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FIXED POINT THEOREMS OF FINITE FAMILY WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract

In this paper, we prove that a multi-step iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the mappings in convex metric spaces. Our results extend and improve the recent result of Kim et al. [9, 10] and many known results.

1 Introduction and preliminaries

Throughout this paper, we assume that X is a metric space, $F(T)$ and $D(T)$ are the set of fixed points and domain of T respectively and $\mathbb{N}$ is the set of all positive integers.

Definition 1.1 ([9, 10]): Let $T\colon D(T) \subset X \to X$ be a mapping.

(1) The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in D(T).$$

(2) The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p), \quad \forall x \in D(T),\ \forall p \in F(T).$$

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(3) The mapping T is said to be asymptotically nonexpansive if there exists a sequence $r_n \in [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, T^n y) \le (1 + r_n)d(x, y), \qquad \forall x, y \in D(T),\ \forall n \in \mathbb{N}.$$

(4) The mapping T is said to be asymptotically quasi-nonexpansive if there exists a sequence $r_n \in [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$d(T^n x, p) \le (1 + r_n)d(x, p), \qquad \forall x \in D(T),\ \forall p \in F(T),\ \forall n \in \mathbb{N}.$$

Remark 1.1: From the definition 1.1, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold.

In 2000, Noor [14] introduced a three step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and auxiliary principle. Glowinski and Le Tallec [6] used three step iterative schemes to find the approximate solution of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown [6] that three step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences. Recently, Xu and Noor [20] introduced and studies a three step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho et al. [4] extended the work of Xu and Noor [20] to the three step iterative scheme with errors in Banach space and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [18] gave weak and strong convergence theorems for a new three step iterative scheme of asymptotically nonexpansive mappings. More recently, Plubtieng et al. [16] introduced three step iterative scheme with errors for three asymptotically nonexpansive mapping and established strong convergence of this scheme to common fixed point of three asymptotically nonexpansive mappings.

In 2004, Kim et al. [10] gave the necessary and sufficient conditions for three-step iterative sequences with errors to converge to a fixed point for asymptotically quasi-nonexpansive mappings in convex metric spaces which generalized and improved his previous result [9].

The purpose of this paper is to study the convergence of multi-step iterative sequences with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. The main result of this paper is also, an extension and improvement of the well known corresponding results in [1] - [5], [7, 8], [11] - [13], [15] - [18] and [20].

For the sake of convenience, we first recall some definitions and notations.

In 1970, Takahashi [19] introduced the concept of convexity in a metric space and the properties of the space.

Definition 1.2:([19]) Let (X, D) be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

X together with a convex structure W is called a *convex metric space*, denoted it by (X, d, W). A nonempty subset K of X is said to be *convex* if $W(x, y, \lambda) \in K$ for all $(x, y, \lambda) \in K \times K \times I$.

Remark 1.2: Every linear normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \\ &\leq \alpha \|u - x\| + \beta \|u - y\| + \gamma \|u - z\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \qquad \forall u \in X. \end{aligned}$$

But there exists some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [19]).

Definition 1.3: (1) Let (X, d, W) be a convex metric space, $T_1, T_2, \ldots, T_N : X \to X$ be N mappings and let $x_1 \in X$ be a given point. Then the sequence $\{x_n\}$ defined by:

$$\begin{aligned} x_{n+1} &= x_n^{(N)} \\ &= W(x_n, T_N^n x_n^{(N-1)}, u_n^{(N)}; \alpha_n^{(N)}, \beta_n^{(N)}, \gamma_n^{(N)}), \\ x_n^{(N-1)} &= W(x_n, T_{N-1}^n x_n^{(N-2)}, u_n^{(N-1)}; \alpha_n^{(N-1)}, \beta_n^{(N-1)}, \gamma_n^{(N-1)}), \\ \ldots &= \ldots \\ \ldots &= \ldots \\ x_n^{(3)} &= W(x_n, T_3^n x_n^{(2)}, u_n^{(3)}; \alpha_n^{(3)}, \beta_n^{(3)}, \gamma_n^{(3)}), \qquad (1.1) \\ x_n^{(2)} &= W(x_n, T_2^n x_n^{(1)}, u_n^{(2)}; \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}), \\ x_n^{(1)} &= W(x_n, T_1^n x_n, u_n^{(1)}; \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}), \end{aligned}$$

is called the multi-step iterative sequence with errors for N mappings $T_1, T_2, \dots, T_N$, where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ are sequences in $[0,1]$ satisfying the following conditions:

