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ON THE SOLVABILITY OF NONLINEAR INTEGRAL EQUATIONS IN LEBESGUE SPACE

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Abstract

In this paper we prove theorems on the existence of integrable and monotonic solutions of nonlinear integral equation in Lebesgue Space. The basic tool used in the proof is the fixed point theorem due to Darbo with respect to the so-called measure of weak noncompactness.

1 Introduction

Linear integral equations are considered as branch of the applications of functional analysis. This branch is of great importance, not only for the specialists in this field but also for those whose interest lies in the other branaches of mathematics with special reference to mathematical physics.

The most frequently investigated integral equation in nonlinear functional analysis are the Hammerstein equation and the Urysohn equation. these equations have been studied in several papers and monograph (see for example Krasnoselskii et al ([11] and Appell [1, 2])

The aim of this paper is to prove theorem on existence of solutions of the nonlinear integral equation

$$y(t) = g(t) + \lambda(t) \int_0^t k(t,s)\, f(s, y(\phi(s)))\, ds, \qquad t \in (0,1) \tag{1}$$

The considered nonlinear integral equation in this paper is the general form of Hammerstein integral equation. These kinds of nonlinear integral equations appear in many applications. For instance, it can be applied to solve many problems in physic, engineering and economics. Also many problems considered in the theory of partial differential equation lead us to nonlinear integral equations of the type mentioned in J. Banas and K. Goebel [6] .

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2 Notation and auxiliary facts

Definition: Let E be a measurable set and $f\ (x)$ is a real function defined on E.

We say that $f\ (x)$ is Lebesgue measurable or briefly, measurable on E if for each real number k the values $x \in E$ for which $f\ (x) \succ k$ is measurable.

Let $L^1\,(a,b)$ denote the space of Lebesgue integral functions on the interval (a,b) with the stander form

$$\|y\| = \int_a^b |\ y\ (\ t\ )|\, dt\ .$$

For simplicity, we shall consider the space $L^1 = L^1\,(0,1)$.

This section is mainly devoted to recall some auxiliary result which will be needed further on. Denoted by M be the set of all functions which are measurable on $(0,1)$ with the metric

$$d\ (x,y) = \inf.\{a + meas.\,(t : |x\ (\ t\ ) - y\ (\ t\ )| \succeq a) : a \succ 0\},$$

then M becomes a complete metric space. Moreover, it is well known that the convergence in measure coincides with convergence generated by the metric d (see Dunford and Schwarts |10|).

The convergence in measure of a sequence $\{x_n\} \sqsubset L^1$does not imply the weak convergence of this sequennce and conversely. Nevertheless, we have the following result.

Lemma : If a sequence $\{x_n\} \sqsubset L^1$converges weakly to $x \in L^1$ and is compact in measure then it converges in measure to x [11].

Apart from this we recall that a sequence $\{x_n\} \sqsubset L^1$ is convergent strongly (that is, in the norm of L^1)to x if and only if it converges in measure to x and is weakly compact.

We will present notes about the linear integral operator.

Consider the integral operator.

$$(Ky)\,(t) = \int_0^1 k\,(t,s)\,y\,(s)\,ds, \qquad t \in (0,1),$$

where x is assumed to be a function of L^1 and $k\ :\ (0,1)\ \times (0,1) \longrightarrow R$ is assumed to be measurable with respect to both its variables.

Note that there are no necessary and sufficient conditions for the linear integral operator K , generated by the kernel $k\,(t,s)$ to be self mapping of the space L^1, but if the operator K satisfies the following conditions :

There exists a positive constant C such that

$$ess.\sup_{t\,\in\,(0,1)} \int_0^1 |\ k\ \ (t,s\ )|\ ds \preceq C\ , \qquad ess.\sup_{s\,\in\,(0,1)} \int_0^1 |\ k\ \ (t,s\ )|\ ds \preceq C\ ,$$

then K maps the space L^1 into itself [10] and so it will continuous on L^1 [11].

