Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **23:3** (2009), 37–49 DOI:10.2298/FIL0903037B

SOME FURTHER RESULTS ON LAHIRI

Abhijit Banerjee

Abstract

With the aid of weighted sharing of values we prove a result on the uniqueness of meromorphic functions sharing three values. Our result will improve and supplement some earlier results of Lahiri [6], Yi [21] and some very recent results of both the present author [2] and Chen, Shen, Lin [4]. We also exhibit some examples to show that our result is best possible. In the application part of our result we will show that some results obtained by Chen, Shen, Lin [4] are wrong by obtaining the actual results.

1 Introduction and Definitions.

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

We use I to denote any set of infinite linear measure of $0 < r < \infty$, not necessarily the same at each occurrence.

Definition 1.1. *[2] Let s is a non negative integer. We denote by $\overline{N}(r, a; f \mid= s)$ the reduced counting function of those a-points of f whose multiplicity is exactly s, for $a \in \mathbb{C} \cup \{\infty\}$.*

Definition 1.2. *[6, 8] Let s be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r, a; f \mid\geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once.*

Progress to investigate the uniqueness of meromorphic functions that share three values has been remarkable during the last few decades and naturally several elegant results have been obtained in this aspect. {c.f. [1]-[4], [6]-[7], [9]-[23]}.

Ozawa [13] investigated the influence of the distribution of zeros on the uniqueness problem of entire function and proved the following.

2000 *Mathematics Subject Classifications*. 30D35.
Key words and Phrases. Meromorphic function, uniqueness, weighted sharing.
Received: January 21, 2008
Communicated by Dragan S. Djordjević

Theorem A. *Let f and g be two entire function of finite order such that they share 0, 1 CM and $2\delta(0; f) > 1$, then either $f \equiv g$ or $f \cdot g \equiv 1$.*

Ueda [15] removed the order restriction in *Theorem A* and extended it to meromorphic functions.

In 1989 G.Brosch [3] improved the result of Ueda.

In 1998 H.X.Yi [20] improved all the previous results and proved the following.

Theorem B. *Let f and g share $1, \infty, 0$ CM. If*

$$\limsup_{\substack{r \longrightarrow \infty \\ r \in I}} \frac{N(r, 0; f \mid= 1) + N(r, \infty; f \mid= 1) - \frac{1}{2} m(r, 1; g)}{T(r, f)} < \frac{1}{2}$$

then $f \equiv g$ or $f \cdot g \equiv 1$.

To state the next results we have to introduce the notion of gradation of sharing known as weighted sharing which measure how close a shared value is to being shared CM or to being shared IM.

Definition 1.3. *[6, 8] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $1 + k$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2001 with the notion of weighted sharing of values the following two results were proved in [6].

Theorem C. *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, $(\infty, 0)$ and $(1, \infty)$. If*

$$N(r, 0; f \mid= 1) + 4\overline{N}(r, \infty; f) < (\lambda + 0(1))T(r)$$

for $r \in I$ and $0 < \lambda < \frac{1}{2}$ then either $f \equiv g$ or $f \cdot g \equiv 1$.

Theorem D. *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, (∞, ∞) and $(1, \infty)$. If*

$$N(r, 0; f \mid= 1) + N(r, \infty; f \mid= 1) < (\lambda + 0(1))T(r)$$

for $r \in I$ and $0 < \lambda < \frac{1}{2}$ then either $f \equiv g$ or $f \cdot g \equiv 1$.

In 2003 Yi [21] improved *Theorem C* and *Theorem D* and proved the following three theorems.

Theorem E. *Let f and g share $(0,1),(\infty,0),(1,5)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{N(r,0;f\mid=1)+3\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{1.1}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$.

Theorem F. *Let f and g share $(0,1),(\infty,0),(1,3)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{1.2}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$.

Theorem G. *Let f and g share $(0,1),(\infty,2),(1,6)$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}}\frac{N(r,0;f\mid=1)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$.

We now state some more definitions.

Definition 1.4. *[2] Let f and g be two nonconstant meromorphic functions such that f and g share (a,k) where $a\in\mathbb{C}\cup\{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ $(\overline{N}_L(r,a;g))$ the counting function of those a-points of f and g where $p>q$ $(q>p)$, by $\overline{N}_E^{(k+1}(r,a;f)$ the counting function of those a-points of f and g where $p=q\geq k+1$ each point in these counting functions is counted only once. In the same way we can define $\overline{N}_E^{(k+1}(r,a;g)$.*
Clearly $\overline{N}_E^{(k+1}(r,a;f)=\overline{N}_E^{(k+1}(r,a;g)$

Definition 1.5. *[6, 8] Let f, g share a value IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*
Clearly $\overline{N}_*(r,a;f,g)=\overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g)=\overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)$.

Recently the present author [2] has improved *Theorem E* and *Theorem F* by weakening the conditions (1.1) and (1.2) as follows.

