $a \in \mathcal{A}$ we define left and right multiplication operators by setting, for each $x \in \mathcal{A}$,

$$L_a(x) = ax; \;\; R_a(x) = xa; \tag{3.1}$$

more generally if $a \in \mathcal{A}^n$ and $b \in \mathcal{A}^n$ are n-tuples the elementary operator $L_a \circ R_b : \mathcal{A} \to \mathcal{A}$ is defined by setting, for each $x \in \mathcal{A}$,

$$(L_a \circ R_b)(x) = \sum_{j=1}^{n} a_j x b_j. \tag{3.2}$$

The same operator $T = L_a \circ R_b$ can be given by many different pairs of tuples a and b: the minimum possible n is sometimes called the "length" of the operator. There is algebraic isomorphism between the linear space of elementary operators on $\mathcal{A}$ and the tensor product $\mathcal{A} \otimes \mathcal{A}$: thus if there is an involution $* : \mathcal{M} \to \mathcal{M} \subseteq \mathcal{A}$ it is possible to successfully define an involution on the subspace of those elementary operators $L_a \circ R_b$ for which $(a, b) \in \mathcal{M}^n \times \mathcal{M}^n$ by setting

$$(L_a \circ R_b)^* = L_{a^*} \circ R_{b^*}, \tag{3.3}$$

where we write for example $(a_1, a_2, \cdots, a_n)^* = (a_1^*, a_2^*, \cdots, a_n^*)$ if $a \in \mathcal{A}^n$. The most important examples of elementary operators are the "mixed derivation" $L_a - R_b$ for single elements $a, b \in \mathcal{A}$ and the products $L_a R_b$; Duggal [6] has looked in particular at the operator $L_a R_b - I$.

When $\mathcal{A} = B(\mathcal{X})$ for a Banach space $\mathcal{X}$ then an involution $* : \mathcal{M} \to \mathcal{M} \subseteq B(\mathcal{X})$ gives rise to a dual involution $* : \mathcal{M}^\dagger \to \mathcal{M}^\dagger = \{x^\dagger : x \in \mathcal{M}\} \subseteq B(\mathcal{X}^\dagger)$ defined by setting

$$(x^\dagger)^* = (x^*)^\dagger, \; (x \in \mathcal{M}). \tag{3.4}$$

In this section we consider the relationship between the Fuglede property for tuples $a \in \mathcal{A}^n$, $b \in \mathcal{A}^n$ and $L_a \circ R_b \in B(\mathcal{A})$: For example Duggal [7] has obtained the result if $\mathcal{A} = B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ and if a, b in $\mathcal{A}$ are normal and c, d^* are hyponormal, then

$$ac - ca = bd - db = 0 \Longrightarrow L_a R_b - L_c R_d \text{ Fuglede.} \tag{3.5}$$

Theorem 3.1 *If $\mathcal{M} \subseteq \mathcal{A} \subseteq B(\mathcal{X})$ for a Banach space $\mathcal{X}$ and if $a, b \in \mathcal{A}$ then*

$$a \in \mathcal{M} \text{ Fuglede} \Longleftrightarrow L_a \in B(\mathcal{A}) \text{ Fuglede}; \tag{3.6}$$

$$b^\dagger \in \mathcal{M}^\dagger \text{ Fuglede} \Longleftrightarrow R_b \in B(\mathcal{A}) \text{ Fuglede}; \tag{3.7}$$

$$a \in \mathcal{M} \text{ Fuglede and } b^\dagger \in \mathcal{M}^\dagger \text{ Fuglede} \implies L_a R_b \in B(\mathcal{A}) \text{ Fuglede}. \tag{3.8}$$

Proof. If $x \in \mathcal{A}$ is arbitrary then $ax = 0 \Longleftrightarrow \forall \xi \in \mathcal{X} : ax\xi = 0$ and if a is Fuglede it follows $a^* x\xi = 0$ so $a^* x = 0$: thus L_a is also Fuglede. Conversely if $x \in \mathcal{X}$ and $\varphi \in X^\dagger$ are arbitrary and if L_a is Fuglede, we obtain the following implication

$$L_a(\varphi \otimes x) = 0 \Longrightarrow \varphi \otimes a^* x = (L_a)^*(\varphi \otimes x).$$

In particular $ax = 0$ then (3.6) holds for all $\varphi \in \mathcal{X}^\dagger$, giving $a^* x = 0$ by Hahn-Banach.

