Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

Filomat **23:3** (2009), 135–153 DOI:10.2298/FIL0903135P

GREEN'S FORMULA AND THE HARDY-STEIN IDENTITIES

Miroslav Pavlović*

Abstract

This is a collection of some known and some new facts on the holomorphic and the harmonic version of the Hardy-Stein identity as well as on their extensions to the real and the complex ball. For example, we prove that if f is holomorphic on the unit disk $\mathbb{D}$, then

$$\|f\|_{H^p} \asymp |f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2(1-|z|)\,dA(z), \qquad (\dagger)$$

where H^p is the p-Hardy space, which improves a result of Yamashita [Proc. Amer. Math. Soc. **75** (1979), no. 1, 69–72]. An extension of (†) to the unit ball of $\mathbb{C}^n$ improves results of Beatrous an Burbea [Kodai Math. J. **8** (1985), 36–51], and of Stoll [J. London Math. Soc. (2) **48** (1993), no. 1, 126–136]. We also prove the analogous result for the harmonic Hardy spaces. The proofs of known results are shorter and more elementary then the existing ones, see Zhu [*Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005, Ch. IV]. We correct some constants in that book and in a paper of Jevtić and Pavlović [Publ. Inst. Math. (Beograd) (N.S.) **64(78)** (1998), 36–52].

1 Introduction

In the simplest case Green's theorem states that if $g \in C^2(\mathbb{D})$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then

$$\frac{d}{dr}\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta = \frac{1}{2r}\int_{|z|<r} \Delta g(z)\,dA(z), \quad 0 < r < 1, \qquad (1)$$

or, what is the same,

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta = g(0) + \frac{1}{2}\int_{|z|<r} \Delta g(z)\log\frac{r}{|z|}\,dA(z), \quad 0 < r < 1, \qquad (2)$$

*Suported by Ministarstvo nauke Srbije, Project ON144010

2000 *Mathematics Subject Classifications.* 32A37, 32A36, 30D55.

Key words and Phrases. Hardy spaces, harmonic Hardy spaces, Hardy-Stein identities, Green's formula invariant Laplacian.
Received: March 26, 2009
Communicated by Ivan Jovanović

where dA is the normalized Lebesgue measure on $\mathbb{D}$, and Δ is the ordinary Laplacian,

$$\Delta g(z) = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4\frac{\partial^2 g}{\partial z \partial \bar{z}}, \quad z = x + iy.$$

Although the proof of (1) and (2) is short and elementary they have important consequences in the theory of Hardy spaces. The first one was proved by Hardy in [4]. In order to state it let $H(\mathbb{D})$ denote the class of all functions holomorphic in $\mathbb{D}$ and, for $0 < p < \infty$, define the Hardy class H^p by

$$H^p = \{f \in H(\mathbb{D}) : \|f\|_p \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty\},$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}.$$

For basic properties of H^p we refer to [3].

Theorem 1.A ($0 < p < \infty$). (i) *If $f \in H(\mathbb{D})$, then the function $r \mapsto M_p(r, f)$ is of class C^1 on the interval $(0, 1)$ and*

$$\frac{d}{dr}M_p^p(r, f) = \frac{p^2}{2r}\int_{|z|<r} |f(z)|^{p-2}|f'(z)|^2 \, dA(z), \tag{3}$$

and

$$M_p^p(r, f) = |f(0)|^p + \frac{p^2}{2}\int_{|z|<r} |f(z)|^{p-2}|f'(z)|^2 \log\frac{r}{|z|} \, dA(z).$$

(ii) *A function $f \in H(\mathbb{D})$ belongs to H^p if and only if*

$$H_p(f) := \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|} \, dA(z) \; < \infty,$$

and we have

$$\|f\|_p^p = |f(0)|^p + \frac{p^2}{2}H_p(f). \tag{4}$$

The analogous fact for harmonic functions was proved by P. Stein [24]. To state it denote by $h(\mathbb{B})$ the class of all real-valued functions harmonic in $\mathbb{D}$, and by h^p the harmonic Hardy class,

$$h^p = \{f \in H(\mathbb{D}) : \|f\|_p \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty\}.$$

Theorem 1.B ($1 < p < \infty$). (i) *If $u \in h(\mathbb{D})$, then the function $r \mapsto M_p(r, u)$, $0 < r < 1$, is of class C^1 and*

$$\frac{d}{dr}M_p^p(r, f) = \frac{p\,(p-1)}{2r}\int_{|z|<r} |u(z)|^{p-2}|\nabla u(z)|^2 \, dA(z), \tag{5}$$

and

$$M_p^p(r,u) = |u(0)|^p + \frac{p(p-1)}{2} \int_{|z|<r} |u(z)|^{p-2} |\nabla u(z)|^2 \log \frac{r}{|z|} \, dA(z). \tag{6}$$

(ii) *A function $u \in h(\mathbb{D})$ belongs to h^p if and only if*

$$S_p(u) := \int_{\mathbb{D}} |u(z)|^{p-2} |\nabla u(z)|^2 \log \frac{1}{|z|} \, dA(z) \ < \infty,$$

and we have

$$\|u\|_p^p = |u(0)|^p + \frac{p\,(p-1)}{2} S_p(u). \tag{7}$$

The function $u(x,y) = x$ shows that (7) is not valid for $p \le 1$.

Relations (7) and (4) are known as the Hardy-Stein identities (see [6, p. 41]).

Solving a problem of Bohr and Landau, Hardy [4] used Theorem 1.A to extend the Hadamard three circle theorem to the case of the integral means, while P. Stein [24] used Theorems 1.A and 1.B to prove the famous theorem of M. Riesz [23]: If $f = u + iv$ is holomorphic in $\mathbb{D}$, and if $u \in h^p$, where $1 < p < \infty$, then $f \in H^p$, and

$$\|f\|_p \le C_p \|u\|_p \quad (\text{if } f(0) = u(0)).$$

Indeed, if $1 < p \le 2$, then the Riesz theorem follows from (4) and (7) immediately because $|\nabla u| = |f'|$ and $|f|^{p-2} \le |u|^{p-2}$. Then a duality argument proves the Riesz theorem in the case $p \ge 2$.

As another application of (7) one can give a quick proof (see [18]) of the following sharpened variant of the Littlewood-Paley inequality [12]:

$$\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|)^{p-1} \, dA(z) \le C_p (\|u\|_p^p - |u(0)|^p), \quad p \ge 2.$$

(Usually, $\|u\|_p^p$ instead of $\|u\|_p^p - |u(0)|^p$ stands.)