Theorem H. *Let f and g be two nonconstant meromorphic functions sharing* $(0,1)$, $(\infty,0)$, $(1,5)$. *If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}} \frac{N(r,0;f\mid=1)+3\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{1.3}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$

Theorem I. *Let f and g be two nonconstant meromorphic functions sharing* $(0,1)$, $(\infty,0)$, $(1,3)$. *If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}} \frac{N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{1.4}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$

Recently Chen, Shen and Lin [4] have got the following two theorems that improved *Theorem E*, *Theorem F* and *Theorem G* in a different direction from that in [2].
Following two theorems are their main results.

Theorem J. *Let f and g be two nonconstant meromorphic functions sharing* $(0,1)$, $(\infty,0)$, $(1,m)$ *where $m(\geq 2)$ is an integer or infinity. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}} \frac{N(r,0;f\mid=1)+\frac{2(m+1)}{m-1}\overline{N}(r,\infty;f)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{1.5}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$

Theorem K. *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, (∞,k), $(1,m)$, where k and m are positive integers or infinity satisfying $(m-1)(km-1)>(1+m)^2$. If*

$$\limsup_{\substack{r\longrightarrow\infty\\ r\in I}} \frac{N(r,0;f\mid=1)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2} \tag{1.6}$$

then either $f\equiv g$ or $f\cdot g\equiv 1$

It can be easily seen that the condition $(m-1)(km-1)>(1+m)^2$ is equivalent to $(m-1)(k-1)>4$ and so is symmetric in m and k. Obviously m, k both must be ≥ 2. Next we consider the followoing examples.

Example 1.1. *Let $f=\frac{4e^z(e^z-1)^2}{(e^z+1)^4}$, $g=-\frac{(e^z-1)^4}{4e^z(e^z+1)^2}$. Clearly f, g share $(0,1)$, $(\infty,1)$ and $(1,\infty)$. Here $T(r,f)=4T(r,e^z)+O(1)$, $T(r,g)=4T(r,e^z)+O(1)$ and $N(r,0;f\mid=1)=N(r,\infty;f\mid=1)=0$, $\overline{N}(r,\infty;f)\sim T(r,e^z)$, $N(r,1;g)\sim 4T(r,e^z)$ but neither $f\equiv g$ nor $fg\equiv 1$.*

Example 1.2. *Let* $f = \frac{1}{(e^z-1)^2}$, $g = \frac{1}{(e^z-1)}$. *Clearly* f, g *share* $(0,\infty)$, $(\infty,0)$ *and* $(1,\infty)$. *Here* $T(r,f) = 2T(r,e^z)+O(1)$, $T(r,g) = T(r,e^z)+O(1)$, $N(r,0;f\mid=1) = N(r,\infty;f\mid=1) = 0$, $N(r,1;g) \sim T(r,e^z)$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

Example 1.3. *Let* $f = \frac{1}{(1-e^z)^3}$, $g = \frac{e^{2z}}{3(e^z-1)}$. *Clearly* f, g *share* $(0,\infty)$, $(\infty,0)$ *and* $(1,\infty)$. *Here* $T(r,f) = 3T(r,e^z)+O(1)$, $T(r,g) = 2T(r,e^z)+O(1)$, $N(r,0;f\mid=1) = N(r,\infty;f\mid=1) = 0$, $N(r,1;g) \sim 2T(r,e^z)$, $\overline{N}_L(r,1;g) = 0$, *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

In the above three examples though the condition (1.6) is satisfied, the conclusion of *Theorem K* ceases to hold. So the condition $(m-1)(k-1) > 4$ is necessary in *Theorem K*.

Hence it is a natural query to explore the situation for $k < 2$ and at the same time to investigate the case $(m-1)(k-1) \leq 4$ with $m \geq 2$ in *Theorem K*.

The above discussion is the motivation of this paper. We will not only provide an affirmative solution in the above direction but also improve all the theorems discussed till now by deriving a generalized result at the cost of modification of condition (1.6). Following theorem is the main result of the paper.

Theorem 1.1. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(0,1)$, (∞,k), $(1,m)$ *where* $m \geq 2$. *If*

$$\limsup_{\substack{r\to\infty,\\ r\in I}} \frac{\left\{\begin{array}{c} N(r,0;f\mid=1)+\overline{N}(r,\infty;f)+\frac{m+3}{m-1}\overline{N}(r,\infty;f\mid\geq k+1) \\ -\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g) \end{array}\right\}}{T(r,f)} < \frac{1}{2} \tag{1.7}$$

then either $f \equiv g$ *or* $f \cdot g \equiv 1$

Remark 1.1. *If* $k = 0$ *then the condition (1.7) can be rewritten by the weaker condition (1.5) and so* Theorem 1.1 *improves* Theorem J. *If* $k = 0$ *and* $m = 5$ *or* $k = 0$ *and* $m = 3$ *in* Theorem 1.1 *then from (1.7) we get the weaker conditions (1.3) and (1.4) respectively, and consequently* Theorem 1.1 *improves* Theorem H *and* Theorem I. *Since the conditions of* Theorem 1.1 *imply the weaker conditions (1.6) and* $(m-1)(km-1) > (1+m)^2$, *thus* Theorem 1.1 *improves* Theorem K.