In the sequel the following theorem, due to Krzyz [12] gives the necessary and sufficient condition that the linear operator K maps functions being non-increasing on $(0,1)$ into function of the same kind.

Theorem 1. Let $k\ :\ (0,1) \times (0,1) \longrightarrow R_+$ be a measurable function generating the linear operator K acting from L^1 into L^1. If for every $p \in [o,1]$ and for all $t_{1,}\ t_2\ \in [o,1]$ the implication holds

$$t_1 \prec t_2 \Longrightarrow \int_0^b k(t_1, s)\, ds \succeq \int_0^b k(t_2, s)\, ds ,$$

then the operator K transforms the set of positive and non-increasing functions from L^1 into itself .

The following sufficient condition will be more convenient for our purposes.

Theorem 2. Let X be a bounded subset of $L^1(0,1)$ and suppose that there is a family of measurable subset $\{\Omega_c\}_{o \preceq c \preceq 1}$ of the interval $[o,1]$ such that $meas.\ \Omega_c = c$. If for any $c \in [o,1]$ and for any $y \in X$, $y(t_1) \preceq y(t_2)$, $(t_1 \in \Omega_c,\ t_2 \notin \Omega_c)$ then the set X is compact in measure [11] .

Now, let us assume that $f = f\ (t,x)$ and $f\ : (0,1) \times R \longrightarrow R$ satisfies Caratheodory conditions i.e. it is measurable in t for any x and continuous in x for almost all t . Then to every function $y(t)$ being measurable on the interval $(0,1)$ we may assign the function $(F y)(t) = f\ (t, y(t))$, $t \in (0,1)$. The operator defined in such way is called the superposition operator.

Notice that this operator is one of the simplest and most important operator investigated in nonlinear functional analysis ([1] , [11]) .

Theorem 3. The superposition operator F maps continuously the space $L^1(0,1)$ into itself if and only if $|f\ (t,\ s)| \preceq a(t) + b\,|x|$ for all $t \in (0,1)$ and $x \in R$, where $a(t)$ is a function from L^1 and b is a non-negative constant.

Now, let S denotes an arbitrary Banach space and let X be a nonempty and bounded subset of S . Moreover, denote by B_r the closed ball in S centered at θ and with radius r .

Let us recall the notion of the measure of weak and strong noncompactness defined by De Blasi [9] and Hausdorff [6], respectively in the following way:

$\beta(X) = \inf.\{r \succ 0$ *there exists a weakly compact subset* Y *of* E *such that* $X \sqsubset Y + B_r\}$,

$\chi(X) = \inf\{r \succ 0$ *there exists a compact subset* Y *of* E *such that* $X \sqsubset Y + B_r\}$.

The functions $\beta(X)$ and $\chi(X)$ possess several useful properties which may be found in [9] and [7] . The convenient and handly formula for the function $\beta(X)$ in the space L^1 was gives by Appell and De pascale [3]:

$$\beta(X) = \lim_{c \longrightarrow 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |y(t)|\, dt : D \sqsubset (0,1),\ meas.\ D \leq \varepsilon \right] \right\} \right\},$$

where the symbol $meas.D$ stands for Lebesgue measure of a subset D. The two measures $\beta(X)$ and $\chi(X)$ are connected in the case when X is compact in measure as in the following theorem.

Theorem 4. Let X be an arbitrary nonempty and bounded subset of $L^1(0,1)$ if X is compact in measure then $\beta(X) = \chi(X)$.

Now, we recall the fixed point theorem due to Darbo [8] .

Theorem 5. Let Q be a nonempty, closed, convex and bounded subset of S and let $T : Q \longrightarrow Q$ be a continuous operator having the property that there is a constant $k \in (0,1)$ such that $\chi(TX) \leq k\,\chi(X)$ for any nonempty subset X of Q , then T has at least one fixed point in the Q.