$\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\forall i = 1, 2, \dots, N$ and $\forall n \in \mathbb{N}$,

and $\{u_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ are bounded sequences in X.

In (1.1), if $N = 3$, $T_1 = T_2 = T_3 = T$, $x_n^{(2)} = y_n$, $x_n^{(1)} = z_n$, $u_n^{(3)} = u_n$, $u_n^{(2)} = v_n$, $u_n^{(1)} = w_n$, $\alpha_n^{(3)} = a_n$, $\alpha_n^{(2)} = \bar{a}_n$, $\alpha_n^{(1)} = \hat{a_n}$, $\beta_n^{(3)} = b_n$, $\beta_n^{(2)} = \bar{b}_n$, $\beta_n^{(1)} = \hat{b_n}$, $\gamma_n^{(3)} = c_n$, $\gamma_n^{(2)} = \bar{c}_n$ and $\gamma_n^{(1)} = \hat{c_n}$, then scheme (1.1) reduces to the three-step iterative scheme with errors for one mapping defined by Kim et al. [10].

2 Main results

In order to prove our main result, we will first prove the following lemma.

Lemma 2.1: Let (X, d, W) be a convex metric space, $T_1, T_2, \dots, T_N \colon X \to X$ be N asymptotically quasi-nonexpansive mappings satisfying $\sum_{n=1}^{\infty} r_n < \infty$ where $\{r_n\}$ is the sequence appeared in Definition 1.1, and $F = \bigcap_{i=1}^{N} F(T_i)$ be a nonempty set. For a given $x_1 \in X$, let $\{x_n\}$ be the multi-step iterative sequences with errors defined by (1.1). Then

(**a**) $d(x_{n+1}, p) \le (1 + r_n)^N d(x_n, p) + t_n^{(N)}$, $\forall p \in F$, $n \in \mathbb{N}$,

where $t_n^{(N)} = \beta_n^{(N)}(1 + r_n) t_n^{(N-1)} + \gamma_n^{(N)} d(u_n^{(N)}, p)$ such that $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$.

(**b**) there exists a constant $M > 0$ such that

$d(x_m, p) \le M.d(x_n, p) + M.\sum_{j=n}^{m-1} t_j^{(N)}$, $\forall p \in F$, $m > n$.

Proof: (**a**) Since T_i $(i = 1, 2, \dots, N)$ is asymptotically quasi-nonexpansive. Let $p \in F = \bigcap_{i=1}^{N} F(T_i)$, we have

$$\begin{aligned}
d(x_n^{(1)}, p) &= d(W(x_n, T_1^n x_n, u_n^{(1)}; \alpha_n^{(1)}, \beta_n(1), \gamma_n^{(1)})) \\
&\le \alpha_n^{(1)} d(x_n, p) + \beta_n^{(1)} d(T_1^n x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p) \\
&\le \alpha_n^{(1)} d(x_n, p) + \beta_n^{(1)} (1 + r_n) d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p) \\
&\le \alpha_n^{(1)} (1 + r_n) d(x_n, p) + \beta_n^{(1)} (1 + r_n) d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p) \\
&\le (\alpha_n^{(1)} + \beta_n^{(1)})(1 + r_n) d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p) \\
&= (1 - \gamma_n^{(1)})(1 + r_n) d(x_n, p) + \gamma_n^{(1)} d(u_n^{(1)}, p) \\
&\le (1 + r_n) d(x_n, p) + t_n^{(1)} \qquad (2.1)
\end{aligned}$$