Towards (3.7), if $xb = 0$ then

$$\forall \varphi \in \mathcal{X}^\dagger : b^\dagger(\varphi x) = \varphi x b,$$

giving if $b^\dagger$ is Fuglede

$$\varphi x b^* = (b^*)^\dagger(\varphi x) = (b^\dagger)^*(\varphi x) = 0$$

and hence by Hahn-Banach's theorem $R_b^* x = xb^* = 0$. Conversely if $b^\dagger \varphi = 0 \in \mathcal{X}^\dagger$ then for arbitrary $x \in \mathcal{X}$ we have $(\varphi \otimes x)b = 0$ and hence if R_b is Fuglede $(\varphi \otimes x)b^* = 0$. Since $x \in \mathcal{X}$ is arbitrary it follows $\varphi b^* = (b^\dagger)^* \varphi = 0$.

Finally for (3.8) suppose $L_a(xb) = (L_a R_b)x = 0$: if $a \in \mathcal{A}$ and $L_a \in B(\mathcal{A})$) are Fuglede, this yields $R_b(a^* x) = a^*(xb) = 0$. Also if $b^\dagger$ and R_b are Fuglede, we get $(L_a R_b)^*(x) = R_b^*(a^* x) = 0$. ■

Proposition 3.2 *If $A, B \in \mathcal{M} \subseteq B(\mathcal{X})$ with the involution defined by (1.6) then,*

(i) A Fuglede $\Leftrightarrow L_A$ Fuglede

(ii) $B^\dagger$ Fuglede $\Leftrightarrow R_B$ Fuglede

(iii) A, $B^\dagger$ are Fuglede $\Rightarrow M_{A,B}$ Fuglede.

Proof. If $\mathcal{A} = B(\mathcal{X})$ where $\mathcal{X}$ is a Banach spacec and $\mathcal{M} = \mathcal{H} + i\mathcal{H}$ is equipped with the involution $*$ in the sense of (1.6) then we can check easily that $(\mathcal{M})^\dagger = \mathcal{H}^\dagger + i\mathcal{H}^\dagger$ and the dual involution $\star$ of $*$ is given by

$$\forall h, k \in \mathcal{H} : \left(h^\dagger + ik^\dagger\right)^\star = h^\dagger - ik^\dagger. \tag{3.9}$$

The results follow immediately from the Theorem 3.1. ■

Let $\mathcal{A}$ be a Banach algebra with unit 1.

Theorem 3.3 *([1, 8, 11]).*
For $a, b \in \mathcal{A}$ we have the following statements.

(i) a, b *hermitian elements* $\Rightarrow L_a, R_b$ *hermitian operators* $\Rightarrow \delta_{a,b}$ *hermitian*

(ii) a, b *normal elements* $\Rightarrow L_a, R_b$ *normal operators* $\Rightarrow \delta_{a,b}$ *normal*

(iii) *if* $\mathcal{A} = B(\mathcal{X})$*;* a *normal* $\Rightarrow a$ *Fuglede.*

As a consequence, if $a = h + ik$ is normal and $b \in \mathcal{A}$, then

$$[a, b] = 0 \Leftrightarrow [h, b] = [k, b] = 0.$$

Proposition 3.4 *If* a, b *are normal operators in* $B(\mathcal{X})$ *and* x *any element in* $\mathcal{A} = B(\mathcal{X})$*, then*

$$M^2_{a,b}x = 0 \Rightarrow M_{a,b}x = 0.$$

Proof. If a, b are hermitian operators then we can check easily that, for arbitrary $r \in \mathbb{R}$ and all $x \in \mathcal{A}$, $\|x\| = \left\|e^{ira}xe^{irb}\right\|$. Let

$$\begin{aligned} e^{ira} &= 1 + ira + K_a : K_a = \sum_{n=2}^{\infty} \frac{(ira)^n}{n!} \\ e^{irb} &= 1 + irb + K_b : K_b = \sum_{n=2}^{\infty} \frac{(irb)^n}{n!}. \end{aligned}$$