3 Main results

In this section we shall investigate the nonlinear integral equation (1). For convenience the Hammerstein operator

$$(\ H\ y\ )(\ t\ ) = \int_0^t k\,(t,s)\,f\ \ (s,y\ \ (s))\,ds.$$

Will be written as product $H = K\ F$ of the superposition operator

$$(\ Fy\ )(\ t\ ) = f\ \ (\ t,\ y\ \ (\ t\ ))$$

and the linear integral operator

$$(K\ y\ )(\ t\ ) = \int_0^t k\,(\ t,\ s\ )\,y\,(s)\ \ ds.$$

Then the equation (1) has the form $y = T\ y$ where

$$T\ \ y\ = g\ + \lambda\ K\ F\ y \qquad (2)$$

Now, we formulate the assumption under which the equation (1) will be invesigated. Namely, we assume the following :

(i) The function $\lambda, g \in L^1$ is a.e. positive non-increasing on the interval $(0,1)$,

(ii) f :$(0,1) \times R \longrightarrow R_+$ satisfies Caratheodory conditions and there exist a function $a\,(t) \in L^1$and a

non-negative constant b such that $f\ \ (\ t,x\ ) \ \le\ a\ (t)\ + b\,|\ x\ |$, for all $t \in (o,1)$ and $x \in R$.

Moreover, $f\ \ (\ t,x\ )$ is assumed to be non-increasing on the set $(0,1)\times R \longrightarrow R$ with respect to t

and non-decreasing with respect to x ,

(iii) $k\ :\ (0,1) \times (0,1) \longrightarrow R$ is measurable with respect to both variables and such that the integral

operator K maps L^1 into itself,

(iv) For every $p \in (0,1)$ and for all $t_{1,}\ t_2 \in (0,1)$ the following implication holds true

$$t\ _1\ \ \prec\ t\ _2\ \Longrightarrow\ \int_0^p\ k\,(t_1\ ,\ \mathrm{s})\,ds\ \ \succeq\ \ \int_0^p\ k\,(t_2\ ,\ \mathrm{s})\,ds\ ,$$

(v) $\phi : (0,1) \longrightarrow (0,1)$ is an increasing absolutely continuous and there exists constant $C \succ 0$

such that $\varphi'(\ t\ ) \succeq C$ for almost all $t \in (0,1)$,

(vi) $\frac{b\ \|\ \lambda\ \|\ \|\ k\ \|}{C} \prec 1.$

Then we can prove the following theorem:

Theorem 6. Under the above assumptions $(\ i\ ) \Longrightarrow (vi\ )$ the equation (1) has at least one solution $y \in L^1$which is a.e. non-increasing on the interval $(0,1)$.

Proof: Let us take an arbitrary $y \in L^1$, then according to the assumption (ii) , (iii) ,and (v) we have $T\ y \in L^1$, where T is the operator defined in (2) .

Moreover, we get

$$\|Ty\| = \int_0^1 \left| g(t) + \lambda(t) \int_0^1 k(t,s)\ f\ (s, y(\phi(s)))\ ds \right| dt$$

$$\preceq \ \| g \| + \| \lambda \| \| k \| \int_0^1 |f\ (s, y\ (\phi(s)))\ ds|\, dt$$

$$\preceq \ \| g \| + \| \lambda \| \| k \| \int_0^1 \{a\ (s) + b\ |y\ (\phi(s))|\}\, ds$$

$$\preceq \ \| g \| + \| \lambda \| \| k \| \| a \| + \ \frac{b \| \lambda \| \| k \|}{C} \int_0^1 |y\ (\phi(s))|\, \phi' ds$$

$$\preceq \ \| g \| + \| \lambda \| \| k \| \| a \| + \frac{b \| \lambda \| \| k \|}{C} \|y\| .$$

From the above estimate we conclude the operator T transform the ball B_r into itself, where

$$r = \frac{\| g \| + \| a \| \| \lambda \| \| k \|}{1 - b \| \lambda \| \| k \|} .$$

Now, let Q_r denoted the subset of B_r consisting of all functions being positive and a.e. non-increasing on $(0,1)$. The set Q_r is obviously nonempty, bounded, convex, closed and compact in measure, this is since, the convexity of Q_r is for if $y_1, y_2 \in Q_r$, then $\|y_i\| \preceq r$, $(i = 1,2)$.