For various generalizations and applications of the Hardy-Stein identity, see [29, Section 4] and [7, 8, 9, 10, 11, 13, 14, 17, 25, 27].

The following fact is a consequence of Theorem 1.A(ii) (see [28]).

Theorem 1.C ($0 < p < \infty$). *A function $f \in H(\mathbb{D})$ belongs to H^p if and only if*

$$\widehat{H}_p(f) := \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 (1-|z|^2) \, dA(z) \ < \infty.$$

In the case $p \ge 2$, we can state somewhat more, namely

$$H_p(f) \asymp \widehat{H}_p(f), \quad f \in H^p.\ ^{\dagger} \tag{8}$$

†We write $A(s) \asymp B(s)$ to indicate that there is a positive constant C independent of s such that $1/C \le A(s)/B(s) \le C$. Throughout the paper $s = u$ or $s = f$. Also, we use the letter C to denote a positive constant, independent of u and f, whose value may vary from one occurence to another.

This can easily be proved by using the subharmonicity of $|f|^{p-2}|f'|^2$ $(p \geq 2)$. However if $p < 2$, it is not clear whether (8) is valid or not. Furthermore, it is not clear whether the following weaker form of (8) holds for $p < 2$:

$$\|f\|_p^p \asymp |f(0)|^p + \widehat{H}_p(f), \quad f \in H^p.$$

Namely, we cannot apply the closed graph theorem because we do not know whether the quantity on the right-hand side is a norm (or even a quasinorm).

In the case of harmonic functions the situation is even worse: it is not clear whether the relation

$$\|u\|_p^p \asymp |u(0)|^p + \widehat{S}_p(u)$$

holds for any $p > 1$ (not only for $1 < p < 2$). It is surprising that nobody considered this questions. Here we prove:

Theorem 1.1 $(1 < p < \infty)$**.** *A function $u \in h(\mathbb{D})$ belongs to h^p if and only if*

$$\widehat{S}_p(u) := \int_{\mathbb{D}} |u(z)|^{p-2}|\nabla u(z)|^p(1-|z|^2)\, dA(z) \ < \infty.$$

Moreover, we have $|u(0)|^p + \widehat{S}_p(u) \asymp \|u\|_p^p$ $(1 < p < 2)$, and $\widehat{S}_p(u) \asymp S_p(u)$ $(p \geq 2)$.

The analogous theorem, in a generalized form, for holomorphic functions reads as follows.

Theorem 1.2 $(0 < p < \infty)$**.** *Let $\gamma \geq 0$. A function $f \in H(\mathbb{D})$ belongs to H^p if and only if*

$$\widehat{H}_{p,\gamma}(f) := \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^p|z|^\gamma(1-|z|^2)\, dA(z) \ < \infty.$$

Moreover, we have $|f(0)|^p + \widehat{H}_{p,\gamma}(f) \asymp \|f\|_p^p$ $(0 < p < 2)$ and $\widehat{H}_{p,\gamma}(f) \asymp H_p(f)$ $(p \geq 2)$.

The proof of Theorem 1.1 (resp. 1.2) is in Section 2 (resp. 3). An extension of Theorem 1.1 to the case of the unit ball in $\mathbb{R}^n$ is in Section 4. In Section 5 we use Theorem 1.2 prove a similar theorem in the case of several complex variables.

2 Proof of Theorem 1.1

Lemma 2.1 $(1 < p < \infty)$**.** *If $u \in h(\mathbb{D})$, then*

$$\|u\|_p^p = \int_{\mathbb{D}} |u|^p\, dA + \frac{p\,(p-1)}{4}\widehat{S}_p(u).$$

Proof. This is proved by multiplying (5) with r^2, then using partial integration on the left hand side and the Fubini's theorem on the right. For the (more general) case of $\mathcal{M}$-harmonic functions, see [7, Theorem 1.7]. $\square$

Lemma 2.2. *[19] If $u \in h(\mathbb{D})$ and $0 < \alpha, \beta < \infty$, then there exists a constant $C = C_{\alpha,\beta}$ such that*

$$\sup_{|z|<1/4} |u(z)|^\alpha |\nabla u(z)|^\beta \le C \int_{|z|<1/2} |u(z)|^\alpha |\nabla u(z)|^\beta \, dA(z).$$

In the terminology of [20], the lemma says that the function $|u|^\alpha |\nabla u|^\beta$ is quasi-nearly subharmonic. A function $g \ge 0$, defined and locally integrable on $\mathbb{D}$, is quasi-nearly subharmonic if there exists a constant C such that

$$g(z) \le \frac{C}{R^2} \int_{D_R(z)} g \, dA$$

whenever $D_R(z) : \{w : |w - z| < R\}$ is contained in $\mathbb{D}$.

Now we can prove Theorem 1.1 in the case $p \ge 2$. The inequality $\widehat{S}_p(u) \le CS_p(u)$, is an immediate consequence of (7) and the inequality $1 - t \le \log(1/t)$, $t > 0$. To prove the reverse inequality we use Lemma 2.2 with $\alpha = p - 2$, $\beta = 2$. It follows that

$$\begin{aligned}
S_p(u) = &\int_{|z|<1/4} |u(z)|^{p-2} |\nabla u(z)|^2 \log(1/|z|) \, dA(z) \\
&+ \int_{1/4<|z|<1} |u(z)|^{p-2} |\nabla u(z)|^2 \log(1/|z|) \, dA(z) \\
\le\ & C \sup_{|z|<1/4} |u(z)|^{p-2} |\nabla u(z)|^2 \int_{|z|<1/4} \log(1/|z|) \, dA(z) \\
&+ C \int_{1/4<|z|<1} |u(z)|^{p-2} |\nabla u(z)|^2 (1 - |z|) \, dA(z) \\
\le\ & C \int_{|z|<1/2} |u(z)|^{p-2} |\nabla u(z)|^2 \, dA(z) + C \, \widehat{S}_p(u) \\
\le\ & C \, \widehat{S}_p(u),
\end{aligned}$$

which was to be proved. (We have used Lemma 2.2 and the fact that $\int_{\mathbb{D}} \log(1/|z|) \, dA(z) < \infty$.)

