EVALUATION OF A POLYNOMIAL BY MEANS OF MATHEMATICAL SPECTRA OF M. PETROVIĆ

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Abstract

In this paper we propose new spectral methods for the exact evaluation of polynomials with integer, fixed point real and fractional coefficients. The new methods are based on Besout's proposition and the division of polynomials by the use of mathematical spectra. Some examples are given to illustrate the presented methods. We also give the implementation of the proposed methods in MATHEMATICA.

1   Introduction

In the field of research of minimizing the number of arithmetic operations in the exact evaluation of polynomials, Konstantin Orlov used the spectral method. The theory of mathematical spectra was introduced by M.Petrovich ([1]). K.Orlov ([2], [3]) and J.Madić ([4], [5], [6]) discovered some of its practical applications.

Definition 1.1. *Let*

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_i \in \mathbb{Z}, i = 0, \ldots, n \tag{1}$$

be a polynomial with integer coefficients. The **spectral rhythm** H *is the smallest integer that satisfies the condition*

$$10^H > 2 \cdot \max_{i=0,\ldots,n} \{|a_i|\}, \tag{2}$$

$S = P(10^H)$ *is called the* **spectral value***, and the* **mathematical spectrum** *of the polynomial is the ordered pair of integers* (S, H).

In the spectral representation of a polynomial, each coefficient is coded by H digits of the spectral value S. Since the coefficients can be negative, we have to provide a double number of places, so the factor 2 appears in (2).

The programs for the implementation of mathematical spectra as large numbers are given in [5] (preprocessor in Fortran) and [6] (the interpreter in Lisp).

The mathematical spectrum of a polynomial is a large number. It can be used for the representation of the polynomial in the computer memory. Solving a problem by means of mathematical spectra one performs some spectral operations on the spectra (addition, subtraction, multiplication and division). These aggregate operations minimize the overall number of operations.

We use the programming package MATHEMATICA for the implementation of the proposed spectral operations. The main reason is that MATHEMATICA supports computations with large numbers. MATHEMATICA works with exact large numbers up to billions of digits, as well as polynomials with millions of terms [7].

In this paper we consider an application of mathematical spectra in the evaluation of polynomials. First, we propose a method for the evaluation of polynomials with integer coefficients, and then we use the proposed method to deal with polynomials with rational coefficients. In Section 2 we recall the known results that are fundamental for the methods proposed in this paper. In Section 3 we propose a method for the evaluation of a polynomial with integer coefficients and give its implementation in MATHEMATICA. Section 4 deals with polynomials with fixed point real and fractional coefficients, and also gives methods for the evaluation and implementation in MATHEMATICA.

2   Division of polynomials by means of mathematical spectra

Division of polynomials by means of mathematical spectra is considered in [4].

Division of two polynomials with integer coefficients gives the quotient $Q(x)$ and the remainder $R(x)$ that have, in the general case, rational coefficients. Lemma 2.1 introduces an integer factor by which the dividend should be multiplied to ensure integer coefficients of Q and R. Lemma 2.3 estimates the theoretical maximum of the absolute value of the coefficients of Q and R. This result is used to define the rhythm in the division process. Theorem 2.5 is used for the division of two polynomials with integer coefficients. The proofs of the two lemmas and the theorem can be found in [4].

Lemma 2.1. *Let A and B be the polynomials with integer coefficients given by*

$$A(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \quad a_i \in \mathbb{Z}$$
$$B(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m, \quad b_i \in \mathbb{Z} \tag{3}$$
$$\text{such that } b_0 \neq 0 \text{ and } n \geq m \geq 0.$$

Let f be the factor

$$f = b_0^{n-m+1}. \tag{4}$$

When the polynomial $f \cdot A$ is divided by B, the quotient Q and the remainder R have integer coefficients.

