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ON STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS VIA NON-LINEAR INTEGRAL CONTRACTORS I

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Abstract

The aim of this paper is to study the existence and uniqueness of solutions for a general stochastic integrodifferential equation of the Ito type, by using the concept of non-linear bounded random integral contractors, which includes the Lipschitz condition as a special case. The method applied in this consideration follows partially the basic ideas of the contractor theory introduced earlier by Altman [1, 2] and Kuo [6]. It is also shown that the Lipschitz condition and the condition based on a bounded random integral contractor for the coefficients of the considered equation, in general, cannot be compared.

1 Introduction

Several phenomena in life and sciences, especially in mechanics, engineering and, since recently, in finance, have been found to depend on random excitations. It therefore seems natural that current trend in describing and studying these phenomena is focused on the use of stochastic mathematical models rather than deterministic ones. Having in mind that in many cases random excitations are of the Gaussian white noise type, which is mathematically described as a formal derivative of the Brownian motion, all such phenomena are mathematically modelled and essentially represented by complex stochastic differential equations of the Ito type. Obviously, the interest oh researchers is usually focused on conditions guaranteeing the existence, uniqueness and bifurcational behavior of solutions to these equations.

For example, the behavior of a non-linear dynamical system can be represented by the following differential equation

$$\ddot{y} + f(t, \dot{y}, y) = g(t, \dot{y}, y) \cdot \xi(t, \omega), \quad t \geq 0,$$

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where $\xi(t,\omega)$ is a Gaussian white noise perturbation and $\omega \in \Omega$ are random events. Since $\xi(t,\omega) = \dot{w}(t,\omega)$, where $w(t,\omega)$ is a Brownian motion, this equation can be transformed into the following stochastic system,

$$
\begin{aligned}
dy(t) &= x(t)\,dt \\
dx(t) &= -f\Big(t, x(t), c + \int_0^t x(s)\,ds\Big)dt + g\Big(t, x(t), c + \int_0^t x(s)\,ds\Big)dw(t),
\end{aligned}
$$

where $y(0) = c$ and ω is usually omitted, as we will do throughout the paper. The second equation in this system is the stochastic integrodifferential equation of the Ito type, which is a special case of the equation considered in the present paper,

$$
\begin{aligned}
dx(t) = F\Big(t, x(t), &\int_0^t f_1(t,s,x(s))\,ds, \int_0^t f_2(t,s,x(s))\,dw(s)\Big)dt \qquad (1)\\
&+G\Big(t, x(t), \int_0^t g_1(t,s,x(s))\,ds, \int_0^t g_2(t,s,x(s))\,dw(s)\Big)dw(t),\\
&t \in [0,T],\ x(0) = x_0 \text{ a.s.}
\end{aligned}
$$

Here $w = (w(t), t \geq 0)$ is a scalar Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $(\mathcal{F}_t, t \geq 0)$ of non-decreasing sub-σ-algebras of $\mathcal{F}$ ($\mathcal{F}_t = \sigma\{w(s), 0 \leq s \leq t\}$), x_0 is a random variable independent of w, the functions $F : [0,T] \times R^3 \to R$, $G : [0,T] \times R^3 \to R$, $f_i : J \times R \to R$, $g_i : J \times R \to R$, $i = 1,2$, where $J = \{(s,t) : 0 \leq s \leq t \leq T\}$, are assumed to be Borel measurable on their domains. The process $x = (x(t), t \in [0,T])$ is a strong solution to Eq. (1) provided it is adapted to $(\mathcal{F}_t, t \geq 0)$, all the Lebesgue and Ito integrals in the integral form of Eq. (1) are well defined, and Eq. (1) holds a.s. for each $t \in [0,T]$. We restricted ourselves to one-dimensional case; the multi-dimensional one is analogous and is not difficult by itself, but involves a complex notation.

Eq. (1) was studied earlier by many authors, first of all by Murge and Pachpatte [10, 11]. A somewhat simpler form of this equation, that is, linear with respect to Lebesgue and Ito integrals, was presented in paper [3] by Berger and Mizel. The basic existence-and-uniqueness theorem under classical conditions was proved in the above cited papers: Let $E|x_0|^2 < \infty$ and the functions F, G, f_i, g_i, $i = 1,2$ be globally Lipschitzian and satisfy the linear growth condition, i.e., let there exist a constant $L > 0$ such that for all $(t,s) \in J$ and $(x,y,z), (x',y',z') \in R^3$,

$$
\begin{aligned}
&|F(t,x,y,z) - F(t,x',y',z')| \leq L(|x-x'| + |y-y'| + |z-z'|), \qquad (2)\\
&|f_i(t,s,x) - f_i(t,s,x')| \leq L|x-x'|,\ i = 1,2
\end{aligned}
$$

$$
\begin{aligned}
&|F(t,x,y,z)| \leq L(1 + |x| + |y| + |z|), \qquad (3)\\
&|f_i(t,s,x)| \leq L(1 + |x|),\ i = 1,2,
\end{aligned}
$$

and analogously for G, g_1, g_2. Then, Eq. (1) has a unique a.s. continuous and $\mathcal{F}_t$-adapted solution $x(t)$ satisfying $E\sup_{t\in[0,T]}|x(t)|^2 < \infty$.