Now, for $0 \preceq t \preceq 1$ let $y = t\, y_1 + (1-t)\, y_2$, then we have

$$\| y \| \preceq t\ \| y_1\| + (1-t)\ \| y_2\| \preceq t\ r + (1-t)\ r = \ r\ ,$$

it means that $y \in Q_r$ and so Q_r is convex. For closeness of Q_r , let (y_n) be a strong convergent sequence of elements in Q_r and it converges to y , then the sequence (y_n) converges in measure to y [4]. By using Vitali theorem [13] there exists a subsequence (y_{k_n}) of (y_n) which converges to y a.e. on $(o,1)$ and y will be non-increasing a.e. on $(0,1)$ which means that $y \in Q_r$ and so Q_r is closed.

Finally, the compactness in measure of Q_r can be deduced by using Theorem 2 considering that $\Omega_c = c$ for any $c \in (0,1)$.

Further, let us take an arbitrary function $y \in Q_r$ then $y(\phi)$ is a.e. non-increasing and positive on the interval $(0,1)$. By using the assumption (ii) $Fy(\phi)$ is also a.e. non- increasing and positive on $(0,1)$. The image $KFy\ (\phi)$ by the operator K is also of the same kind that is due to Theorem 1 and the assumption $(iii), (iv)$.

Next, in view of the assumption (ii) again we deduce that $T\ y$ is a.e. non-increasing and positive on the interval $(0,1)$.Therefore, given that $T\ : B_r \longrightarrow B_r$, we conclude that T is a self mapping of the set Q_r

Now, we observe that the assumption (ii) in conjunction with Theorem 3 and the continuity of the opoerator K on the space L^1allows us to deduce that the operator T is continuous on the set Q_r .

In what follows take a nonempty set $X \sqsubset Q_r$ and fix $\varepsilon \succ 0$. Further, let $D \sqsubset (0,1)$ be such that meas $D \preceq \varepsilon$. Then, for an arbitrary $x \in X$ in view of our assumptions we obtain

$$\int_D (Ty)(t)\, dt = \int_D |\ g(t)|\, dt + \int_D \left| \lambda(t) \int_0^1 k(t,s)\, f\ (s, y(\phi(s)))\, ds \right| dt$$

$$\preceq \|g\|_{L^1(D)} + \int_D |\lambda(t)| \left| \int_0^1 k(t,s)\, f\ (s, y(\phi(s)))\, ds \right| dt$$

$$\preceq \|g\|_{L^1(D)} + \int_D |\lambda(t)| \, |KF\ (y\ (\phi))| \ dt$$
$$\preceq \|g\|_{L^1(D)} + \|B\|_{L^1(D)} \|K\|_D \int_D \ |a + b\ |y\,(\phi\,(s))\ ||\ ds$$

$$\preceq \|g\|_{L^1(D)} + \|B\|_{L^1(D)} \|K\|_D \left[\|a\|_{L^1(D)} + \frac{b}{C} \int_D |y\ (\phi\,(s))| \, \acute{\phi}(s)\, ds \right]$$

where the operator K maps the space $L^1(D)$ into itself and continuous.

Also, the symbol $\|K\|_D$ stands for the norm of the operator $K : L^1(D) \longrightarrow L^1(D)$.