Remark 2.2. *Instead of multiplying the dividend A by the factor f defined by (4), we require its coefficients to have the form*

$$a_i = b_0^{n-m+1} \cdot a_i', \quad i = 0, \ldots, n-m, \text{ for some integers } a_i'. \tag{5}$$

Lemma 2.3. *Let A and B be the polynomials with integer coefficients given by (3) and let the coefficients of A be of the form (5). Let a_{max}, b_{max}, q_{max} and r_{max} be the largest coefficients by absolute value of A, B, their quotient Q and remainder R, respectively. Then we have*

$$|q_{max}| \leq |a_{max}| \, (|b_{max}| + 1)^{n-m}$$
$$|r_{max}| \leq |a_{max}| \, (|b_{max}| + 1)^{n-m++1}.$$

Definition 2.4. *Let (SA, H) and (SB, H) be the spectra of the sequences of integers $a_i, i = 0, \ldots, n$ and $b_j, j = 0, \ldots, m$ according to the same rhythm H defined as the smallest integer that satisfies the condition*

$$10^H > 2 \cdot a_{max} \cdot (b_{max} + 1)^{n-m+1} \tag{6}$$

where $a_{max} = \max_i\{|a_i|\}$, $b_{max} = \max_i\{|b_i|\}$, $n \geq m \geq 0$, $b_0 \neq 0$, and a_i are of the form (5). We define

$$SQ = \frac{SA}{SB} \quad \text{and} \quad SR = SA - SB \cdot SQ. \tag{7}$$

*The ordered pair (SQ, H) is called the **spectral quotient** and the ordered pair (SR, H) is called the **spectral remainder** of the spectral division (SA, H) by (SB, H).*

Theorem 2.5. *Let A and B be the polynomials with integer coefficients given by (3) and let the coefficients of A be of the form (5). Let (SA, H) and (SB, H) be their spectra according to the rhythm H obtained from (6).*
Let (SQ', H) and (SR', H) be their spectral quotient and spectral remainder given

by (7). They determine the polynomials Q' and R'.
If Q and R are the quotient and the remainder of the division of A by B, and (SQ, H) and (SR, H) their spectra, respectively, then we have

1. $SQ = SQ'$, $SR = SR'$ *if* $dg(R') < dg(B)$
2. $SQ = SQ' + 1$, $SR = SR' - SB$ *if* $dg(R') = dg(B), SR' \cdot SB > 0$
3. $SQ = SQ' - 1$, $SR = SR' + SB$ *if* $dg(R') = dg(B), SR' \cdot SB < 0$

where $dg(P)$ denotes the degree of polynomial P.

3   Evaluation of polynomials with integer coefficients

For the evaluation of polynomials with integer coefficients we apply Besout's proposition. It states that the value of a polynomial P for the argument $x = r$ is equal to the remainder of the division of P by the binomial B with $B(x) = x - r$, that is,

$$P(r) = P(x) - Q(x) \cdot B(x), \quad B(x) = x - r, \ Q(x) = \frac{P(x)}{x - r}.$$

The spectral analogon is given in the next proposition. The mathematical spectrum (SP, H) is the coded value of the polynomial P. The mathematical spectrum (SB, H) of the binomial B is the coded value of the argument r (this is some kind of complement of the desired value r).

$$B(x)|_{x=10^H} = (x - r)|_{x=10^H} = 10^H - r.$$

Proposition 3.1. *Let (SP, H) and (SB, H) be the spectra of the polynomial P in (1) and the binomial B with $B(x) = x - r$, both with integer coefficients.*
Let $SQ' = SP/SB$ and $SR' = SP - SQ' \cdot SB$ be their spectral quotient and spectral remainder.
The joint rhythm H is the smallest integer that satisfies the condition

$$10^H > 2 \cdot a_{max} \cdot (b_{max} + 1)^n \tag{8}$$

where $a_{max} = \max\{|a_i|\}$ and $b_{max} = \max\{|b_i|, |r|\}$.
Then the value $P(r)$ of the polynomial P at r is

1. $P(r) = SR'$,   *if* $dg(R') = 0$
2. $P(r) = SR' - SB$ *if* $dg(R') = 1, SR' > 0$
3. $P(r) = SR' + SB$ *if* $dg(R') = 1, SR' < 0$