The focus of our analysis in the present paper is to study the existence and uniqueness of the solution to Eq. (1) under some non-classical conditions, that is, by using the concept of a random integral contractor which includes the Lipschitz condition as a special case.

2 Formulation of the problem and main results

The concept of integral contractors was introduced by Altman [1, 2] for studying some different classes of deterministic equations in Banach spaces. Later, this approach was appropriately extended by Kuo [6] to the notion of random integral contractors for stochastic differential equations of the Ito type, and also for special classes of stochastic integral and integrodifferential equations in various functional spaces. In particular, we highlight papers [4, 5, 6, 8, 9, 13, 12, 14] and esspecially paper [8] by Mao treating stochastic differential-functional equations with semimartingales. The important fact is that in all these papers, with a partial exception of [9], the considered equations were linear with respect to Lebesgue and Ito integrals, so the regular integral contractor was defined as a solution to any linear functional equation. However, in the present paper we study the non-linear case, which makes it difficult for us to introduce notions and present conditions guaranteeing the existence and uniqueness of the solution to Eq. (1). For that reason, the main aim in this paper is to introduce the notion of a non-linear bounded random integral contractor, so that the non-linearity of the Lebesgue and Ito integrals in Eq. (1) could be exceeded, and then to prove the existence and uniqueness of the solution.

In the remainder, let us denote that $\mathcal{C}$ is a collection of scalar stochastic processes, defined on $[0,T]$, continuous almost surely and adapted to the filtration $(\mathcal{F}_t, t \geq 0)$.

For reasons of notational simplicity, let us introduce the following operators: For each $x \in \mathcal{C}$,

$$(A_1x)(t) := \int_0^t f_1(t,s,x(s))\,ds, \quad (A_2x)(t) := \int_0^t f_2(t,s,x(s))\,dw(s)$$
$$(B_1x)(t) := \int_0^t g_1(t,s,x(s))\,ds, \quad (B_2x)(t) := \int_0^t g_2(t,s,x(s))\,dw(s).$$

Likewise, let us denote that

$$F[x(t)] = F\big(t, x(t), (A_1x)(t), (A_2x)(t)\big),$$
$$G[x(t)] = G\big(t, x(t), (B_1x)(t), (B_2x)(t)\big).$$

In accordance with this, Eq. (1) can be written in a shorter integral form

$$x(t) = x_0 + \int_0^t F[x(s)]\,ds + \int_0^t G[x(s)]\,dw(s), \quad t \in [0,T]. \tag{4}$$

Let

$$\Phi : [0,T] \times R^3 \to R, \qquad \Gamma : [0,T] \times R^3 \to R,$$
$$\Phi_i : J \times R \to R, \qquad \Gamma_i : J \times R \to R, \quad i = 1, 2,$$

be measurable mappings, bounded in the sense that there exist positive constants $\alpha, \beta, \alpha_i, \beta_i,\ i = 1, 2$, such that for every $(t, x, u, v) \in [0, T] \times R^3$, $(t, s, x) \in J \times R$, $y \in R$,

$$|\Phi(t,x,u,v)\cdot y| \le \alpha\,|y|, \quad |\Gamma(t,x,u,v)\cdot y| \le \beta\,|y|, \qquad (5)$$
$$|\Phi_i(t,s,x)\cdot y| \le \alpha_i\,|y|, \quad |\Gamma_i(t,s,x)\cdot y| \le \beta_i\,|y|,\ i=1,2.$$

We also introduce the following operators: For every $x, y \in \mathcal{C}$,

$$((\tilde\Phi_1 x)y)(t) := \int_0^t \Phi_1(t,s,x(s))\,y(s)\,ds,$$
$$((\tilde\Phi_2 x)y)(t) := \int_0^t \Phi_2(t,s,x(s))\,y(s)\,dw(s), \qquad (6)$$
$$((\tilde\Gamma_1 x)y)(t) := \int_0^t \Gamma_1(t,s,x(s))\,y(s)\,ds,$$
$$((\tilde\Gamma_2 x)y)(t) := \int_0^t \Gamma_2(t,s,x(s))\,y(s)\,dw(s),$$

and denote that

$$\Phi[x(t),y(t)] = \Phi\big(t, x(t), ((\tilde\Phi_1 x)y)(t), ((\tilde\Phi_2 x)y)(t)\big),$$
$$\Gamma[x(t),y(t)] = \Gamma\big(t, x(t), ((\tilde\Gamma_1 x)y)(t), ((\tilde\Gamma_2 x)y)(t)\big).$$

We are now able to introduce the following non-linear operator A: For every $x, y \in \mathcal{C}$,

$$((Ax)y)(t) := y(t) + \int_0^t \Phi[x(s),y(s)]\,y(s)\,ds \qquad (7)$$
$$+ \int_0^t \Gamma[x(s),y(s)]\,y(s)\,dw(s), \quad t \in [0,T].$$

Clearly, $(Ax)y \in \mathcal{C}$.