Suppose, for hermitian a, b and $x \in \mathcal{A}$ that $M^2_{a,b}x = 0$, then

$$a^n x b^m = 0 (m, n \geq 2).$$

Hence, $K_a x K_b = 0$ and therefore, we can leave in the expansion of $\left\|e^{ira}xe^{irb}\right\|$:

$$\begin{aligned} \|x\| &= \left\|e^{ira}x(1 + irb) + (1 + ira)xe^{irb} - (1 + ira)x(1 + irb)\right\| \\ &= \left\|r^2 axb - ir\,(ax + xb) - x + e^{ira}x(1 + irb) + (1 + ira)xe^{irb}\right\|, \end{aligned}$$

for all $r > 0$.
Consequently if

$$\left\|r^2 axb\right\| \leq \left\|ir\,(ax + xb) - x + e^{ira}x(1 + irb) + (1 + ira)xe^{irb}\right\|$$

then,

$$\|axb\| \leq \frac{1}{r^2}\left[r\,\|ax + xb\| + \|x\| + \|x(1 + irb)\| + \|(1 + ira)x\|\right] \tag{3.10}$$

If not, we have

$$\left\|r^2 axb\right\| \leq \left\|ir\,(ax + xb) - x + e^{ira}x(1 + irb) + (1 + ira)xe^{irb}\right\| + \|x\|$$

and

$$\|axb\| \leq \frac{1}{r^2}\left[r\,\|ax + xb\| + 2\,\|x\| + \|x(1 + irb)\| + \|(1 + ira)x\|\right]. \tag{3.11}$$

From the equations (3.10) and (3.11), we conclude that $axb = 0$.

If a, b are normal elements with $a = h_1 + ik_1, b = h_2 + ik_2$. Then, by Theorems (3.1) and (3.3), L_a, R_b are Fuglede operators and so it follows from $a^2xb^2 = 0$ that

$$a^{*2}xb^{*2} = aa^*xb^2 = a^2xb^{*2} = a^{*2}xb^2 = a^2xbb^* = 0.$$

Hence,

$$(a^* \pm a)^2x(b^* \pm b)^2 = 0.$$

Using the first case, we get

$$(a^* \pm a)x(b^* \pm b) = 0.$$

This yields

$$h_1xh_2 = h_1xk_2 = h_1xh_2 = k_1xh_2 = k_1xk_2 = 0.$$

Therefore $axb = 0$. ■

Corollary 3.5 *If a, b are normal operators in $B(\mathcal{X})$ then*

$$\ker M_{a,b} \cap ran\ M_{a,b} = \{0\}.$$

Proposition 3.6 *If $\mathcal{A} = B(\mathcal{X})$ where $\mathcal{X}$ is a Banach space and $T \in B(\mathcal{X})$ is a normal operator, then T is a natural operator.*

Proof. Let $\mathcal{X}^\dagger$ be the dual of $\mathcal{X}$ and $T^\dagger$ be the dual adjoint of $T \in B(\mathcal{X})$. With respect to the involution (1.6) and its dual (3.9), we have that $T^\dagger$ is normal. So that $T^\dagger$ and T are Fuglede operators and by duality we get $\overline{ran\ T} = \overline{ran\ T^*}$. Using [8] we get $\ker T \cap ran\ T^* = \{0\}$. Thus, $\ker TT^* = \ker T^*$ which means that T is a natural operator. ■

Consequently for $T \in B(\mathcal{X})$, we have

$$T \text{ normal} \Rightarrow T \text{ Fuglede} \Rightarrow T \text{ reduced} \quad (3.12)$$

$$T \text{ normal} \Rightarrow T \text{ natural}. \quad (3.13)$$

In what follows we show that the elmentary operator L_aR_b induced by hermitians elements is not necessarily a hermitian operator.

Lemma 3.7 *[14], Let T be a bounded linear operator on $B(H)$, for a Hilbert space H. Then T is hermitian if and only if there exist two self-adjoints operators A, $B \in B(H)$ such that $T = L_A + \delta_B$.*

Proposition 3.8 *Let $A, B \in B(H)$ be a self-adjoints operators. If A and B are not scalar operators then $M_{A,B}$ is not hermitian operator.*

Proof. If $M_{A,B}$ is a hermitian operator, then by Lemma 3.7, $M_{A,B} = L_{AB} + \delta_R$ where R is a self-adjoint operator. Hence,

$$\forall X \in B(H) : AXB - ABX = XBA - BXA.$$

Therefore,

$$\forall X \in B(H) : A(XB - BX) - (XB - BX)A = 0.$$

Thus,

$$A\delta_B - \delta_B A = 0.$$

Which means that $\delta_A \delta_B = \delta_I$ (I denotes the identity operator), by [16] it follows that either A or B is a scalar. Contradiction to our assumptions. ■

Remark 3.9 *Theorem 3.3, showed that the hermitian and normal properties are preserved for L_A and R_B and their sum but not preserved for the product $L_A R_B$ (Proposition 3.8). However, Theorem 3.1, showed that the Fuglede property is preserved for L_A, R_B, their sum and their product for an arbitrary involution.*

Let $\mathcal{A}$ be a Banach algebra with unit e and E be the elementary operator defined on $\mathcal{A}$ by $E = M_{a_1,b_1} + M_{a_2,b_2}$.