Proof. We apply Theorem 2.5. Now we have $B(x) = x - r$, that is, $b_0 = 1$, so the factorization (5) is obsolete. Here $m = 1$, so the (6) reduces to (8). Also, $dg(B) = 1$, so $dg(R') < dg(B)$ reduces to $dg(R') = 0$. By Besout's proposition $P(r)$ is equal to the remainder of the division of $P(x)$ by $B(x) = x - r$, that is, equal to SR, which we compute in the same way as in Theorem 2.5.   □

Remark 3.2. *From Proposition 3.1 we can see that we perform the following operations*

(1)   *calculation of the appropriate rhythm,*

(2)   *calculation of the mathematical spectra (SP, H) and (SB, H),*

(3)   *one division of spectral values and the determination of the remainder of this division (operations with large numbers),*

(4)   *determination of the degree of a simple binomial,*

(5)   *one addition or subtraction of spectral values (optional).*

Example 3.3. We illustrate Proposition 3.1 by calculating the value of the polynomial

$$P_4(x) = -23x^4 + 7x^3 - 74x^2 + 2x - 9 \quad \text{for } x = -7.$$

The spectral values of polynomials $P_4(x)$ and $B(x) = x + 7$, with the appropriate rhythm $H = 6$ are

$SP =$ -22 999993 000073 999998 000009
$SB =$ 1 000007.

The spectral quotient and remainder are

$SQ = SP/SB$ $=$- 22 999832 001249 991248
$SR = SP - SQ \cdot SB =$ -61273.

Therefore, the value of the polynomial is $P_4(-7) = SR = -61273$.

Implementation of the method given in Proposition 3.1

Input: Polynomial *poly* in the form of list
             Integer r, value of the argument
Output: Value of the polynomial $poly(x)$ for $x = r$

Since all the computations need polynomial coefficients, rather than the polynomial itself, we represent the polynomial in the form of list. The following function makes the list from the polynomial coefficients. To compute the spectrum, we need that the starting coefficient be the one with the higher exponent, so we reverse the order of the elements in the list.

```
PolyCoeffList[poly_] := Module[{l},
  l = Reverse[CoefficientList[poly, x]];
  Return[l]];
```

First, we need a function that computes the joint rhythm H of the given polynomial and the binomial B with $B(x) = x - r$. We use the formula in (8).

```
PolyBinRhythm[a_List, b_List] := Module[{n, lim, H = 1},
  n = Length[a] - 1;
  lim = 2*Max[Abs[a]]*(Max[Abs[b]] + 1)^n;
  While[10^H <= lim, H++];
  Return[H]];
```

Then we compute the spectra of *poly* and $B(x)$, according to the joint rhythm H. The spectral value codes each coefficient by H digits, so it appears as a number in the basis 10^H. A negative coefficient appears to "borrow" 1 from the previous digit. If the first coefficient is negative, the spectral value itself is negative.

```
Spectrum[a_List, H_Integer] := Module[{basis, S},
   basis = Power[10, H];
   S = FromDigits[a, basis];
   Return[S]];
```

We also need to know the degree of the polynomial which corresponds to the spectral remainder (SR', H). Here we use the fact that the maximum degree of R' is 1 and the absolute value of the first coefficient is less than or equal 1.

```
SimpleBinomialDegreeFromSpectrum[{S_, H_}] :=
  Module[{digits, degree},
    If[S < 10^H/2, degree = 0, degree = 1];
    Return[degree]];
```

Now we are ready to perform the method given in Proposition 3.1.

```
Proposition1[poly_List, r_Integer] :=
 Module[{H, SP, SB, SQp, SRp, signSRp, dgRp, result},
  H = PolyBinRhythm[poly, {1, -r}];
  SP = Spectrum[poly, H];
  SB = Spectrum[{1, -r}, H];
  SQp = IntegerPart[SP/SB];
  SRp = SP - SB*SQp;
  If[SRp >= 0, signSRp = 1, signSRp = -1];
  dgRp = SimpleBinomialDegreeFromSpectrum[{SRp, H}];
  If[dgRp == 0,
    result = SRp,
    If[dgRp == 1,
      If[signSRp > 0,
        result = SRp - SB,
        result = SRp + SB],]];
  Return[result]]
```

4   Evaluation of polynomials with rational coefficients

We consider two cases, polynomials with fixed point real coefficients and with fractional coefficients. We reduce both cases to the one with integer coefficients by multiplying the polynomials by an appropriate factor.