For the proof in the case $1 < p \le 2$, we need another two lemmas. The first one is due to Hardy and Littlewood [5].

Lemma 2.3. *If $0 < p < \infty$, then*

$$\int_{\mathbb{D}} |u|^p \, dA \asymp |u(0)|^p + \int_{\mathbb{D}} |\nabla u(z)|^p (1 - |z|^2)^p \, dA(z), \quad u \in h(\mathbb{D}). \tag{9}$$

Lemma 2.4. *Let $1 < p \le 2$, and let $u \in h(\mathbb{D})$. If*

$$K_p(u) := \int_{\mathbb{D}} |u(z)|^{p-2} |\nabla u(z)|^2 (1 - |z|^2)^2 \, dA(z) < \infty,$$

then $\int_{\mathbb{D}} |u|^p \, dA < \infty$*, and there exists a constant* $C = C_p$ *such that*

$$\int_{\mathbb{D}} |u|^p \, dA \le C(|u(0)|^p + K_p(u)). \tag{10}$$

Remark 2.5. *The inequality* $|u(0)|^p + K_p(u) \le C \int_{\mathbb{D}} |u|^p \, dA$ *can easily be deduced from* (6)*. This gives a characterization of the Bergman space*

$$hA^p = \{u \in h(\mathbb{D}) : \int_{\mathbb{D}} |u|^p \, dA < \infty\}.$$

For similar characterizations of holomorphic Bergman spaces, see [10, 11, 21, 22].

Proof. Assume first that $u \not\equiv 0$ is harmonic in a neighborhood of the closed disk. Applying the reverse Hölder inequality with the indices $p/(p-2)$, $p/2$ we obtain

$$K_p(u) \ge \left(\int_{\mathbb{D}} |u|^p \, dA\right)^{1-2/p} \left(\int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|^2)^p \, dA(z)\right)^{2/p}. \tag{11}$$

Combining this with (9) we get

$$K_p(u) \ge c(|u(0)|^p + B)^{1-2/p} B^{2/p},$$

where

$$B = \int_{\mathbb{D}} |\nabla u(z)|^p (1-|z|^2)^p \, dA$$

and c is a positive constant depending only on p. Now, in view of (9), it remains to prove that $B \le C(|u(0)|^p + K_p(u))$. If $B \le |u(0)|^p$, then there is nothing to prove. Let $B \ge |u(0)|^p \ne 0$. Then

$$\begin{aligned} K_p(u) &\ge c(|u(0)|^p + B)\left(\frac{B}{|u(0)|^p + B}\right)^{2/p} \\ &\ge c(|u(0)|^p + B)\left(\frac{|u(0)|^p}{|u(0)|^p + |u(0)|^p}\right)^{2/p}, \end{aligned}$$

which gives the desired result under the above hypothesis. If $u(0) = 0$, then the result follows from (11) and (9).

If u is arbitrary, then we apply inequality (10) to the functions $z \mapsto u(\rho z)$; we get

$$\int_{\mathbb{D}} |u(\rho z)|^p \, dA(z) \le C|u(0)|^p + \frac{C}{\rho^2} \int_{|z|<\rho} |u|^{p-2} |\nabla u|^2 (\rho - |z|)^2 \, dA(z),$$

where C is independent of ρ (and u). Now we apply Fatou's lemma on the left hand side and the monotone convergence theorem on the right to conclude the proof of the lemma. □

We continue the proof of Theorem 1.1. Let $1 < p \le 2$. As in the case $p \ge 2$, the inequality $|u(0)|^p + \widehat{Q}_p(u) \le C\|u\|_p^p$ is easy to prove. To prove the reverse inequality we use Lemmas 2.4 and 2.1. We get

$$\begin{aligned}
\|u\|_p^p &\le \int_{\mathbb{D}} |u|^p \, dA + C\,\widehat{Q}_p(u) \\
&\le C|u(0)|^p + C\,K_p(u) + C\,\widehat{Q}_p(u) \\
&\le C|u(0)|^p + 2C\,\widehat{Q}_p(u).
\end{aligned}$$

Thus the proof of Theorem 1.1 is complete.

3 Proof of Theorem 1.2

The validity of the inequality $\widehat{H}_{p,\gamma}(f) \le CH_p(f),\ 0 < p < \infty$, follows from the inequality $t^\gamma(1-t) \le \log(1/t),\ 0 < t < 1$. To prove the rest of the theorem assume first that $p \ge 2$. Then the desired result follows from the following lemma applied to the subharmonic function $h = |f|^{p-2}|f'|^2$.

Lemma 3.1. *If $h \ge 0$ is a function subharmonic in $\mathbb{D}$, $\gamma > 0$, and $\alpha > -1$, then there is a constant C independent of h such that*

$$\int_{\mathbb{D}} \frac{1}{|z|} h(z)(1-|z|)^\alpha \, dA(z) \le C \int_{\mathbb{D}} |z|^\gamma h(z)(1-|z|)^\alpha \, dA(z).$$

Proof. It suffices to prove that

$$\int_{|z|<\varepsilon} \frac{1}{|z|} h(z) \, dA(z) \le C \int\limits_{\varepsilon/2<|z|<3\varepsilon/2} h(z) \, dA(z),$$

where $\varepsilon = 1/2$.

Applying the maximum principle we get

$$\begin{aligned}
\int_{|z|<\varepsilon} \frac{1}{|z|} h(z) \, dA(z) &\le C \sup_{|z|=\varepsilon} h(z) \\
&\le C \sup_{|z|=\varepsilon} \frac{4}{\varepsilon^2} \int\limits_{|w-z|<\varepsilon/2} h(w) \, dA(w) \\
&\le C \sup_{|z|=\varepsilon} \frac{4}{\varepsilon^2} \int\limits_{\varepsilon/2<|w|<3\varepsilon/2} h(w) \, dA(w) \\
&= \frac{4C}{\varepsilon^2} \int\limits_{\varepsilon/2<|w|<3\varepsilon/2} h(w) \, dA(w),
\end{aligned}$$

which was to be proved. $\square$

Consider now the case $0 < p \le 2$. We apply (4) to the function $z^k f(z)$, where k is a positive integer such that

$$kp - 2 \ge 1 \text{ and } 2k - 1 \ge \gamma.$$

We have

$$\begin{aligned}
\|f\|_p^p &= \frac{p^2}{2}\int_{\mathbb{D}} |z|^{k(p-2)}|f(z)|^{p-2}|kz^{k-1}f(z) + z^k f'(z)|^2 \log\frac{1}{|z|}\,dA(z)\\
&\le p^2k^2\int_{\mathbb{D}} |z|^{kp-2}|f(z)|^p \log\frac{1}{|z|}\,dA(z) + p^2\int_{\mathbb{D}} |f|^{p-2}|f'(z)|^2|z|^{2k}\log\frac{1}{|z|}\,dA(z)\\
&\le C\int_{\mathbb{D}} |f|^p\,dA + C\int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2|z|^{\gamma}(1-|z|^2)\,dA(z)\\
&= C\int_{\mathbb{D}} |f|^p\,dA + C\widehat{H}_{p,\gamma}(f).
\end{aligned}$$