where $t_n^{(1)} = \gamma_n^{(1)} d(u_n^{(1)}, p)$. Since $\{u_n^{(1)}\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(1)} < \infty$. It follows from (2.1) that

$$\begin{aligned}
d(x_n^{(2)}, p) &= d(W(x_n, T_2^n x_n^{(1)}, u_n^{(2)}; \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)})) \\
&\leq \alpha_n^{(2)} d(x_n, p) + \beta_n^{(2)} d(T_2^n x_n^{(1)}, p) + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&\leq \alpha_n^{(2)} d(x_n, p) + \beta_n^{(2)} (1 + r_n) d(x_n^{(1)}, p) + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&\leq \alpha_n^{(2)} d(x_n, p) + \beta_n^{(2)} (1 + r_n)[(1 + r_n) d(x_n, p) + t_n^{(1)}] + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&\leq \alpha_n^{(2)} d(x_n, p) + \beta_n^{(2)} (1 + r_n)^2 d(x_n, p) + \beta_n^{(2)} (1 + r_n) t_n^{(1)} \\
&\quad + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&\leq \alpha_n^{(2)} (1 + r_n)^2 d(x_n, p) + \beta_n^{(2)} (1 + r_n)^2 d(x_n, p) + \beta_n^{(2)} (1 + r_n) t_n^{(1)} \\
&\quad + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&\leq (\alpha_n^{(2)} + \beta_n^{(2)})(1 + r_n)^2 d(x_n, p) + \beta_n^{(2)} (1 + r_n) t_n^{(1)} + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&= (1 - \gamma_n^{(2)})(1 + r_n)^2 d(x_n, p) + \beta_n^{(2)} (1 + r_n) t_n^{(1)} + \gamma_n^{(2)} d(u_n^{(2)}, p) \\
&\leq (1 + r_n)^2 d(x_n, p) + t_n^{(2)} \qquad (2.2)
\end{aligned}$$

where $t_n^{(2)} = \beta_n^{(2)}(1+r_n)t_n^{(1)} + \gamma_n^{(2)} d(u_n^{(2)}, p)$. Since $\{u_n^{(2)}\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n^{(2)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$. Similarly, we see that

$$\begin{aligned}
d(x_n^{(3)}, p) &= d(W(x_n, T_3^n x_n^{(2)}, u_n^{(3)}; \alpha_n^{(3)}, \beta_n^{(3)}, \gamma_n^{(3)})) \\
&\leq \alpha_n^{(3)} d(x_n, p) + \beta_n^{(3)} d(T_3^n x_n^{(2)}, p) + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&\leq \alpha_n^{(3)} d(x_n, p) + \beta_n^{(3)} (1 + r_n) d(x_n^{(2)}, p) + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&\leq \alpha_n^{(3)} d(x_n, p) + \beta_n^{(3)} (1 + r_n)[(1 + r_n)^2 d(x_n, p) + t_n^{(2)}] + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&\leq \alpha_n^{(3)} d(x_n, p) + \beta_n^{(3)} (1 + r_n)^3 d(x_n, p) + \beta_n^{(3)} (1 + r_n) t_n^{(2)} \\
&\quad + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&\leq \alpha_n^{(3)} (1 + r_n)^3 d(x_n, p) + \beta_n^{(3)} (1 + r_n)^3 d(x_n, p) + \beta_n^{(3)} (1 + r_n) t_n^{(2)} \\
&\quad + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&\leq (\alpha_n^{(3)} + \beta_n^{(3)})(1 + r_n)^3 d(x_n, p) + \beta_n^{(3)} (1 + r_n) t_n^{(2)} + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&= (1 - \gamma_n^{(3)})(1 + r_n)^3 d(x_n, p) + \beta_n^{(3)} (1 + r_n) t_n^{(2)} + \gamma_n^{(3)} d(u_n^{(3)}, p) \\
&\leq (1 + r_n)^3 d(x_n, p) + t_n^{(3)} \qquad (2.3)
\end{aligned}$$

where $t_n^{(3)} = \beta_n^{(3)}(1+r_n)t_n^{(2)} + \gamma_n^{(3)} d(u_n^{(3)}, p)$. Since $\{u_n^{(3)}\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n^{(3)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(3)} < \infty$. Continuing the above process, we get

$$d(x_{n+1}, p) \leq (1 + r_n)^N d(x_n, p) + t_n^{(N)},$$

where $\{t_n^{(N)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$. This completes the proof of (a).