Definition 2.1 *Let there exist a positive constant K such that for every $x, y \in \mathcal{C}$ the following inequalities hold almost surely:*

$$\big|F[x(t) - ((Ax)y)(t)] - F[x(t)] + \Phi[x(t),y(t)]\cdot y(t)\big|$$
$$\le K\big[||y||_t + \big|(A_1(x-(Ax)y))(t) - (A_1x)(t) + ((\tilde\Phi_1 x)y)(t)\big|$$
$$+\big|(A_2(x-(Ax)y))(t) - (A_2x)(t) + ((\tilde\Phi_2 x)y)(t)\big|\big]$$
$$\big|f_i(t,s,x(s) - ((Ax)y)(s)) - f_i(t,s,x(s)) + \Phi_i(t,s,x(s))\cdot y(s)\big|$$
$$\le K\,||y||_s,\ i=1,2, \qquad (8)$$
$$\big|G[x(t) - ((Ax)y)(t)] - G[x(t)] + \Gamma[x(t),y(t)]\cdot y(t)\big|$$
$$\le K\big[||y||_t + \big|(B_1(x-(Ax)y))(t) - (B_1x)(t) + ((\tilde\Gamma_1 x)y)(t)\big|$$
$$+\big|(B_2(x-(Ax)y))(t) - (B_2x)(t) + ((\tilde\Gamma_2 x)y)(t)\big|\big],$$
$$\big|g_i(t,s,x(s) - ((Ax)y)(s)) - g_i(t,s,x(s)) + \Gamma_i(t,s,x(s))\cdot y(s)\big|$$
$$\le K\,||y||_s,\ i=1,2,$$

where $||y||_t = \sup_{0\le s\le t} |y(s)|$. *Then the set of functions* $\{F, f_1, f_2, G, g_1, g_2\}$ *has a bounded random integral contractor*

$$\Big\{I + \int_0^t \Phi\Big(s, x, \int_0^s \Phi_1\, dr, \int_0^s \Phi_2\, dw(r)\Big) ds \qquad (9)$$
$$+ \int_0^t \Gamma\Big(s, x, \int_0^s \Gamma_1\, dr, \int_0^s \Gamma_2\, dw(r)\Big) dw(s)\Big\}.$$

Definition 2.2 *A bounded random integral contractor (9) is said to be regular if the equation*

$$(Ax)\, y = z \qquad (10)$$

has a solution y *in* $\mathcal{C}$ *for any* x *and* z *in* $\mathcal{C}$.

Let $L_2([0,T]\times\Omega)$ be a collection of stochastic processes in $\mathcal{C}$ such that $P\big\{\int_0^T |x(t)|^2 dt < \infty\big\} = 1$.

Definition 2.3 *The functions* F *and* G *in Eq. (4) are said to be stochastically closed if for any* x *and* x_n *in* $\mathcal{C}$, *such that* $x_n \to x$ *and* $F[x_n] \to y$, $G[x_n] \to z$ *in* $L_2([0,T]\times\Omega)$, *it follows that* $y = F[x]$ *and* $z = G[x]$ *almost surely, for every* $t \in [0,T]$.

It is easy to check that if the functions F, f_1, f_2, G, g_1, g_2 satisfy the global Lipschitz condition (2), then F and G are stochastically closed and the set $\{F, f_1, f_2, G, g_1, g_2\}$ has a trivial integral contractor (9) for $\Phi = \Gamma = \Phi_i = \Gamma_i \equiv 0, i = 1, 2$. Obviously, the converse also holds. Moreover, if the global Lipschitz condition (2) is valid, let us prove that there exists a class of non-trivial bounded integral contractors, but that the converse assumption does not hold.

First, we can prove that

$$\Big\{I + \int_0^t \Phi\Big(s, x, \int_0^s \Phi_1\, dr, 0\Big)\, ds\Big\} \qquad (11)$$

is a bounded integral contractor for $\Phi_2 = \Gamma = \Gamma_1 = \Gamma_2 \equiv 0$. Since the Lipschitz condition (2) and the conditions (5) imply