Therefore it remains to prove that

$$\int_{\mathbb{D}} |f|^p\,dA \le C|f(0)|^p + C\widehat{H}_{p,\gamma}(f). \tag{12}$$

To prove this we use the holomorphic variant of (9) (due to Hardy and Littlewood):

$$\int_{\mathbb{D}} |f|^p\,dA \asymp |f(0)|^p + \int_{\mathbb{D}} |f'|^p(1-|z|^2)^p\,dA(z). \tag{13}$$

Then, as in the proof of Lemma 2.4, we first suppose that f is holomorphic on the closed disk and apply the reverse Hölder inequality to get

$$\begin{aligned}
\widehat{H}_{p,\gamma}(f) &\ge \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^p|z|^{\gamma}(1-|z|^2)^2\,dA(z)\\
&\ge \left(\int_{\mathbb{D}} |f|^p\,dA\right)^{(p-2)/p}\left(\int_{\mathbb{D}} |f'(z)|^p|z|^{\gamma p/2}(1-|z|^2)^p\,dA(z)\right)^{2/p}\\
&\ge c\left(\int_{\mathbb{D}} |f|^p\,dA\right)^{(p-2)/p}\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p\,dA(z)\right)^{2/p},
\end{aligned}$$

where c is a small positive constant. The inequality

$$\int_{\mathbb{D}} |f'(z)|^p|z|^{\gamma p/2}(1-|z|^2)^p\,dA(z) \ge c\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^p\,dA(z),$$

used in the last step is a consequence of Lemma 3.1 applied to the function $h = |f'|^p$. Now we use (13) to prove (12) in the case where f is holomorphic on the closed disk. Finally we apply (12) to the functions $\rho \mapsto f(\rho z)$, and this completes the proof of Theorem 1.2.

4 The case of the real ball

In this section we fix an integer $n \geq 3$ and denote by $\mathbb{B} = \mathbb{B}_n$ the unit ball in $\mathbb{R}^n$. Let $dV = dV_n$ denote the normalized volume measure on $\mathbb{B}$, and let $d\sigma = d\sigma_n$ denote the normalized surface measure on the sphere $\mathbb{S} = \partial\mathbb{B}_n$. For a Borel function u on $\mathbb{B}$, let

$$M_p(r,u) = \left(\int_{\mathbb{S}} |u(ry)|^p \, d\sigma(y)\right)^{1/p}, \quad 0 < r < 1,\ 0 < p \leq \infty.$$

Let $h(\mathbb{B})$ denote the class of all real valued functions harmonic in $\mathbb{B}$. The subclass of $h(\mathbb{B})$ consisting of those u for which

$$\|u\|_p := \sup_{0<r<1} M_p(r,u) \ < \infty$$

will be denoted by $h^p(\mathbb{B})$.

The following assertion is a special case of Green's theorem for functions of n variables.

Lemma 4.1. *If g is a function of class $C^2(\mathbb{B})$, then*

$$\frac{d}{dr}\int_{\mathbb{S}} g(ry)\, d\sigma(y) = \frac{r^{1-n}}{n}\int_{|x|<r} \Delta g(x)\, dV(x) \tag{14}$$

and, equivalently,

$$\int_{\mathbb{S}} g(ry)\, d\sigma(y) = g(0) + \int_{|x|<r} \Delta g(x) G_n(x,r)\, dV(x), \tag{15}$$

$$G_n(x,r) = \frac{|x|^{2-n} - r^{2-n}}{n(n-2)}, \quad x \in \mathbb{B},\ 0 < r < 1.$$

Proof. Let

$$g^{\#}(x) = \int_{\mathcal{O}} g(Ux)\, dU$$

(radialization of g), where dU is the Haar measure on the group, $\mathcal{O}$, of all orthogonal transformations of $\mathbb{R}^n$. Then $g^{\#}(x) = \varphi(|x|)$, where $\varphi \in C^2[0,1)$. Since $(\Delta g^{\#})(x) = (\Delta g)^{\#}(x)$ and

$$\Delta g^{\#}(x) = \varphi''(|x|) + \frac{n-1}{r}\varphi'(|x|),$$

we see, via the formula,

$$\int_{|x|<r} h(x)\, dV(x) = n\int_0^r \rho^{n-1}\, d\rho \int_{\mathbb{S}} h(\rho y)\, d\sigma(y),$$

(integration in polar coordinates) that the proof of (14) reduces to the proof that

$$\varphi'(r) = \frac{r^{1-n}}{n} n \int_0^r \rho^{n-1}\Big(\varphi''(\rho) + \frac{n-1}{\rho} + \varphi'(\rho)\Big)\, d\rho.$$

$\square$

Theorem 4.2 $(1 < p < \infty)$**.** *If* $u \in h(\mathbb{B})$, *then the function* $r \mapsto M_p^p(r,u)$ *is of class* $C^1(0,1)$ *and*

$$\frac{d}{dr}M_p^p(r,u) = \frac{p(p-1)}{n}r^{1-n}\int_{|x|<r}|u(x)|^{p-2}|\nabla u(x)|^2\,dV(x), \tag{16}$$

$$M_p^p(r,u) = |u(0)|^p + p(p-1)\int_{r\mathbb{B}}|u(x)|^{p-2}|\nabla u(x)|^2 G_n(x,r)\,dV(x).$$

Observe that, formally,

$$\lim_{n\to 2}\frac{|x|^{2-n}-r^{2-n}}{n-2} = \log\frac{r}{|x|}.$$

Proof of Theorem 4.2. If $p \ge 2$, then the function $g = |u|^p$ $(u \in h(\mathbb{B}))$ is of class C^2 and we only have to compute $\Delta(|u|^p)$.