(**b**) If $x \geq 0$, then $1 + x \leq e^x$ and $(1+x)^N \leq e^{Nx}$. Therefore from (a) we can obtain that

$$\begin{aligned}
d(x_m, p) &\leq (1 + r_{m-1})^N d(x_{m-1}, p) + t^{(N)}_{m-1} \\
&\leq e^{Nr_{m-1}} d(x_{m-1}, p) + t^{(N)}_{m-1} \\
&\leq e^{Nr_{m-1}}[e^{Nr_{m-2}} d(x_{m-2}, p) + t^{(N)}_{m-2}] + t^{(N)}_{m-1} \\
&\leq e^{N(r_{m-1}+r_{m-2})} d(x_{m-2}, p) + e^{Nr_{m-1}} t^{(N)}_{m-2} + t^{(N)}_{m-1} \\
&\leq e^{N(r_{m-1}+r_{m-2})} d(x_{m-2}, p) + e^{Nr_{m-1}}[t^{(N)}_{m-1} + t^{(N)}_{m-2}] \\
&\leq \dots\dots \\
&\leq \dots\dots \\
&\leq e^{N(r_{m-1}+r_{m-2}+\cdots+r_n)} d(x_n, p) \\
&\quad + e^{N(r_{m-1}+r_{m-2}+\cdots+r_n)}[t^{(N)}_{m-1} + t^{(N)}_{m-2} + \cdots + t^{(N)}_{n}] \\
&\leq e^{N\sum_{j=n}^{m-1} r_j} d(x_n, p) + e^{N\sum_{j=n}^{m-1} r_j}. \sum_{j=n}^{m-1} t^{(N)}_j \\
&\leq M.d(x_n, p) + M. \sum_{j=n}^{m-1} t^{(N)}_j,
\end{aligned}$$

where $M = e^{N\sum_{j=n}^{m-1} r_j}$. This completes the proof of (b).

Lemma 2.1 [12]: Let the number of sequences $\{a_n\}$, $\{b_n\}$ and $\{\lambda_n\}$ satisfy that $a_n \geq 0$, $b_n \geq 0$, $\lambda_n \geq 0$, $a_{n+1} \leq (1 + \lambda_n)a_n + b_n$, $\forall n \in \mathbb{N}$, $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$. Then

(a) $\lim_{n\to\infty} a_n$ exists.

(b) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.1: Let (X, d, W) be a complete convex metric space, $T_1, T_2, \dots, T_N : X \to X$ be N asymptotically quasi-nonexpansive mappings and $F = \bigcap_{i=1}^{N} F(T_i)$ be a nonempty set. For a given $x_1 \in X$, let $\{x_n\}$ be the multi-step iterative sequence with errors defined by (1.1) and $\{r_n\}$, $\{\gamma^{(i)}_n\}$ for all $i = 1, 2, \dots, N$ be sequences satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} r_n < \infty$,

(ii) $\sum_{n=1}^{\infty} \gamma^{(i)}_n < \infty$, for all $i = 1, 2, \dots, N$,

where $\{r_n\}$ is a sequence appeared in Definition 1.1 and $\{\gamma^{(i)}_n\}$ for all $i = 1, 2, \dots, N$ is a sequence appeared in (1.1). Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0.$$