$$\begin{aligned}
&|F[x(t) - ((Ax)y)(t)] - F[x(t)] + \Phi[x(t), y(t)] \cdot y(t)| \\
&\quad \le |F[x(t) - ((Ax)y)(t)] - F[x(t)]| + |\Phi[x(t), y(t)] \cdot y(t)| \\
&\quad \le L\big[|((Ax)y)(t)| + |(A_1(x - (Ax)y))(t) - (A_1x)(t)| \\
&\qquad +|(A_2(x - (Ax)y))(t) - (A_2x)(t)|\big] + \alpha\, |y(t)| \\
&\quad \le L\big[||((Ax)y||_t + |(A_1(x - (Ax)y))(t) - (A_1x)(t) + ((\tilde{\Phi}_1x)y)(t)| \\
&\qquad +|(A_2(x - (Ax)y))(t) - (A_2x)(t)|\big] + (\alpha + L\alpha_1 T)\, ||y||_t \quad \text{a.s.},
\end{aligned}$$

then

$$\begin{aligned}
||(Ax)y||_t &\le \sup_{0\le s\le t}\Big[|y(s)| + \int_0^s |\Phi[x(r), y(r)]\, y(r)|\, dr\Big] \qquad (12)\\
&\le (1 + \alpha T)\, ||y||_t.
\end{aligned}$$

Hence

$$\begin{aligned}
&|F[x(t)-((Ax)y)(t)]-F[x(t)]+\Phi[x(t),y(t)]\cdot y(t)|\\
&\quad\le K\big[||y||_t+|(A_1(x-(Ax)y))(t)-(A_1x)(t)+((\tilde{\Phi}_1x)y)(t)|\\
&\qquad +|(A_2(x-(Ax)y))(t)-(A_2x)(t)|\big].
\end{aligned}$$

that is, F satisfies (8) with the constant $K=L(1+\alpha_1T+\alpha T)+\alpha$. However,

$$|((\tilde{\Phi}_1x)y)(t)|\le\int_0^t|\Phi_1(t,s,x(s))\,y(s)|\,ds\le\alpha_1T\,||y||_t\ \text{ a.s.},\quad \Phi_2\equiv 0,$$

and thus

$$\begin{aligned}
&|f_1(t,s,x(s)-((Ax)y)(s))-f_1(t,s,x(s))+\Phi_1(t,s,x(s))\cdot y(s)|\\
&\quad\le L|((Ax)y)(s)|+\alpha_1\,|y(s)|\le[L(1+\alpha T)+\alpha_1]\cdot||y||_s\ \text{a.s.}
\end{aligned}$$

Since $\Gamma=\Gamma_1=\Gamma_2\equiv 0$, all the relations in (8) are satisfied and, therefore, the set of functions $\{F,f_1,f_2,G,g_1,g_2\}$ has a class of bounded integral contractors (11).

Conversely, if there exists a regular bounded integral contractor (11), it follows from (7) and (10) that the equation

$$y(t)+\int_0^t\Phi[x(s),y(s)]\,y(s)\,ds=z(t),\quad t\in[0,T],$$

has a solution $y\in\mathcal{C}$ for every x and z in $\mathcal{C}$. Then,

$$|z(t)|\le|y(t)|+\int_0^t|\Phi[x(s),y(s)]\,y(s)|\,ds\le(1+\alpha T)\,||y||_t\ \text{a.s.},\ t\in[0,T]. \tag{13}$$

Since $((Ax)y)(t)=z(t)$ a.s., from (8) we derive

$$\begin{aligned}
&\big|F[x(t)-z(t)]-F[x(t)]\big|\\
&\quad\le\big|F[x(t)-z(t)]-F[x(t)]+\Phi[x(t),y(t)]\,y(t)\big|+\big|-\Phi[x(t),y(t)]\,y(t)\big|\\
&\quad\le K\,\big[||y||_t+|(A_1(x-z))(t)-(A_1x)(t)+((\tilde{\Phi}_1x)y)(t)|\\
&\qquad +|(A_2(x-z))(t)-(A_2x)(t)|\big]+\alpha\,|y(t)|\\
&\quad\le\big[K(1+\alpha_1T)+\alpha\big]\big[||y||_t+|(A_1(x-z))(t)-(A_1x)(t)|\\
&\qquad +|(A_2(x-z))(t)-(A_2x)(t)|\big]\ \text{ a.s.},\quad t\in[0,T].
\end{aligned}$$

However, from (13) we see that $||y||_t$ does not have to be reduced with $|z(t)|$ a.s., so that Eq. (4) can have a bounded integral contractor, although the Lipschitz condition, in general, does not have to be satisfied. Therefore, the Lipschitz condition and the one based on the integral contractor *cannot, in general, be mutually compared*. This fact could be a motivation to focus our future analysis on conditions and function spaces in order to obtain some alternative existence-and-uniqueness theorems, as well as to establish relations between them.

We now state the following existence-and-uniqueness theorems using the notion of a bounded random integral contractor.