Let $1 < p \le 2$. We take

$$g = g_{[\varepsilon]} = (u^2+\varepsilon)^{p/2}, \quad \text{where } 0 < \varepsilon \le 1.$$

A simple calculation shows that

$$\Delta g_{[\varepsilon]} = p(u^2+\varepsilon)^{p/2-2}[(p-1)u^2+\varepsilon]\,|\nabla u|^2. \tag{17}$$

On the other hand, by (14),

$$\frac{d}{dr}M_p^p(r,g_{[\varepsilon]}) = \frac{r^{1-n}}{n}\int_{|x|<r}\Delta g_{[\varepsilon]}(x)\,dV(x). \tag{18}$$

Now we let ε tend to 0 and use the fact that

$$\lim_{\varepsilon\to 0}\frac{d}{dr}M_p^p(r,g_{[\varepsilon]}) = \frac{d}{dr}M_p^p(r,u)$$

(because the function $(x,t) \mapsto (u(x)^2+t)^{p/2}$ is of class C^1 on the set $\mathbb{B}\times\mathbb{R}$) together with Fatou's lemma to find from (17) and (18) that the function $|u|^{p-2}|\nabla u|^2$ is integrable on $\{x : |x| < r\}$. Now the inequality

$$\Delta g_{[\varepsilon]} \le p^2|u|^{p-2}|\nabla u|^2$$

via the dominated convergence theorem concludes the proof. □

Applying the "increasing" property of $M_p(\cdot,u)$ (which follows from (16)) together with the monotone convergence theorem we get the following version of the Hardy-Stein identity.

Theorem 4.3 $(1 < p < \infty)$**.** *A function* $u \in h(\mathbb{B})$ *belongs to* $h^p(\mathbb{B})$ *if and only if*

$$S_p(u) := \int_{\mathbb{B}}|u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n}-1)\,dV(x) \ < \infty,$$

and

$$\|u\|_p^p = |u(0)|^p + \frac{p(p-1)}{n(n-2)}S_p(u).$$

The following theorem is a generalization of Theorem 1.1 to several variables.

Theorem 4.4 **($1 < p < \infty$).** *A function $u \in h(\mathbb{B})$ belongs to $h^p(\mathbb{B})$ if and only if*

$$\widehat{S}_p(u) := \int_{\mathbb{B}} |u(x)|^{p-2} |\nabla u(x)|^2 (1 - |x|^2)\, dV(x) \ < \infty,$$

and we have

$$\|u\|_p^p = \int_{\mathbb{B}} |u|^p \, dV + \frac{p(p-1)}{2n} \widehat{S}_p(u). \tag{19}$$

Moreover $S_p(u) \asymp \widehat{S}_p(u)$, when $p \geq 2$, and $\|u\|_p^p \asymp |u(0)|^p + \widehat{S}_p(u)$, when $1 < p \leq 2$.

Proof. Multiplying (16) by r^n, then integrating from $r = 0$ to 1 and using partial integration on the left hand side and the Fubini theorem on the right we get (19). The proof of the rest is identical to that of Theorem 1.1: We have to use the above results together with the n-variables version of Lemma 2.2 (see [19]) and the n-variables version of (9), i.e.

$$\int_{\mathbb{B}} |u|^p \, dV \asymp |u(0)|^p + \int_{\mathbb{B}} |\nabla u(x)|^p (1 - |x|)^p \, dV(x),$$

see [16, 2]. □

5 The ball in $\mathbb{C}^n$

5.1 The radial derivative

In this section we change the meaning of $\mathbb{B}$ and $\mathbb{S}$. We denote by $\mathbb{B} = \mathbb{B}_n$ (resp. $\mathbb{S}$) ($n \geq 2$) the unit ball (resp. unit sphere) in $\mathbb{C}^n$. As above, let dV (resp. $d\sigma$) denote the normalized volume measure on $\mathbb{B}$ (resp. surface measure on $\mathbb{S}$.) Let $H(\mathbb{B})$ denote the class of all functions holomorphic in $\mathbb{B}$, and

$$H^p(\mathbb{B}) = \{f \in H(\mathbb{B}) : \|f\|_p = \sup_{0<r<1} M_p(r, f) < \infty\}.$$

For the basic properties of the (Hardy) space $H^p(\mathbb{B})$ we refer to [29, Ch. IV]. For $f \in H(\mathbb{B})$ let

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j} = \frac{d}{dr} f(rz)\Big|_{r=1}, \quad z = (z_j)_1^n \in \mathbb{C}^n.$$

Using integration in polar coordinates and the formula

$$\int_{\mathbb{S}} h(\zeta)\, d\sigma(\zeta) = \int_{\mathbb{S}} \left(\frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}\zeta)\, d\theta \right) d\sigma(\zeta)$$

(integration by slices) we get from (4) the following result due to Beatrous and Burbea [1].

Theorem 5.D ($0 < p < \infty$). *A function f belongs to $H^p(\mathbb{B})$ if and only if*

$$R_p(f) := \int_{\mathbb{B}} |z|^{-2n} |f(z)|^{p-2} |\mathcal{R}f(z)|^2 \log \frac{1}{|z|}\, dV(z) \ < \infty,$$

and the formula

$$\|f\|_p^p = |f(0)|^p + \frac{p^2}{2n} R_p(f)$$

holds.

Using integration by slices and Theorem 1.2 we can eliminate the factor $|z|^{-2n}$.

Theorem 5.1 ($0 < p < \infty$). *A function f belongs to $H^p(\mathbb{B})$ if and only if*

$$\widehat{R}_p(f) := \int_{\mathbb{B}} |f(z)|^{p-2} |\mathcal{R}f(z)|^2 (1 - |z|^2)\, dV(z) \ < \infty$$

and we have $R_p(f) \asymp \widehat{R}_p(f)$ $(p \geq 2)$, and $\|f\|_p^p \asymp |f(0)|^p + \widehat{R}_p(f)$ $(0 < p < 2)$.

The proof is simple and is left to the interested reader.