Proof: The necessity is obvious. Now, we prove the sufficiency. Suppose that the condition $\liminf_{n\to\infty} D_d(x_n, F) = 0$ is satisfied. Then from Lemma 2.1(a), we have

$$d(x_{n+1}, p) \leq (1 + r_n)^N d(x_n, p) + t_n^{(N)}, \quad \forall p \in F, \quad \forall n \in \mathbb{N}, \tag{2.4}$$

where $t_n^{(N)} = \beta_n^{(N)}(1 + r_n)t_n^{(N-1)} + \gamma_n^{(N)} d(u_n^{(N)}, p)$ such that $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$. From (2.4), we can obtain that

$$D_d(x_{n+1}, F) \leq (1 + r_n)^N D_d(x_n, F) + t_n^{(N)}.$$

Since $\liminf_{n\to\infty} D_d(x_n, F) = 0$, by Lemma 2.2, we have

$$\lim_{n\to\infty} D_d(x_n, F) = 0.$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. By Lemma 2.1(b), there exists a constant $M > 0$ such that

$$d(x_m, p) \leq M.d(x_n, p) + M.\sum_{j=n}^{m-1} t_j^{(N)}, \ \forall p \in F, \ m > n. \tag{2.5}$$

Since $\lim_{n\to\infty} D_d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$, there exists a constant N_1 such that for all $n \geq N_1$,

$$D_d(x_n, F) < \frac{\varepsilon}{4M} \quad \text{and} \quad \sum_{j=N_1}^{\infty} t_j^{(N)} < \frac{\varepsilon}{6M}.$$

We note that there exists $p_1 \in F$ such that $d(x_{N_1}, p_1) < \frac{\varepsilon}{3M}$. It follows that from (2.5) that for all $m > n > N_1$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, p_1) + d(x_n, p_1) \\ &\leq M.d(x_{N_1}, p_1) + M.\sum_{j=N_1}^{m-1} t_j^{(N)} + M.d(x_{N_1}, p_1) + M.\sum_{j=N_1}^{n-1} t_j^{(N)} \\ &< M.\frac{\varepsilon}{3M} + M.\frac{\varepsilon}{6M} + M.\frac{\varepsilon}{3M} + M.\frac{\varepsilon}{6M} \\ &= \varepsilon. \end{aligned} \tag{2.6}$$

Since ε is an arbitrary positive number, (2.6) implies that $\{x_n\}$ is a Cauchy sequence. From the completeness of this space, $\lim_{n\to\infty} x_n$ exists. Let $\lim_{n\to\infty} x_n = p$. It will be proven that p is a common fixed point. Let $\bar{\varepsilon} > 0$. Since $\lim_{n\to\infty} x_n = p$, there exists a natural number N_2 such that for all $n \geq N_2$,

$$d(x_n, p) < \frac{\bar{\varepsilon}}{2(2 + r_1)}. \tag{2.7}$$

$\lim_{n\to\infty} D_d(x_n, F) = 0$ implies that there exists a natural number $N_3 \geq N_2$ such that for all $n \geq N_3$,

$$D_d(x_n, F) < \frac{\bar{\varepsilon}}{3(4 + 3r_1)}. \tag{2.8}$$

Therefore, there exists a $p^* \in F$ such that

$$d(x_{N_3}, p^*) < \frac{\bar{\varepsilon}}{2(4 + 3r_1)}. \tag{2.9}$$

From (2.7) and (2.9), we have for any $i \in I$

$$\begin{aligned}
d(T_i p, p) &\leq d(T_i p, p^*) + d(p^*, T_i x_{N_3}) + d(T_i x_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&= d(T_i p, p^*) + 2d(T_i x_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&\leq (1 + r_1) d(p, p^*) + 2(1 + r_1) d(x_{N_3}, p^*) + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&\leq (1 + r_1)[d(p, x_{N_3}) + d(x_{N_3}, p^*)] + 2(1 + r_1) d(x_{N_3}, p^*) \\
&\quad + d(p^*, x_{N_3}) + d(x_{N_3}, p) \\
&= (2 + r_1) d(x_{N_3}, p) + (4 + 3r_1) d(x_{N_3}, p^*) \\
&< (2 + r_1). \frac{\bar{\varepsilon}}{2(2 + r_1)} + (4 + 3r_1). \frac{\bar{\varepsilon}}{2(4 + 3r_1)} \\
&= \bar{\varepsilon}.
\end{aligned}$$

Since $\bar{\varepsilon}$ is an arbitrary positive number, this implies that $T_i p = p$. Hence $p \in F(T_i)$ for all $i \in I$ and so $p \in F = \bigcap_{i=1}^N F(T_i)$. Thus the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$. This completes the proof.