Theorem 2.1 *Let F and G be stochastically closed and $\int_0^T |F[x_0]|^2 dt < \infty$, $\int_0^T |G[x_0]|^2 dt < \infty$ a.s. Let also the set of functions $\{F, f_1, f_2, G, g_1, g_2\}$ has a bounded random integral contractor (9). Then Eq. (4) has a solution x in $\mathcal{C}$.*

Proof. The proof is based on the following iteration procedures: We define the sequences $\{x_n(t),\, n \geq 0\}$ and $\{y_n(t),\, n \geq 0\}$ in $\mathcal{C}$ such that

$$\begin{aligned} x_0(t) &= x_0 \text{ a.s.} \\ x_{n+1}(t) &= x_n(t) - ((Ax_n)y_n)(t) \qquad (14) \\ &= x_n(t) - y_n(t) - \int_0^t \Phi[x_n(s), y_n(s)]\, y_n(s)\, ds \\ &\quad - \int_0^t \Gamma[x_n(s), y_n(s)]\, y_n(s)\, dw(s), \\ y_n(t) &= x_n(t) - x_0 - \int_0^t F[x_n(s)]\, ds - \int_0^t G[x_n(s)]\, dw(s). \qquad (15) \end{aligned}$$

For simplicity, we shall prove this assertion step by step.

<u>Step 1.</u>

$$E||y||_t^2 \leq a\,\frac{A^n t^n}{n!}, \quad 0 \leq t \leq T, \quad n \in N, \tag{16}$$

where a and A are some generic constants.

Proof. Let us denote that

$$\begin{aligned} a(t) &= F[x_n(t)] - \Phi[x_n(t), y_n(t)]\, y_n(t) - F[x_{n+1}(t)], \qquad (17) \\ a_i(t,s) &= f_i(t,s,x_n(s)) - \Phi_i(t,s,x_n(s))\, y_n(s) - f_i(t,s,x_{n+1}(s)), \quad i = 1,2, \\ b(t) &= G[x_n(t)] - \Gamma[x_n(t), y_n(t)]\, y_n(t) - G[x_{n+1}(t)], \\ b_i(t,s) &= g_i(t,s,x_n(s)) - \Gamma_i(t,s,x_n(s))\, y_n(s) - g_i(t,s,x_{n+1}(s)), \quad i = 1,2. \end{aligned}$$

Then, from (14) and (15) we have

$$\begin{aligned} y_{n+1}(t) &= x_{n+1}(t) - x_0 - \int_0^t F[x_{n+1}(s)]\, ds - \int_0^t G[x_{n+1}(s)]\, dw(s) \\ &= x_n(t) - y_n(t) - \int_0^t \Phi[x_n(s), y_n(s)]\, y_n(s)\, ds \\ &\quad - \int_0^t \Gamma[x_n(s), y_n(s)]\, y_n(s)\, dw(s) \qquad (18) \\ &\quad - x_0 - \int_0^t F[x_{n+1}(s)]\, ds - \int_0^t G[x_{n+1}(s)]\, dw(s) \\ &\equiv \int_0^t a(s)\, ds + \int_0^t b(s)\, dw(s). \end{aligned}$$

If we take x_n instead of x and y_n instead of y in (8), we obtain

$$\begin{aligned}|-a(t)| &= \big|F[x_n(t)-((Ax_n)y_n)(t)] - F[x_n(t)] + \Phi[x_n(t),y_n(t)]\,y_n(t)\big| \\ &\le K\left[||y_n||_t + \left|-\int_0^t a_1(t,s)\,ds\right| + \left|-\int_0^t a_2(t,s)\,dw(s)\right|\right],\end{aligned}$$

and similarly,

$$|-b(t)| \le K\left[||y_n||_t + \left|-\int_0^t b_1(t,s)\,ds\right| + \left|-\int_0^t b_2(t,s)\,dw(s)\right|\right].$$

By applying the usual stochastic integral isometry, Schwarz inequality and Doob inequality [7], we find from (18) that

$$\begin{aligned}E\sup_{0\le s\le t}|y_{n+1}(s)|^2 &\le 2\left[t\int_0^t E|a(s)|^2 ds + 4\int_0^t E|b(s)|^2 ds\right] \\ &\le 2K^2\left\{3t\int_0^t E\left[||y_n||_s^2 + \left|-\int_0^s a_1(s,r)dr\right|^2 + \left|-\int_0^s a_2(s,r)dw(r)\right|^2\right] ds\right. \\ &\quad \left. +4\cdot 3\int_0^t E\left[||y_n||_s^2 + \left|-\int_0^s b_1(s,r)dr\right|^2 + \left|-\int_0^s b_2(s,r)dw(r)\right|^2\right] ds\right\}.\end{aligned} \tag{19}$$

We can estimate these integrals by using (8) and by applying integration by parts, which yields finally

$$E||y_{n+1}||_t^2 = E\sup_{0\le s\le t}|y_{n+1}(s)|^2 \le A\int_0^t E||y_n||_s^2\,ds, \quad n\in N,$$

where A is a generic constant. By repeating integration, it follows that

$$E||y_n||_t^2 \le \frac{A^n}{(n-1)!}\int_0^t (t-s)^{n-1}E||y_0||_s^2\,ds, \quad n\in N.$$

Since $y_0(t) = -\int_0^t F[x_0]\,ds - \int_0^t G[x_0]\,dw(s)$, then

$$E||y_0||_T^2 \le 2\left[T\int_0^T E\big|F[x_0]\big|^2 ds + 4\int_0^T E\big|G[x_0]\big|^2 ds\right] = a,$$

so that

$$E||y_n||_t^2 \le a\,\frac{A^n t^n}{n!}, \quad t\in[0,T],\ n\in N,$$

which proves the first step.