4.1. Evaluation of polynomials with fixed point real coefficients

The next proposition deals with the case of polynomials with fixed point real coefficients and argument.

Proposition 4.1. *Let the coefficients of the polynomial P be real numbers with maximum k decimal places and let the value of the argument be a real number r with l decimal places.*
Let (SP', H) and (SB', H) be the spectra of the polynomials

$$P'(x) = 10^{l \cdot n} \cdot 10^{k} \cdot P(x)$$
$$B'(x) = 10^{l} \cdot (x - r)$$

with the joint rhythm H.
Let $SQ' = SP/SB$ and $SR' = SP - SQ' \cdot SB$ be the spectral quotient and spectral remainder of the spectra (SP', H) and (SB', H).
Then the value of the polynomial $P(r)$ is

1. $10^{-l \cdot n} \cdot 10^{-k} \cdot SR'$,   if $dg(R') = 0$
2. $10^{-l \cdot n} \cdot 10^{-k} \cdot (SR' + SB')$ if $dg(R') = 1, SR' > 0$
3. $10^{-l \cdot n} \cdot 10^{-k} \cdot (SR' - SB')$ if $dg(R') = 1, SR' < 0$

Proof. We reduce this case to the case of integer coefficients, by multiplying each coefficient of $P(x)$ by 10 to the power $l \cdot n + k$, which is the maximal possible number of decimal places, and $B(x)$ by 10^{l}. Then we apply Proposition 3.1.   □

Example 4.2. We illustrate Proposition 4.1 by calculating the value of the polynomial P_2 with

$$P_2(x) = x^2 - 3x + 4 \quad \text{for } x = 0.3.$$

By appropriate transformations of the polynomials P_2 and B with $B(x) = x - 0.3$ we get

$$P_2'(x) = 10^2 \cdot P_2(x) = 100x^2 - 300x + 400$$
$$B'(x) = 10^l \cdot B(x) = 10x - 3.$$

The spectral values of the polynomials P_2' and B' with the joint rhythm $H = 5$ are

$$SP_2' = 99\ 99700\ 00400$$
$$SB' = \qquad 9\ 99997.$$

The spectral quotient and remainder are

$$SQ' = SP_2'/SB' \qquad = 9\ 99973$$
$$SR' = SP_2' - SQ' \cdot SB' = \ \ 00319.$$

The value of the polynomial is $P_2(0.3) = 10^{-2} \cdot SR' = 3.19$.

Implementation of the Proposition 4.1

Input: Polynomial *poly* in the form of a list
            Integer k, maximum of decimal places of *poly* coefficients
            Real r, value of the argument
            Integer l, maximum of decimal places of r
Output: Value of the polynomial $poly(x)$ for $x = r$

```
Proposition2[poly_List, k_Integer, r_Real, l_Integer] :=
 Module[{n, p, b, H, Sp, Sb, SQp, SRp,
         signSRp, dgRp, result, res},
  n = Length[poly] - 1;
  p = Power[10, l*n]*Power[10, k]*poly;
  b = IntegerPart[Power[10, l]*{1, -r}];
  H = PolyBinRhythm[p, b];
  Sp = Spectrum[p, H];
  Sb = Spectrum[b, H];
  SQp = IntegerPart[Sp/Sb];
  SRp = Sp - Sb*SQp;
  If[SRp >= 0, signSRp = 1, signSRp = -1];
  dgRp = SimpleBinomialDegreeFromSpectrum[{SRp, H}];
  If[dgRp == 0,
    result = Power[10, -l*n]*Power[10, -k]*SRp,
    If[dgRp == 1,
      If[signSRp > 0,
        result = Power[10, -l*n]*Power[10, -k]*(SRp - Sb),
        result = Power[10, -l*n]*Power[10, -k]*(SRp + Sb)],]];
  res = N[result, IntegerLength[IntegerPart[-99.56]] + l];
  Return[res]]
```

4.2. Evaluation of polynomials with fractional coefficients

The case of polynomials with fractional coefficients and argument is given in the next proposition.