By using the same method in Theorem 2.1, we can easily obtain the following theorem.

Theorem 2.2: Let (X, d, W) be a complete convex metric space, $T_1, T_2, \dots, T_N : X \to X$ be N quasi-nonexpansive mappings and $F = \bigcap_{i=1}^N F(T_i)$ be a nonempty set. For a given $x_1 \in X$, let $\{x_n\}$ be the multi-step iterative sequence with errors defined by:

$$
\begin{array}{rcl}
x_{n+1} & = & x_n^{(N)} \\
& = & W(x_n, T_N x_n^{(N-1)}, u_n^{(N)}; \alpha_n^{(N)}, \beta_n^{(N)}, \gamma_n^{(N)}), \\
x_n^{(N-1)} & = & W(x_n, T_{N-1} x_n^{(N-2)}, u_n^{(N-1)}; \alpha_n^{(N-1)}, \beta_n^{(N-1)}, \gamma_n^{(N-1)}), \\
\dots & = & \dots \\
\dots & = & \dots \\
x_n^{(3)} & = & W(x_n, T_3 x_n^{(2)}, u_n^{(3)}; \alpha_n^{(3)}, \beta_n^{(3)}, \gamma_n^{(3)}), \qquad (*) \\
x_n^{(2)} & = & W(x_n, T_2 x_n^{(1)}, u_n^{(2)}; \alpha_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}), \\
x_n^{(1)} & = & W(x_n, T_1 x_n, u_n^{(1)}; \alpha_n^{(1)}, \beta_n^{(1)}, \gamma_n^{(1)}),
\end{array}
$$

where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ are sequences in $[0, 1]$ satisfying the following condition:

$\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $\forall i = 1, 2, \dots, N$ and $\forall n \in \mathbb{N}$,

and $\{u_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ are bounded sequences in X.

If $\{\gamma_n^{(i)}\}$ for all $i = 1, 2, \dots, N$ be sequences satisfying the following condition:

(i) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, for all $i = 1, 2, \dots, N$.

Then the iterative sequence$\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0.$$

From Theorem 2.1, we can also obtain the following result for the Banach space.

Theorem 2.3: Let X be a real Banach space, $T_1, T_2, \dots, T_N \colon X \to X$ be N asymptotically quasi-nonexpansive mappings satisfying the condition (i) in Theorem 2.1 and $F = \bigcap_{i=1}^{N} F(T_i)$ be a nonempty set. Let $\{x_n\}$ be the multi-step iterative sequence with errors defined by:

$$x_1 \in X,$$

$$\begin{aligned} x_{n+1} = x_n^{(N)} &= \alpha_n^{(N)} x_n + \beta_n^{(N)} T_N^n x_n^{(N-1)} + \gamma_n^{(N)} u_n^{(N)} \\ x_n^{(N-1)} &= \alpha_n^{(N-1)} x_n + \beta_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + \gamma_n^{(N-1)} u_n^{(N-1)} \\ \ldots &= \ldots \\ \ldots &= \ldots \\ x_n^{(3)} &= \alpha_n^{(3)} x_n + \beta_n^{(3)} T_3^n x_n^{(2)} + \gamma_n^{(3)} u_n^{(3)} \\ x_n^{(2)} &= \alpha_n^{(2)} x_n + \beta_n^{(2)} T_2^n x_n^{(1)} + \gamma_n^{(2)} u_n^{(2)} \\ x_n^{(1)} &= \alpha_n^{(1)} x_n + \beta_n^{(1)} T_1^n x_n + \gamma_n^{(1)} u_n^{(1)} \end{aligned}$$

where $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ for all $i = 1, 2, \ldots, N$ are sequences in $[0, 1]$ satisfying the following condition:

$$\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1, \ \forall i = 1, 2, \ldots, N \text{ and } \forall n \in \mathbb{N},$$

and $\{u_n^{(i)}\}$ for all $i = 1, 2, \ldots, N$ are bounded sequences in X and $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ for all $i = 1, 2, \ldots, N$. Then the iterative sequence $\{x_n\}$ converges to a common fixed point of $\{T_i : i = 1, 2, \ldots, N\}$ if and only if

$$\liminf_{n\to\infty} D_d(x_n, F) = 0.$$

where $D_d(y, S) = \inf\{d(y, s) : s \in S\}$.