Step 2.

$$P\big\{||y_{n+1}||_T > 2^{-n-1}\big\} \le c\,\frac{c_1^n}{(n+1)!}, \tag{20}$$

where c and c_1 are generic constants.

Proof. It follows from (18) that

$$P\{||y_{n+1}||_T > 2^{-n-1}\} \le P\Big\{\int_0^T |a(s)|\,ds > 2^{-n-2}\Big\} + P\Big\{\sup_{0\le t\le T}\Big|\int_0^t b(s)\,dw(s)\Big| > 2^{-n-2}\Big\}. \tag{21}$$

The application of Chebyshev's inequality and (19) yields

$$\begin{aligned}P\Big\{\int_0^T |a(s)|\,ds > 2^{-n-2}\Big\} &\le 2^{2n+4}E\Big(\int_0^T |a(s)|\,ds\Big)^2\\ &\le 2^{2n+4}\cdot T\int_0^T E|a(s)|^2\,ds \le 3\cdot 2^{2n+4}K^2T\Big[\int_0^T E||y_n||_t^2\,dt\\ &\quad +K^2\int_0^T t\int_0^T E||y_n||_s^2\,ds\,dt + K^2\int_0^T\int_0^T E||y_n||_s^2\,ds\,dt\Big]\\ &\le c\,\frac{(4AT)^n}{(n+1)!},\end{aligned}$$

where c is a generic constant. The second term on the right-hand side in (21) can be estimated analogously and, therefore, (20) holds.

Step 3. The sequence $\{x_n\}$ in $\mathcal{C}$ converges almost surely, uniformly in $[0,T]$.

Proof. Let us start from (14) and derive that

$$\begin{aligned}\sup_{0\le t\le T}|x_{n+1}(t)-x_n(t)| &= ||x_{n+1}-x_n||_T\\ &\le ||y_n||_T + \int_0^T \big|\Phi[x_n(s),y_n(s)]\,y_n(s)\big|\,ds\\ &\quad + \Big\|\int_0^t \Gamma[x_n(s),y_n(s)]\,y_n(s)\,dw(s)\Big\|_T.\end{aligned}$$

From the boundedness of the mappings Φ and Γ we find that

$$\begin{aligned}P\Big\{\int_0^T \big|\Phi[x_n(s),y_n(s)]\,y_n(s)\big|\,ds > 2^{-n}\Big\} &\le 2^{2n}T\int_0^T E\big|\Phi[x_n(s),y_n(s)]\,y_n(s)\big|^2\,ds\\ &\le 2^{2n}T\alpha^2\int_0^T E||y_n||_s^2\,ds\\ &\le 2^{2n}T\alpha^2 a\frac{A^nT^{n+1}}{(n+1)!},\end{aligned}$$

$$P\Big\{\Big\|\int_0^t |\Gamma[x_n(s),y_n(s)]\,y_n(s)|\,dw(s)\Big\|_T > 2^{-n}\Big\} \le 4\cdot 2^{2n}\beta^2 a\frac{A^nT^{n+1}}{(n+1)!}.$$

Thus,

$$\begin{aligned}P\{||x_{n+1}-x_n||_T > 3\cdot 2^{-n}\}\\ \le P\{||y_n||_T > 2^{-n}\} + P\Big\{\int_0^T \big|\Phi[x_n(s),y_n(s)]\,y_n(s)\big|\,ds > 2^{-n}\Big\}\\ +P\Big\{\Big\|\int_0^t \Gamma[x_n(s),y_n(s)]\,y_n(s)\,dw(s)\Big\|_T > 2^{-n}\Big\}\\ \le c\cdot\frac{c_1^n}{n!},\end{aligned}$$

where c and c_1 are generic constants. Since

$$\sum_{n=1}^{\infty} P\{||x_{n+1}-x_n||_T > 3\cdot 2^{-n}\} < \infty,$$

the application of Borel-Cantelli's lemma yields that for all large enough n,

$$\sup_{0\le t\le T} |x_{n+1}(t)-x_n(t)| \le 3\cdot 2^{-n} \text{ almost surely.}$$

Therefore, the sequence $\{x_n(t)\}$ converges almost surely, uniformly in $[0,T]$.