Now applying the theorem on integration by substitution for Lebesgue integrals we can rewrite the last estimation in the following way:

$\int_D |(Ty)(t)|\, dt \preceq \|g\|_{L^1(D)} + \|a\|_{L^1(D)} \|B\|_{L^1(D)} \|K\|_D + \frac{b\,\|B\|_{L^1(D)}\|K\|_D}{C}$ $\int_{\phi(D)} |y\ (t)|\, dt.$

Further, taking into account the equality

$$\lim_{\varepsilon \longrightarrow 0} \left\{ \sup \left[\int_D g\,(t)\, dt + \|a\|_{L^1(D)} \|B\|_{L^1(D)} \|K\|_D : D \sqsubset (0,1)\,, meas.D \leq \varepsilon \right] \right\} = 0,$$

and keeping in mind the absolute continuity of the function ϕ we obtain

$$\beta\,(T\ Y) \preceq \frac{b\,\|B\|_{L^1(D)}\|K\|_D}{C} \beta\ (Y)\,,$$

where β is the De Blasě measure of weak noncompactness.

In view of the properties of the set Q_r established before and Theorem (4) we can rewrite the last inequality in the following form

$$\chi\,(T\ Y) \preceq \frac{b\,\|B\|_{L^1(D)}\|K\|_D}{C} \chi\ (Y)\,,$$

where χ is the Hausdorf measure of noncompact.

The last inequality together with assumption (vi) enables us to apply Theorem 5. This completes the proof.

Example. Consider the integral equation

$$y\ (\ t\ ) = \ \frac{1-t}{1+t} \ + \ \frac{1}{1+t} \ \int_0^1 e^{-t+s} \tan^{-1}\left(\frac{s\ |\ y+2\ |}{s\ +\ 2}\right) ds \quad \text{..........} \qquad (3)$$

In this case we have $g\ (\ t\ ) = \frac{1-t}{1+t}$, is a.e. positive non-increasing function on $(0,1)$, where

$$\acute{g}\ (\ t\ ) = \frac{-2}{(1+t)^2} \prec 0 \quad \text{for} \quad t \in (o,1)\,,$$

$\lambda\ (\ t\ ) = \frac{1}{1+t}$ is also a.e. positive non-increasing function on $(0,1)$, so condition (i) is satisfied.

$k\ (\ t,\ s\ ) = e^{-t\ +\ s}$ is continuous and so measurable function for all $t\ , s$.

For $x \in L^1$, we can see that the linear operator

$$(K\ x\ )\,(\ t\ ) = \int_0^1 k\ (\ t,\ s\ )\, x\ (s\ )\, ds,$$

Transform $L^1(0,1)$ into itself.

Indeed,

$$\int_o^1 \left| \ \int_0^1 \ k\,(\ t,\ s\ )\,|\ x\ \ (s\ )|\, ds \right| \prec e^t \int_o^1 |\ x\ \ (s\ )|\, ds,$$

and hence condition (iii) is satisfied.

Next, $f\ \ (t,y) = \ \tan^{-1}\left(\frac{t\ |\ y+2\ |}{t\ +\ 2}\right)$ is continuous function for y and measurable for t as well as

$$| \ f \ \ (t,y)| = \left|\tan^{-1}\left(\frac{t \mid y+2 \mid}{t+2}\right)\right| = \tan^{-1}\left(\frac{t \mid y+2 \mid}{t+2}\right) \prec \frac{t \ (y+2)}{t+2}$$
$\prec \frac{1}{3} \mid y \mid \ +2t$

So, $a\ (\ t\ ) = 2t \in L^1\,(0,1\ )$and $b = \frac{1}{3} \succ 0$ hence we get (ii).

Also, condition (iv) is satisfied, when $b = \frac{1}{3}$, $\mid \lambda \mid = \frac{1}{1+\ t} \prec 1, \| \ k \ \| \prec e$, $\phi\,(t\ ) = t$ and $\acute{\phi}\,(\ t\ ) = 1$, $C = 1$.

Then, we have

$$\frac{b \ \| \ \lambda \ \|\| \ K \ \|}{C} \prec \frac{e}{3} \prec 1.$$

So, all conditions of Theorem (6) are satisfied, hence the integral equation (3) has a non-increasing solution in $L^1\,(0,1\ )$.

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