Pluriharmonic functions

A real-valued function $u \in C^2(\mathbb{B})$ is said to be pluriharmonic if each of the one-variable functions $\lambda \mapsto u(z_0 + \lambda z)$ $(|z_0| + |z| < 1,\ \lambda \in \mathbb{D})$ is harmonic in $\mathbb{D}$. Integration by slices together with Theorem 1.B shows that the following formula holds:

$$\|u\|_p^p = |u(0)|^p + \frac{p(p-1)}{2n} \int_{\mathbb{B}} |z|^{-2n} |u(z)|^{p-2} |\mathcal{R}u(z)|^2 \log \frac{1}{|z|}\, dV(z). \tag{20}$$

Here

$$\mathcal{R}u(z) = \frac{d}{dr} u(rz)\Big|_{r=1}.$$

If $f \in H(\mathbb{B})$, then $u = \operatorname{Re} f$ is pluriharmonic and hence Theorem 5.D and (20) together with the identity $\mathcal{R}f = \mathcal{R}u$, show that the inequality

$$\|f\|_p^p - |f(0)|^p \leq C_p (\|u\|_p^p - |u(0)|^p) \quad (1 < p \leq 2)$$

holds. However this fact can be deduced from the one-variable case by integration by slices.

5.2 The euclidean gradient

For $f \in H(\mathbb{B})$, let Df denote the holomorphic euclidean gradient of f,

$$Df(z) = (\partial f / \partial z_j)_{j=1}^n.$$

In various calculations the formulas

$$|\nabla |f|^2| = 2|f|\,|Df|, \quad f \in H(\mathbb{B}), \tag{21}$$

and

$$\Delta(|f|^2) = 4|Df|^2, \quad f \in H(\mathbb{B}), \tag{22}$$

play an important role.

The following was proved in [1].

Theorem 5.E ($0 < p < \infty$). *A function $f \in H(\mathbb{B})$ belongs to $H^p(\mathbb{B})$ if and only if*

$$E_p(f) := \int_{\mathbb{B}} |f(z)|^{p-2}|Df(z)|^2(|z|^{2-2n} - 1)\, dV(z) \ < \infty$$

and the formula

$$\|f\|_p^p = |f(0)|^p + \frac{p^2}{4n(n-1)} E_p(f) \tag{23}$$

holds.

Proof. In this case the formula (15) reads:

$$\int_{\mathbb{S}} g(r\zeta)\, d\sigma(\zeta) = g(0) + \int_{|z|<r} \Delta g(z) G_{2n}(z,r)\, dV(z).$$

Now we take $g = (|f|^2 + \varepsilon)^{p/2}$, $0 < \varepsilon < 1$. We have, by (21) and (22),

$$\begin{aligned}\Delta g &= \frac{p}{2}\Big(\frac{p}{2} - 1\Big)(|f|^2 + \varepsilon)^{p/2-2}|\nabla(|f|^2)|^2 + \frac{p}{2}(|f|^2+\varepsilon)^{p/2-1}\Delta(|f|^2)\\ &= p(p-2)(|f|^2+\varepsilon)^{p/2-2}|f|^2|Df|^2 + 2p(|f|^2+\varepsilon)^{p/2-1}|Df|^2\\ &= p(|f|^2+\varepsilon)^{p/2-2}[(p-2)|f|^2 + 2|f|^2 + 2\varepsilon]|Df|^2,\end{aligned}$$

whence

$$\Delta g = p(|f|^2+\varepsilon)^{p/2-2}[p|f|^2 + 2\varepsilon]|Df|^2. \tag{24}$$

If $p \geq 2$, then the function $|f|^p$ is of class C^2 and we can take $\varepsilon = 0$ to get

$$\Delta(|f|^p) = p^2|f|^{p-2}|Df|^2.$$

If $0 < p < 2$, we proceed in a similar way as in the proof of Theorem 4.2; we omit the details. Thus we have proved that

$$M_p^p(\rho, f) = |f(0)|^p + p^2 \int_{|z|<\rho} |f(z)|^{p-2}|Df(z)|^2 G_{2n}(z,r)\, dV(z). \tag{25}$$

Now we let $\rho \to 1$ and apply the increasing property of $M_p(\cdot, f)$ on the left hand side and the monotone convergence theorem on the right. This completes the proof. $\square$

Theorem 5.2 ($0 < p < \infty$). *A function $f \in H(\mathbb{B})$ belongs to $H^p(\mathbb{B})$ if and only if*

$$\widehat{E}_p(f) := \int_{\mathbb{B}} |f(z)|^{p-2}|Df(z)|^2(1 - |z|^2)\, dV(z) \ < \infty,$$

and we have $E_p(f) \asymp \widehat{E}_p(f)$ $(p \geq 2)$, and $\|f\|_p^p \asymp |f(0)|^p + \widehat{E}_p(f)$ $(0 < p < 2)$.

Proof. Inequality $\widehat{E}_p(f) \leq CE_p(f)$ follows from the elementary inequality $1 - t^2 \leq C(t^{2-n} - 1)$. The reverse inequality follows from Theorem 5.1 and the inequality $|\mathcal{R}f(z)| \leq |Df(z)|$. $\square$

5.3 The invariant gradient

Let $\mathrm{Aut}(\mathbb{B})$ denote the automorphism group of $\mathbb{B}$, i.e. the set of all bijective holomorphic maps $\varphi : \mathbb{B} \mapsto \mathbb{B}$. For each $a \in \mathbb{B}$ there exists a unique automorphism $\varphi_a \in \mathrm{Aut}(\mathbb{B})$ such that $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $(\varphi_a \circ \varphi_a)(z) = z$ for all $z \in \mathbb{B}$. If f is holomorphic in $\mathbb{B}$, we define the invariant gradient of f,

$$\widetilde{D}f(z) = D(f \circ \varphi_z)(0).$$

In the one-dimensional case we have

$$\widetilde{D}f(z) = -(1 - |z|^2)f'(z).$$

The quantity $|\widetilde{D}f|$ is Möbius invariant, which means that

$$|\widetilde{D}(f \circ \varphi)| = |(\widetilde{D}f) \circ \varphi|$$

for $\varphi \in \mathrm{Aut}(\mathbb{B})$ (see [29] and [15]).

