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SOME FIXED POINT THEOREMS ON ORDERED UNIFORM SPACES

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Abstract

In this paper, we introduce an order relation on uniform spaces and utilize this relation to prove some fixed point theorems for single and multi valued mappings in ordered uniform spaces.

1 Introduction

There exists considerable literature of fixed point theory dealing with results on fixed or common fixed points in uniform space (e.g. [1]-[4], [8]-[15]). But the majority of these results are proved for contractive or contractive type mappings (notice from the cited references). Recently, Aamri and El Moutawakil [2] have introduced the concept of E-distance function on uniform spaces and utilize it to improve some well known results of the existing literature involving both E-contractive or E-expansive mappings. In this paper, we introduce a partial ordering on uniform spaces utilizing E-distance function and use the same to prove a fixed point theorem for single-valued non-decreasing mappings on (thus obtained) ordered uniform spaces. Similar results are also proved for multi-valued mappings.

With a view to have a possibly self contained presentation, we recall some relevant definitions and properties from the foundation of uniform spaces. For further informations and details, one is referred to the book by Bourbaki [5]. We call a pair (X, ϑ) to be a uniform space which consists of a non-empty set X together with an uniformity ϑ wherein the latter begins with a special kind of filter on $X \times X$ whose all elements contain the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$ then x and y are said to be V-close. Also a sequence $\{x_n\}$ in X, is said to be a Cauchy sequence w.r.t. uniformity ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $m, n \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ.

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A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to diagonal Δ of X i.e. $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V = V^{-1} = \{(y, x) : (x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-close and then one may assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they are naturally interpreted with respect to the topological space $(X, \tau(\vartheta))$.

2 Preliminaries

We shall require the following definitions and a lemma in the sequel.

Definition 1 ([2]). *Let (X, ϑ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an E-distance if*

(p_1) For any $V \in \vartheta$ there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$, for some $z \in X$, imply $(x, y) \in V$,

(p_2) $p(x, y) \leq p(x, z) + p(z, y)$, $\forall x, y, z \in X$.

One can find some examples of E-distances in [2]. Especially, one can infer from Example 1 (contained in [2]) that every metric d on a uniform space (X, ϑ) is an E-distance.

The following lemma embodies some useful properties of E-distance.

Lemma 1 ([1], [2]). *Let(X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:*

(a) If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

(b) If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X, ϑ).

Let (X, ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2 ([1], [2]). *Let (X, ϑ) be a uniform space and p be an E-distance on X.Then*

(i) X said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim\limits_{n\to\infty} p(x_n, x) = 0$,

(ii) *X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,*

(iii) *$f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(fx_n, fx) = 0$; and*

(iv) *$f : X \to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} fx_n = fx$ with respect to $\tau(\vartheta)$.*

Remark 1 ([2]). *Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 1 (b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness.*

3 Main results

We begin with the following lemma whose metric version is available in [6].

Lemma 2. *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\phi : X \to \mathbb{R}$. Define the relation "$\preceq$" on X as follows;*

$$x \preceq y \iff x = y \text{ or } p(x, y) \leq \phi(x) - \phi(y).$$

Then "$\preceq$" is a (partial) order on X induced by ϕ.

Proof. For all $x \in X$, $x = x$, therefore $x \preceq x$ i.e. "$\preceq$" is reflexive. Again for $x, y \in X$, let $x \preceq y$ and $y \preceq x$, then

$$x = y \text{ or } p(x, y) \leq \phi(x) - \phi(y)$$

and

$$y = x \text{ or } p(y, x) \leq \phi(y) - \phi(x).$$

Therefore either $x = y$ or $p(x, y) + p(y, x) = 0$. If $p(x, y) + p(y, x) = 0$, than $p(x, y) = 0$ and $p(y, x) = 0$ yielding thereby $p(x, x) = 0$ which in view of Lemma 1 (a) gives $x = y$. Thus "$\preceq$" is anti-symmetric. Now for $x, y, z \in X$, let $x \preceq y$ and $y \preceq z$, then

$$x = y \text{ or } p(x, y) \leq \phi(x) - \phi(y)$$

and

$$y = z \text{ or } p(y, z) \leq \phi(y) - \phi(z).$$

Now we distinguish the following four cases.

Case 1. If $x = y$ and $y = z$, then $x = z$ i.e. $x \preceq z$,

Case 2. If $x = y$ and $p(y, z) \leq \phi(y) - \phi(z)$, then $p(x, z) \leq \phi(x) - \phi(z)$ i.e. $x \preceq z$,

Case 3. If $p(x, y) \leq \phi(x) - \phi(y)$ and $y = z$, then $p(x, z) \leq \phi(x) - \phi(z)$ i.e. $x \preceq z$, Case 4. If $p(x,y) \leq \phi(x) - \phi(y)$ and $p(y,z) \leq \phi(y) - \phi(z)$, then

$$\begin{aligned} p(x,z) &\leq p(x,y) + p(y,z) \\ &\leq \phi(x) - \phi(y) + \phi(y) - \phi(z) \\ &= \phi(x) - \phi(z) \end{aligned}$$

i.e. $x \preceq z$. Therefore "$\preceq$" is transitive.