Proposition 4.3. *Let the coefficients of the polynomial $P(x)$ and the argument $x = r$ be the fractions*

$$a_i = \frac{p_i}{q_i}, \qquad q_i > 0,\ i = 0, \ldots, n$$

$$r = \frac{p}{q}, \qquad q > 0.$$

Let

$P' = q^n \cdot LCM \cdot P$
$B' = q \cdot B$ *with* $B(x) = x - r$

be the transformed polynomials P and B and their spectra be (SP', H) and (SB', H) with the joint rhythm H.

Let (SQ', H) and (SR', H) be the spectral quotient and the spectral remainder of these spectra.
Then the value $P(r)$ of the polynomial P at r is

  1. $q^{-n} \cdot (LCM)^{-1} \cdot SR'$,   *if $dg(R') = 0$*
  2. $q^{-n} \cdot (LCM)^{-1} \cdot (SR' + SB')$ *if $dg(R') = 1, SR' > 0$*
  3. $q^{-n} \cdot (LCM)^{-1} \cdot (SR' - SB')$ *if $dg(R') = 1, SR' < 0$*

where LCM denotes the least common multiplier of the denominators q_i, $i = 0, \ldots, n$.

Proof. We reduce this case to the case of integer coefficients, by multiplying each coefficient of P by $q^n \cdot LCM$, and B by q. Then we apply Proposition 3.1. $\square$

Example 4.4. We illustrate Proposition 4.3 by calculating the value of the polynomial P_2 with

$$P_2(x) = \frac{2}{3}x^2 - \frac{1}{1}x + \frac{3}{5} \quad \text{for } x = \frac{3}{4}.$$

By appropriate transformations of the polynomials P_2 and B with $B(x) = x - \frac{3}{4}$ with $LCM = 30$, we get

$P_2'(x) = 4^2 \cdot 30 \cdot P_2(x) = 320x^2 - 240x + 288$
$B'(x) = 4 \cdot B(x) \quad\;\; = 4x - 3.$

The spectral values of the polynomials $P_2'(x)$ and $B'(x)$, with the joint rhythm $H = 5$ are

$SP_2' = 319\ 99760\ 00288$
$SB' = \quad\quad\quad\; 3\ 99997.$

The spectral quotient and remainder are

$SQ' = SP_2'/SB' \quad\quad = 80\ 00000$
$SR' = SP_2' - SQ' \cdot SB' = \quad\; 288.$

The value of the polynomial is

$$P_2\left(\frac{3}{4}\right) = 4^{-2} \cdot 30^{-1} \cdot 288 = \frac{288}{16 \cdot 30} = \frac{3}{5}.$$

Implementation of Proposition 4.3

  Input: Polynomial *poly* in the form of a list,
      fractional coefficients
      Fraction r, value of the argument
  Output: Value of the polynomial *poly(x)* for $x = r$

```
Proposition3[poly_List, r_Rational] :=
 Module[{n, p, q, qi, lcm, Pp, Bp, H, SPp, SBp,
         SQp, SRp, signSRp, dgRp, result},
```

```
n = Length[poly] - 1;
p = Numerator[r];
q = Denominator[r];
qi = Denominator[poly];
lcm = Apply[LCM, qi];
Pp = Power[q, n]*lcm*poly;
Bp = {q, -p};
H = PolyBinRhythm[Pp, Bp];
SPp = Spectrum[Pp, H];
SBp = Spectrum[Bp, H];
SQp = IntegerPart[SPp/SBp];
SRp = SPp - SBp*SQp;
If[SRp >= 0, signSRp = 1, signSRp = -1];
dgRp = SimpleBinomialDegreeFromSpectrum[{SRp, H}];
If[dgRp == 0,
  result = Power[q, -n]*Power[lcm, -1]*SRp,
  If[dgRp == 1,
    If[signSRp > 0,
      result = Power[q, -n]*Power[lcm, -1]*(SRp - SBp),
      result = Power[q, -n]*Power[lcm, -1]*(SRp + SBp)],]];
Return[result]]
```

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