Let $d\tau$ denote the Möbius invariant measure on $\mathbb{B}$,

$$d\tau(z) = \frac{dV(z)}{(1 - |z|^2)^{n+1}}.$$

Theorem 5.F **($0 < p < \infty$).** *If $f \in H(\mathbb{B})$ and $0 < r < 1$, then*

$$\begin{aligned} M_p^p(r, f) &= |f(0)|^p + p^2 \int_{|z|<r} |f(z)|^{p-2} |\widetilde{D}f(z)|^2 \widetilde{G}(z, r)\, d\tau(z) \\ &= |f(0)|^p + \frac{p^2}{2n} \int_0^r \frac{(1 - t^2)^{n-1}}{t^{2n-1}}\, dt \int_{|z|<t} |f(z)|^{p-2} |\widetilde{D}f(z)|^2\, d\tau(z), \end{aligned} \tag{26}$$

where

$$\widetilde{G}(z, r) = \frac{1}{2n} \int_{|z|}^r \frac{(1 - t^2)^{n-1}}{t^{2n-1}}\, dt.$$

Remark 5.3. *This theorem is stated in [29, Ex. 4.8] with*

$$M_p^p(r, f) = |f(0)|^p + \left(\frac{p}{2}\right)^2 \int_{|z|<r} |f(z)|^{p-2} |\widetilde{D}f(z)|^2 d\tau(z) \int_{|z|}^r \frac{(1 - t^2)^{n-1}}{t^{2n-1}}\, dt,$$

which does not agree with (26)*.*

For the proof we need the "invariant" Green formula:

Lemma 5.4. *If $g \in C^2(\mathbb{B})$, then*

$$\int_{\mathbb{S}} g(r\zeta)\, d\sigma(\zeta) = g(0) + \int_{|z|<r} \widetilde{\Delta}g(z) \widetilde{G}(z, r)\, d\tau(z), \quad 0 < r < 1,$$

where $\widetilde{\Delta}$ denotes the invariant Laplacian,

$$\widetilde{\Delta}g(z) = \Delta(g \circ \varphi_z)(0).$$

Proof. See e.g. [15, Lemma 2.5]. □

Proof of Theorem 5.F. Now we have to compute $|\widetilde{\Delta} g|$, where

$$g(z) = (|f|^2 + \varepsilon)^{p/2}$$

(see [12], where the case of $\mathcal{M}$-harmonic functions was considered, or [29], [26], [25]). We can use (24) and the definition of $\widetilde{\Delta}$ and $\widetilde{D}$, to get

$$\widetilde{\Delta} g = p(|f|^2 + \varepsilon)^{p/2-2}[p|f|^2 + 2\varepsilon]|\widetilde{D} f|^2.$$

Then, to finish the proof, we proceed as in the proof of Theorems 4.2 and 5.E. □

As a consequence of the above theorem we have the following result due to Stoll [25].

Theorem 5.G ($0 < p < \infty$). *A function $f \in H(\mathbb{B})$ belongs to $H^p(\mathbb{B})$ if and only if*

$$I_p(f) := \int_{\mathbb{B}} |f(z)|^{p-2} |\widetilde{D} f(z)|^2 \widetilde{G}(z,1)\, d\tau(z) \ < \infty,$$

and the formula

$$|f\|^p = |f(0)|^p + p^2 I_p(f) \tag{27}$$

holds.

In the following theorem the singularity induced by $\widetilde{G}(z,1)$ is eliminated.

Theorem 5.5 ($0 < p < \infty$). *A function $f \in H(\mathbb{B})$ belongs to $H^p(\mathbb{B})$ if and only if*

$$\widehat{I}_p(f) := \int_{\mathbb{B}} |f(z)|^{p-2} |\widetilde{D} f(z)|^2 (1 - |z|^2)^{-1}\, dV(z) \ < \infty.$$

Moreover we have $I_p(f) \asymp \widehat{I}_p(f)$ $(p \geq 2)$ and $\|f\|_p^p \asymp |f(0)|^p + \widehat{I}_p(f)$ $(0 < p < 2)$.

Proof. The inequality $\widehat{I}_p(f) \leq C I_p(f)$ $(p > 0)$ follows from the inequality

$$(1 - |z|^2)^{-1}\, dV(z) \leq C\, \widetilde{G}(z,1)\, d\tau(z).$$

In the other direction, we can use the inequality

$$|\mathcal{R} f(z)| \leq (1 - |z|^2)^{-1} |\widetilde{D} f(z)|$$

together with Theorem 5.1. □

5.4 Some more formulas

Let $f \in H(\mathbb{B})$ and $0 < p < \infty$, $n \geq 2$.

As a consequence of Theorem 5.F we have:

$$\frac{d}{dr} M_p^p(r,f) = \frac{p^2 r^{1-2n}(1-r^2)^{n-1}}{2n} \int_{|z|<r} |f(z)|^{p-2} |\widetilde{D} f(z)|^2 \, d\tau(z). \tag{28}$$

Multiplying (28) by r^{2n-2}, and then integrating from 0 to 1 we get

$$\|f\|_p^p = \int_{\mathbb{B}} |f|^p \, dV + \Big(\frac{p}{2n}\Big)^2 \int_{\mathbb{B}} |f(z)|^{p-2} |\widetilde{D} f(z)|^2 (1-|z|^2)^{-1} \, dV(z). \tag{29}$$

We can rewrite (25) in the form

$$M_p^p(r,f) = |f(0)|^p + \frac{p^2}{2n} \int_0^r t^{1-2n} \, dt \int_{|z|<t} |f(z)|^{p-2} |Df(z)|^2 dV(z). \tag{30}$$

As a consequence we have

$$\frac{d}{dr} M_p^p(r,f) = \frac{p^2 r^{1-2n}}{2n} \int_{|z|<r} |f(z)|^{p-2} |Df(z)|^2 \, dV(z). \tag{31}$$

In the case $n = 1$, both (28) and (31) reduce to (3).