Now we give some examples.

Example 1. *Consider $X = [0, \infty)$ equipped with usual metric $d(x,y) = |x - y|$. Define the functions p and ϕ as $p(x,y) = y$ and $\phi(x) = x$. Now for $x, y \in X$, we have*

$$\begin{aligned} x \preceq y &\iff x = y \text{ or } p(x,y) \leq \phi(x) - \phi(y) \\ &\iff x = y \text{ or } y \leq x - y \\ &\iff x = y \text{ or } 2y \leq x. \end{aligned}$$

which show that $1 \npreceq 2$, $2 \preceq 1$ and for all $x \in X$, $x \preceq 0$.

Example 2. *Let X and d be the same as in Example 1. Define $p(x,y) = |x - y|$ and $\phi(x) = -x$. Then we have the usual order on X.*

Our main result proved for single-valued mappings runs as follows.

Theorem 1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X such that X is S-complete and $\phi : X \to \mathbb{R}$ be a function which is bounded below. If "$\preceq$" is the partial order induced by ϕ. and $f : X \to X$ is a p-continuous or $\tau(\vartheta)$-continuous non-decreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X.*

Proof. Consider a point $x_0 \in X$ satisfying $x_0 \preceq fx_0$. Now, define a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for $n = 1, 2, \cdots$. Since f is non-decreasing we have $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$ i.e. the sequence $\{x_n\}$ is non-decreasing. By the definition of "$\preceq$" ,we have $\cdots \phi(x_2) \leq \phi(x_1) \leq \phi(x_0)$ i.e. the sequence $\{\phi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Since ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy i.e. for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_m) - \phi(x_n)| < \varepsilon$. Since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \leq \phi(x_n) - \phi(x_m)$. Therefore,

$$\begin{aligned} p(x_n, x_m) &\leq \phi(x_n) - \phi(x_m) \\ &= |\phi(x_m) - \phi(x_n)| \\ &< \varepsilon. \end{aligned}$$

which shows that (in view of Lemma 1 (c)) that $\{x_n\}$ is p-Cauchy sequence. By the S-completeness of X, there is $z \in X$ such that $\lim\limits_{n\to\infty} p(x_n, z) = 0$. Consequently, by the p-continuity of f, we have $\lim\limits_{n\to\infty} p(fx_n, fz) = 0$. Therefore $\lim\limits_{n\to\infty} p(x_n, z) =$

$0 = \lim\limits_{n\to\infty} p(fx_n, fz)$ and so from Lemma 1 (a), we have $fz = z$. The proof is similar when f is $\tau(\vartheta)$-continuous since S-completeness implies p-Cauchy completeness (Remark 1). ∎

If we assume that $\phi(X)$ is compact in $\mathbb{R}$ instead of boundedness of ϕ in Theorem 1, we can have the following theorem.

Theorem 2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X such that X is S-complete and $\phi : X \to \mathbb{R}$ be a function such that $\phi(X)$ is compact.If "$\preceq$" is the partial order induced by ϕ and $f : X \to X$ is a p-continuous or $\tau(\vartheta)$-continuous non-decreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X.*

Example 3. *Let $X = \{a, b, c\}$ and $\vartheta = \{V \subset X \times X : \Delta \subset V\}$. Define $p : X \times X \to \mathbb{R}^+$ as $p(x, x) = 0$ for all $x \in X$, $p(a, b) = p(b, a) = 1, p(a, c) = p(c, a) = 2$ and $p(b, c) = p(c, b) = 3$. By a routine calculation,one can show that (X, ϑ) is a Hausdorff uniform space and p is an E-distance as it is a metric on X. Next, define $\phi : X \to \mathbb{R}, \phi(a) = 3, \phi(b) = 2$ and $\phi(c) = 1$. Since $p(a, b) = 1 = \phi(a) - \phi(b)$, therefore $a \preceq b$. Again as $p(a, c) = 2 = \phi(a) - \phi(c)$, therefore $a \preceq c$. But as $p(b, c) = p(c, b) = 3 \nleq |\phi(b) - \phi(c)|$, therefore $b \npreceq c$ and $c \npreceq b$ which shows that this ordering is partial and hence X is a partially ordered uniform space. Define $f : X \to X$ as $f(a) = b, f(b) = b$ and $f(c) = c$, then by a routine calculation one can verify that all the conditions of Theorem 1 or Theorem 2 are satisfied and f has a fixed point. Notice that $p(f(b), f(c)) = p(b, c)$ which shows that f is neither $E-$contractive nor E-expansive, therefore the results of [2] are not applicable in the context of this example.Thus, this example demonstrates the utility of our results.*

There stands a possibility to prove Caristi type theorems on Hausdorff uniform space and one such result is already available in Aamri, Bennani and El Moutawakil [1]. But, in what follows, we prove a multi-valued versions of the preceding theorem which is essentially inspired and related to those contained in [7].