Step 4. Let x_∞ be a limit of the sequence $\{x_n\}$. Then $x_\infty\in\mathcal{C}$ and $x_n\to x_\infty$ in $L_2([0,T]\times\Omega)$.

Proof. It follows from Step 3 that x_∞ is in $\mathcal{C}$ since the sequence $\{x_n\}$ in $\mathcal{C}$ converges to x_∞ almost surely. We deduce from (14) that

$$\begin{aligned}|x_{n+1}(t)-x_n(t)|^2 \le 3\,|y_n(t)|^2 + 3\,\Big|\int_0^t \Phi[x_n(s),y_n(s)]\,y_n(s)\,ds\Big|^2\\ +3\,\Big|\int_0^t \Gamma[x_n(s),y_n(s)]\,y_n(s)\,dw(s)\Big|^2.\end{aligned}$$

Then, (16) yields

$$\begin{aligned}E||x_{n+1}-x_n||_t^2 \le 3\left[E||y_n||_t^2 + \alpha^2 t\int_0^t E||y_n||_s^2\,ds + 4\beta^2\int_0^t E||y_n||_s^2\,ds\right]\\ \le 3aA^n\left[\frac{t^n}{n!} + \alpha^2\frac{t^{n+2}}{(n+1)!} + 4\beta^2\frac{t^{n+1}}{(n+1)!}\right].,\end{aligned}$$

which implies that

$$\int_0^T E||x_{n+1}-x_n||_t^2\,dt \to 0,\ n\to\infty,$$

and, therefore, $x_n\to x_\infty$ in $L_2([0,T]\times\Omega)$.

Step 5. Let $U_n(t)=\int_0^t F[x_n(s)]\,ds,\ \ V_n(t)=\int_0^t G[x_n(s)]\,dw(s),\ n\in N,$

$$U(t)=\int_0^t F[x_\infty(s)]\,ds,\ \ V(t)=\int_0^t G[x_\infty(s)]\,dw(s).$$

Then $U_n \to U$, $V_n \to V$ in $L_2([0,T]\times\Omega)$.

Proof. By applying (5) and (16), we find from (17) that

$$\begin{aligned}
E\int_0^T &\big|F[x_{n+1}(t)] - F[x_n(t)]\big|^2\,dt \\
&\le 2\,E\int_0^T \big|F[x_{n+1}(t)] - F[x_n(t)] + \Phi[x_n(t), y_n(t)]\,y_n(t)\big|^2\,dt \\
&\quad +2\,E\int_0^T \big|\Phi[x_n(t), y_n(t)]\,y_n(t)\big|^2\,dt \\
&\le 2\left[\int_0^T E|-a(t)|^2\,dt + \int_0^T \alpha\,E||y_n||_t^2\,dt\right] \\
&\le c\,\frac{A^nT^{n+1}}{(n+1)!} \to 0,\ n\to\infty,
\end{aligned}$$

where c is a generic constant. Therefore, $\{F[x_n]\}$ is a Cauchy sequence in $L_2([0,T]\times\Omega)$, which implies that $F[x_n]\to F[x_\infty]$ in $L_2([0,T]\times\Omega)$ since $x_n\to x_\infty$ in $L_2([0,T]\times\Omega)$ and since the coefficients of Eq. (4) are stochastically closed in the sense of Definition 2.2. Hence

$$\begin{aligned}
\int_0^T E|U_n(t)-U(t)|^2\,dt &= \int_0^T E\left|\int_0^t \big(F[x_n(s)] - F[x_\infty(s)]\big)\,ds\right|^2 dt \\
&\le T^2\int_0^T E\big|F[x_n(t)] - F[x_\infty(t)]\big|^2 dt \to 0,\ n\to\infty,
\end{aligned}$$

and, therefore, $U_n\to U$ in $L_2([0,T]\times\Omega)$.

Since $V_n\to V$ in $L_2([0,T]\times\Omega)$ is based on the same computation, the proof of Step 5 becomes complete.

Finally, we will complete the proof of Theorem 2.1. By taking $L_2([0,T]\times\Omega)$ limits on the both sides in (15) and by applying the conclusions of Steps 1, 4 and 5, we observe for all $t\in[0,T]$ that

$$x_\infty(t) = \int_0^t F[x_\infty(s)]\,ds + \int_0^t G[x_\infty(s)]\,dw(s) \quad \text{a.s.} \tag{22}$$

Consequently, x_∞ is the solution to Eq. (4) since the processes on the both sides of (22) are continuous almost surely for all $t\in[0,T]$, which completes the proof. □

Theorem 2.2 *Let the functions F, f_1, f_2, G, g_1, g_2 satisfy the assumptions of Theorem 2.1 and the bounded random integral contractor (9) is regular. Then the solution x to Eq. (4) in $\mathcal{C}$ is unique.*