Multiplying (31) by r^{2n-2} and then integrating from $r = 0$ to $r = 1$ we get

$$\|f\|_p^p = \int_{\mathbb{B}} |f|^p \, dV + \frac{p^2}{4n} \int_{\mathbb{B}} |f(z)|^{p-2} |Df(z)|^2 (1-|z|^2) \, dV(z). \tag{32}$$

In the case $n = 1$, both (29) and (32) reduce to

$$\|f\|_p^p = \int_{\mathbb{D}} |f|^p \, dA + \frac{p^2}{4} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 (1-|z|^2) \, dA(z).$$

Remark 5.6. *Inequalities* (25) *and* (30) *can be proved by an application of* (23) *to the functions* $z \mapsto f(rz)$. *On the other hand,* (26) *cannot be deduced from* (27) *in such a way.*

Finally, we note three formulas (see [1]) that can be deduced from the one-variable formulas by integration by slices:

$$\frac{d}{dr} M_p^p(r,f) = \frac{p^2}{2nr} \int_{|z|<r} |z|^{-2n} |f|^{p-2} |\mathcal{R} f|^2 \, dV(z),$$

$$\|f\|_p^p = \int_{\mathbb{B}} |f|^p \, dV + \frac{p^2}{4n^2} \int_{\mathbb{B}} |f(z)|^{p-2} |\mathcal{R} f(z)|^2 |z|^{-2n} (1-|z|^{2n}) \, dV(z),$$

$$M_p^p(r,f) = |f(0)|^p + \frac{p^2}{2n} \int_{|z|<r} |z|^{-2n} |f(z)|^{p-2} |\mathcal{R} f(z)|^2 \log \frac{r}{|z|} \, dV(z).$$

Remark 5.7. *The identities* (23)*,* (25)*,* (26)*,* (27)*,* (28)*,* (29)*,* (31) *and* (32) *were stated in [12] without proofs but with wrong constants. In his book [29], Zhu states* (27) *in the form*

$$\|f\|_p^p = |f(0)|^p + \frac{p^2}{4} I_p(f),$$

see p. 127, Theorem 4.23. The misprint is on the first line of p. 128, where the factor 4 *is omitted.*

References

[1] F. Beatrous and J. Burbea, *Characterizations of spaces of holomorphic functions in the ball*, Kodai Math. J. **8** (1985), 36–51.

[2] O. Djordjević and M. Pavlović, *Equivalent norms on Dirichlet spaces of polyharmonic functions on thee unit ball in* $\mathbf{R}^n$, Bol. Soc. Mat. Mexicana **13** (2007), no. 3, 307–319.

[3] P. L. Duren, *Theory of* H^p *spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York, 1970.

[4] G. H. Hardy, *The mean value of the modulus of an analytic function*, Lond. M. S. Proc. **14** (1915), no. 2, 269–277.

[5] G. H. Hardy and J. E. Littlewood, *Some properties of conjugate functions*, J. Reine Angew. Math. **167** (1932), 405–423.

[6] W. K. Hayman, *Multivalent functions*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 48, Cambridge University Press, Cambridge, 1958.

[7] M. Jevtić and M. Pavlović, *Littlewood-Paley type inequalities for* M*-harmonic functions*, Publ. Inst. Math. (Beograd) (N.S.) **64(78)** (1998), 36–52, 50th anniversary of the Mathematical Institute, Serbian Academy of Sciences and Arts (Belgrade, 1996).

[8] H. O. Kim and E. G. Kwon, *Weighted subspaces of Hardy spaces*, Canad. J. Math. **40** (1988), no. 5, 1074–1083.

[9] E. G. Kwon, *On the Hardy-Stein identity*, Comm. Korean Math. Soc. **6** (1991), no. 1, 97–99.

[10] E. G. Kwon, *A characterization of Bloch space and Besov space*, J. Math. Anal. Appl. **324** (2006), no. 2, 1429–1437.

[11] E. G. Kwon, *Quantities equivalent to the norm of a weighted Bergman space*, J. Math. Anal. Appl. **338** (2008), no. 2, 758–770.

[12] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series. II*, Proc. London math. Soc. **42** (1936), 52–89.

[13] D. H. Luecking, *A new proof of an inequality of Littlewood and Paley*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 887–893.

[14] M. Mateljević and M. Pavlović, *Multipliers of H^p and BMOA*, Pac. J. Math. **146** (1990), no. 1, 71–84.

[15] M. Pavlović, *Inequalities for the gradient of eigenfunctions of the invariant Laplacian in the unit ball*, Indag. Math. (N.S.) **2** (1991), no. 1, 89–98.

[16] M. Pavlović, *Decompositions of L^p and Hardy spaces of polyharmonic functions*, J. Math. Anal. Appl. **216** (1997), 499–509.

[17] M. Pavlović, *A Littlewood-Paley theorem for subharmonic functions*, Publ. Inst. Math. (Beograd) (N.S.) **68(82)** (2000), 77–82.

[18] M. Pavlović, *A short proof of an inequality of Littlewood and Paley*, Proc. Amer. Math. Soc. **134** (2006), no. 12, 3625–3627 (electronic).

[19] M. Pavlović, *Lipschitz conditions on the modulus of a harmonic function*, Rev. Mat. Iberoam. **23** (2007), no. 3, 831–845.

[20] M. Pavlović and J. Riihentaus, *Classes of quasi-nearly subharmonic functions*, Potential Anal. **29** (2008), no. 1, 89–104.

[21] M. Pavlović and K. Zhu, *New characterizations of Bergman spaces*, Ann. Acad. Sci. Fenn., Math. **33** (2008), no. 1, 87–99.

[22] M. Pavlović and K. Zhu, *Addendum to "New characterizations of Bergman spaces"*, Ann. Acad. Sci. Fenn., Math. **34** (to appear).

[23] M. Riesz, *Sur les fonctions conjuguées*, Math. Z. **27** (1927), 218–244.

[24] P. Stein, *On a theorem of M. Riesz*, J. Lond. Math. Soc. **8** (1933), 242–247.

[25] M. Stoll, *A characterization of Hardy spaces on the unit ball of $\mathbf{C}^n$*, J. London Math. Soc. (2) **48** (1993), no. 1, 126–136.

[26] M. Stoll, *Invariant potential theory in the unit of ball of $\mathbf{C}^n$*, London Mathematical Society Lecture Note Series. 199. Cambridge: Cambridge University Press. x, 173 p., 1994.

[27] M. Stoll, *The Littlewood-Paley inequalities for Hardy-Orlicz spaces of harmonic functions on domains in $\mathbb{R}^n$*, Aikawa, Hiroaki (ed.) et al., Potential theory in Matsue. Selected papers of the international workshop on potential theory, Matsue, Japan, August 23–28, 2004. Tokyo: Mathematical Society of Japan. Advanced Studies in Pure Mathematics 44, 363-376 (2006), 2006.

[28] S. Yamashita, *Criteria for functions to be of Hardy class H^p*, Proc. Amer. Math. Soc. **75** (1979), no. 1, 69–72.

[29] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005.

Faculty of Mathematics, Studentski trg 16, 11001 Belgrade, p.p. 550, Serbia
E-mail: pavlovic@matf.bg.ac.yu