The following result generalizes Rosenblum's Theorem [11].

Proposition 3.10 *If $(a_1, a_2), (b_1, b_2)$ are 2-tuples of commuting normal elements in $\mathcal{A}^2$, then E is a Fuglede operator.*

Proof. If $a_1 x b_1 = a_2 x b_2$, for $x \in \mathcal{A}$ then by induction, $a_1^n x b_1^m = a_2^n x b_2^m$, for all $n, m \in \mathbb{N}$. Hence,

$$\exp(a_1) x \exp(b_1) = \exp(a_2) x \exp(b_2) \tag{3.14}$$

Let $a_i = h_i + ik_i$ and $b_i = v_i + iu_i$, $i = 1, 2$ where h_i, k_i, v_i and $u_i \in \mathcal{H}_{\mathcal{A}}$. Set

$$c_i = \exp(a_i - a_i^*),\ d_i = \exp(b_i - b_i^*),\ \ i = 1, 2. \tag{3.15}$$

Then,

$$c_i = \exp(2ik_i), d_i = \exp(2iu_i) \text{ and } \|c_i\| = \|d_i\| = 1,\ i = 1, 2. \tag{3.16}$$

By (3.14) and $[a_1, a_2] = [b_1, b_2] = 0$, we get

$$x = \exp(-a_1) \exp(a_2) x \exp(b_2) \exp(-b_1). \tag{3.17}$$

From equations (3.15), (3.17), we obtain

$$c_1 c_2^{-1} x d_2^{-1} d_1 = \exp(-a_1^*) \exp(a_2^*) x \exp(b_2^*) \exp(-b_1^*)$$

and by (3.16),

$$\|\exp(-a_1^*) \exp(a_2^*) x \exp(b_2^*) \exp(-b_1^*)\| \le \|x\|. \tag{3.18}$$

Let f be the function from $\mathbb{C}$ to $\mathcal{A}$ defined by

$$f(z) = \exp[z(a_2^* - a_1^*)]x \exp[z(b_2^* - b_1^*)].$$

Clearly f is an entire function and by (3.18) f is bounded on the whole field $\mathbb{C}$. So by Liouville's Theorem, f is a constant function on $\mathbb{C}$.
Hence, for all $z \in \mathbb{C}, f(z) = f(0) = x$. Therefore

$$\exp[z(a_2^* - a_1^*)]x \exp[z(b_2^* - b_1^*)] = x, \text{ for all } z \in \mathbb{C}$$

and

$$\exp(za_1^*)x \exp(zb_1^*) = \exp(za_2^*)x \exp(zb_2^*), \text{ for all } z \in \mathbb{C}.$$

Thus

$$\sum_{n,k=0}^{\infty} \frac{z^{n+k}}{n!k!} \left(a_1^{*n} x b_1^{*k} - a_2^{*n} x b_2^{*k}\right) = \sum_{m=0}^{\infty} \frac{z^m}{n!k!} \sum_{n+k=m} \left(a_1^{*n} x b_1^{*k} - a_2^{*n} x b_2^{*k}\right) = 0.$$

Finally, we get for all $(n,k) \in \mathbb{N}^2$, $a_1^{*n} x b_1^{*k} = a_2^{*n} x b_2^{*k}$.
In particular, for $n = k = 1$, $x \in \ker E^*$. ■

The following corollary generalizes the result given by Brooke, Brush and Pearson [5]

Corollary 3.11 *Let $(a_1, a_2), (b_1, b_2)$ be 2-tuples of commuting hermitian elements in $\mathcal{A}^2$ and $\lambda \in \mathbb{C}$. If $a_1 x b_1 = \lambda a_2 x b_2 \neq 0$, for certain element $x \in \mathcal{A}$ then $\lambda \in \mathbb{R}$.*

In particular, for $b_1 = a_2$ and $a_1 = b_2 = a$, if $ax = \lambda xa \neq 0$, then $\lambda \in \mathbb{R}$.

Proof. From the previous proposition we get $a_1 x b_1 = \overline{\lambda} a_2 x b_2 = \lambda a_2 x b_2$. Hence $\left(\overline{\lambda} - \lambda\right) a_2 x b_2 = 0$. Thus $\overline{\lambda} = \lambda$.

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■

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