Proof: Since X is a Banach space, it is a complete convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) := \alpha x + \beta y + \gamma z$, for all $x, y, z \in X$ and for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$. Therefore, the conclusion of Theorem 2.3 can be obtained from Theorem 2.1 immediately.

Remark 2.1: (1) Theorem 2.1 and 2.2 are two new convergence theorems of multi-step iterative sequences with errors for finite family of nonlinear mappings in convex metric spaces. These two theorems generalize and improves the corresponding results of [11]- [13], [1]- [3] and [5, 7, 8, 15, 17, 20].

(2) Theorem 2.3 generalizes and improves the corresponding results of Kim et al. [9], Liu [12, 13], Shahzad and Udomene [17], Khan and Takahashi [7], Khan and Ud-din [8] and Xu and Noor [20].

(3) Theorem 2.1, 2.2 and 2.3 also extend Theorem 2.1, 2.2 and 2.4 of Kim et al. [10] to the case of multi-step iteration scheme and finite family of mappings considered in this paper.

References

[1] S.S. Chang, *Some results for asymptotically pseudo-contractive mapping and asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 129, No.3 (2001), 845-853.

[2] S.S. Chang, *Iterative approximation problem of fixed points for asymptotically nonexpansive mappings in Banach spaces*, Acta Math. Appl. 24, No.2 (2001), 236-241.

[3] S.S. Chang, *On the approximating problem of fixed points for asymptotically nonexpansive mappings*, Indian J. Pure and Appl. 32, No.9 (2001), 1-11.

[4] Y.J. Cho, H. Zhou and G. Guo, *Weak and strong convergence theorems for three step iterations with errors for asymptotically nonexpansive mappings*, Computers and Math. with Appl. 47(2004), 707-717.

[5] M.K. Ghosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, J. Math. Anal. Appl. 207(1997), 96-103.

[6] R. Glowinski and P. Le Tallec, *Augmented Lagrangian Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia (1989).

[7] S.H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Jpn. 53(2001), 143-148.

[8] S.H. Khan and H. Fukhar-ud-din, *Weak and strong convergence of a scheme with errors for two nonexpansive mappings*, Nonlinear Analysis 61(2005), 1295-1301.

[9] J.K. Kim, K.H. Kim and K.S. Kim, *Convergence theorems of modified three-step iterative sequences with mixed errors for asymptotically quasi-nonexpansive mappings in Banach spaces*, PanAmerican Math. Jour. 14(1)(2004), 45-54.

[10] J.K. Kim, K.H. Kim and K.S. Kim, *Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces*, Nonlinear Anal. Convex Anal. RIMS Vol. 1365 (2004), pp. 156-165.

[11] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl. 259(2001), 1-7.

[12] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. 259(2001), 18-24.

[13] Q.H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member of uniformly convex Banach spaces*, J. Math. Anal. Appl. 266(2002), 468-471.

[14] M.A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. 251(2000), 217-229.

[15] W.V. Petryshyn and T.E. Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl. 43(1973), 459-497.

[16] S. Plubtieng, R. Wangkeeree and R. Punpaeng, *On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 322(2006), 1018-1029.

[17] N. Shahzad and A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications, Volume 2006, Article ID 18909, pages 1-10.

[18] S. Suantai, *Weak and strong convergence criteria of Noor iteration for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. in press.

[19] W. Takahashi, *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep. 22(1970), 142-149.

[20] B.L. Xu and M.A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 267(2002), No.2, 444-453.

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