Let X be a topological space and $\preceq$ be a partial order on X. Let 2^X denote the family of all non-empty subsets of X.

Definition 3 ([7]). *Let A, B be two non-empty subsets of X such that the relations between them(i.e.between A and B) are defined as follows:*

(r_1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.

(r_2) If for every $b \in B$, there exists $a \in A$ such that $a \preceq b$, then $A \prec_2 B$.

(r_3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 2 ([7]). *The relations $\prec_1$ and $\prec_2$ form different relations between A and B. For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\preceq$ be usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$ but if we interchange the roles of A and B (i.e. $A = [0, 1]$, $B = [0, \frac{1}{2}]$), then $A \prec_2 B$ while $A \not\prec_1 B$.*

Remark 3 ([7]). *The relations $\prec_1$, $\prec_2$ and $\prec$ are reflexive and transitive, but are not anti-symmetric. For instance, let $X = \mathbb{R}$, $A = [0,3]$, $B = [0,1] \cup [2,3]$ and $\preceq$ be usual order on X, then $A \prec B$ and $B \prec A$, but $A \neq B$. Hence, they are not partial orders on 2^X.*

Definition 4 ([7]). *A multi-valued operator $T : X \to 2^X$ is said to be order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X, u_n \to u_0, v_n \to v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.*

We can define the order closed operator on uniform space using E-distance function in the following way.

Definition 5. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. A multi-valued operator $T : X \to 2^X$ is called p-order closed if for monotone sequences $\{u_n\}, \{v_n\} \subset X$, $\lim_{n\to\infty} p(u_n, u_0) = 0$, $\lim_{n\to\infty} p(v_n, v_0) = 0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.*

Now we prove some fixed point theorems for multi-valued maps.

Theorem 3. *Let (X, ϑ) be a Hausdorff uniform space equipped with an E-distance p such that X is S-complete and $\phi : X \to \mathbb{R}$ is bounded below . If "$\preceq$" is the partial order induced by ϕ. $F : X \to 2^X$ which is p-order closed operator with $\{x_0\} \prec_1 Fx_0$ for some $x_0 \in X$ and for all $x, y \in X$, $x \preceq y \Longrightarrow Fx \prec_1 Fy$ (i.e. F is non-decreasing with respect to $\prec_1$), then F has a fixed point in X.*

Proof. Since Fx is non-empty for all $x \in X$, therefore there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. Since $Fx_0 \prec_1 Fx_1$, therefore there exists $x_2 \in Fx_1$ such that $x_1 \preceq x_2$. Continuing this process, one gets a non-decreasing sequence $\{x_n\}$, which satisfies $x_{n+1} \in Fx_n$. By the definition of "$\preceq$", we have $\cdots \leq \phi(x_2) \leq \phi(x_1) \leq \phi(x_0)$ i.e. the sequence $\{\phi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Since ϕ is bounded from below, $\{\phi(x_n)\}$ is convergent and hence it is Cauchy i.e. for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\phi(x_m) - \phi(x_n)| < \varepsilon$. Since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \leq \phi(x_n) - \phi(x_m)$. Therefore,

$$\begin{aligned} p(x_n, x_m) &\leq \phi(x_n) - \phi(x_m) \\ &= |\phi(x_m) - \phi(x_n)| \\ &< \varepsilon, \end{aligned}$$

which shows that (in view of Lemma 1 (c)) that $\{x_n\}$ is p-Cauchy sequence. By the S-completeness of X, there is $z \in X$ such that $\lim_{n\to\infty} p(x_n, z) = 0$. Consequently, we have $z \in Fz$ as F is p-order closed and $x_{n+1} \in Fx_n$. ∎

Similarly, we can prove the following theorem.

Theorem 4. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X such that X is S-complete and $\phi : X \to \mathbb{R}$ is bounded above. If "$\preceq$" is the partial order induced by ϕ. $F : X \to 2^X$ which is p-order closed operator with $Fx_0 \prec_2 \{x_0\}$ for some $x_0 \in X$ and for all $x, y \in X$, $x \preceq y \Longrightarrow Fx \prec_2 Fy$ (i.e. F is non-decreasing with respect to $\prec_2$), then F has a fixed point in X.*

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