Proof. Let x_1 and x_2 be two solutions to Eq. (4). Since the bounded random integral contractor (9) is regular, the equation

$$(Ax_1)y = x_1 - x_2$$

has a solution $y \in \mathcal{C}$, that is,

$$\begin{aligned} y(t) &+ \int_0^t \Phi[x_1(s), y(s)]\, y(s)\, ds + \int_0^t \Gamma[x_1(s), y(s)]\, y(s)\, dw(s) \\ &= x_1(t) - x_2(t) \\ &= \int_0^t \big(F[x_1(s)] - F[x_2(s)]\big)\, ds + \int_0^t \big(G[x_1(s)] - G[x_2(s)]\big)\, dw(s). \end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned} y(t) = &- \int_0^t \big(F[x_2(s)] - F[x_1(s)] + \Phi[x_1(s), y(s)]\, y(s)\big)\, ds \\ &- \int_0^t \big(G[x_2(s)] - G[x_1(s)] + \Gamma[x_1(s), y(s)]\, y(s)\big)\, dw(s). \end{aligned} \tag{24}$$

Since $E||y||_t^2$ is generally not finite, we use the truncation:

$$I_N(t) = \begin{cases} 1, & ||y||_s \leq N,\ 0 \leq s \leq t, \\ 0, & \text{otherwise.} \end{cases}$$

Then $I_N \in \mathcal{C}$ and $I_N(t) = I_N(t) \cdot I_N(s) \cdot I_N(r)$ for $0 \leq r \leq s \leq t \leq T$. Now, (24) implies

$$\begin{aligned} &EI_N(t)||y||_t^2 \\ &\quad \leq 2\Big[t\, E \int_0^t I_N(s) \big|F[x_2(s)] - F[x_1(s)] + \Phi[x_1(s), y(s)]\, y(s)\big|^2 ds \\ &\qquad + 4\, E \int_0^t I_N(s) \big|G[x_2(s)] - G[x_1(s)] + \Gamma[x_1(s), y(s)]\, y(s)\big|^2 ds\Big]. \end{aligned}$$

From (8) we find that

$$\begin{aligned} &\big|F[x_2(s)] - F[x_1(s)] + \Phi[x_1(s), y(s)]\, y(s)\big| \\ &\quad \leq K\, \big[||y||_s + \big|(A_1x_2)(s) - (A_1x_1)(s) + ((\tilde{\Phi}_1x_1)y)(s)\big| \\ &\qquad + \big|(A_2x_2)(s) - (A_2x_1)(s) + ((\tilde{\Phi}_2x_1)y)(s)\big|\big], \end{aligned} \tag{25}$$

and similarly for G. From now on, we observe that

$$\begin{aligned} &EI_N(t)||y||_t^2 \\ &\quad \leq 6K^2 \Big\{t\, E \int_0^t I_N(s) \big[\, ||y||_s^2 + \big|(A_1x_2)(s) - (A_1x_1)(s) + ((\tilde{\Phi}_1x_1)y)(s)\big|^2 \\ &\qquad + \big|(A_2x_2)(s) - (A_2x_1)(s) + ((\tilde{\Phi}_2x_1)y)(s)\big|^2\big]\, ds \\ &\qquad + 4E \int_0^t I_N(s) \big[\, ||y||_s^2 + \big|(B_1x_2)(s) - (B_1x_1)(s) + ((\tilde{\Gamma}_1x_1)y)(s)\big|^2 \\ &\qquad + \big|(B_2x_2)(s) - (B_2x_1)(s) + ((\tilde{\Gamma}_2x_1)y)(s)\big|^2\big]\, ds\Big\}. \end{aligned}$$

To estimate the right-hand side in the previous relation, we will proceed analogously to (19). By omitting details, we obtain finally

$$\begin{aligned} EI_{N}(t)||y||_t^2 \le 6K^2\Big\{ t\int_0^t EI_{N}(s)||y||_s^2\,ds \\ + 2K^2 t\int_0^t I_{N}(s)s\int_0^s EI_{N}(r)||y||_r^2\,dr\,ds \\ + 4\int_0^t EI_{N}(s)||y||_s^2\,ds \\ + 8K^2\int_0^t I_{N}(s)\int_0^s EI_{N}(r)||y||_r^2\,dr\,ds\Big\} \\ \le c\int_0^t (1+t-s)EI_{N}(s)||y||_s^2\,ds \\ \le c\,(1+T)\int_0^t EI_{N}(s)||y||_s^2\,ds, \end{aligned}$$

where c is a constant. To close the proof, we apply the well-known the Gronwall-Bellman lemma and conclude that $EI_N(t)||y||_t^2 = 0$ for all $t \in [0,T]$. The application of the Lebesgue monotone convergence theorem implies that $E||y||_t^2 = \lim_{N\to\infty} EI_N(t)||y||_t^2 = 0$, $t \in [0,T]$ and, therefore, $y(t) = 0$ almost surely for all $t \in [0,T]$. Finally, from (23) it follows that $x_1 = x_2$ almost surely, which completes the proof. □

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