We also consider the following examples.

Example 1.4. *Let* $f = e^z - e^{2z}$, $g = e^{-z} - e^{-2z}$. *Clearly* f, g *share* $(0,\infty)$, (∞,∞) *and* $(1,\infty)$. *Here* $N(r,0;f\mid=1) \sim T(r,e^z)$, $\overline{N}(r,\infty;f) = 0$, $\overline{N}_L(r,1;g) = 0$. *Again* $T(r,f) = 2T(r,e^z)+O(1)$, $T(r,g) = 2T(r,e^z)+O(1)$, $N(r,1;g) \sim 2T(r,e^z)$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

Example 1.5. *Let* $f = \frac{e^{2z}}{e^z-1}$, $g = \frac{1}{e^z(1-e^z)}$. *It is easy to see that* f, g *share* $(0,\infty)$, (∞,∞) *and* $(1,\infty)$. *Here* $N(r,0;f\mid=1) = 0$, $N(r,\infty;f\mid=1) = \overline{N}(r,\infty;f) \sim T(r,e^z)$, $\overline{N}_L(r,1;g) = 0$. *Also* $T(r,f) = 2T(r,e^z)+O(1)$, $T(r,g) = 2T(r,e^z)+O(1)$, $N(r,1;g) \sim 2T(r,e^z)$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

Example 1.6. *Let* $f = e^z + 1$, $g = e^{-z} + 1$. *Clearly* f, g *share* $(0, \infty)$, (∞, ∞) *and* $(1, \infty)$. *Here* $N(r, 0; f \mid= 1) \sim T(r, e^z)$, $\overline{N}(r, \infty; f) = 0$, $\overline{N}_L(r, 1; g) = 0$. *Again* $T(r, f) = T(r, e^z) + O(1)$, $T(r, g) = T(r, e^z) + O(1)$, $N(r, 1; g) = 0$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

Example 1.7. *Let* $f = \frac{1}{1-e^z}$, $g = \frac{-e^z}{(1-e^z)}$.

From *Example 1.1* and *Examples 1.4-1.7* it is easy to verify that the conditions (1.7) and (1.6) in *Theorem 1.1* and *Theorem K* respectively are sharp.

Following examples show that in *Theorem 1.1* and *Theorem K* the sharing $(0, 1)$ can not be relaxed to $(0, 0)$.

Example 1.8. *Let* $f = (1 - e^z)^3$, $g = \frac{3(e^z-1)}{e^{2z}}$. *Clearly* f, g *share* $(0, 0)$, (∞, ∞) *and* $(1, \infty)$. *Here* $N(r, 0; f \mid= 1) = \overline{N}(r, \infty; f) = 0$, $N(r, 1; g) \sim 2T(r, e^z) + O(1)$, $\overline{N}_L(r, 1; g) = 0$. *Also* $T(r, f) = 3T(r, e^z) + O(1)$, $T(r, g) = 2T(r, e^z) + O(1)$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

Example 1.9. *Let* $f = (e^z - 1)^2$, $g = (e^z - 1)$. *Clearly* f, g *share* $(0, 0)$, (∞, ∞) *and* $(1, \infty)$. *Here* $N(r, 0; f \mid= 1) = \overline{N}(r, \infty; f) = 0$, $N(r, 1; g) \sim T(r, e^z)$, $\overline{N}_L(r, 1; g) = 0$. *Also* $T(r, f) = 2T(r, e^z) + O(1)$, $T(r, g) = T(r, e^z) + O(1)$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

Example 1.10. *Let* $f = \frac{(1-e^z)^3}{1-3e^z}$, $g = \frac{4(1-e^z)}{1-3e^z}$. *Clearly* f, g *share* $(0, 0)$, (∞, ∞) *and* $(1, \infty)$. *Here* $N(r, 0; f \mid= 1) = 0$, $\overline{N}(r, \infty; f) \sim T(r, e^z)$, $N(r, 1; g) \sim T(r, e^z) + O(1)$, $\overline{N}_L(r, 1; g) = 0$. *Also* $T(r, f) = 3T(r, e^z) + O(1)$, $T(r, g) = T(r, e^z) + O(1)$ *but neither* $f \equiv g$ *nor* $fg \equiv 1$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Henceforth we shall denote by H, ϕ_1, ϕ_2 the following three functions.

$$H = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1},$$

$$\phi_1 = \frac{f'}{f-1} - \frac{g'}{g-1} \quad \text{and} \quad \phi_2 = \frac{f'}{f} - \frac{g'}{g}.$$

Lemma 2.1. *[6, 11] If* f, g *share* $(0, 0), (1, 0), (\infty, 0)$ *then*

$$\begin{aligned} (i)\ \ T(r, f) &\le 3T(r, g) + S(r, f), \\ (ii)\ \ T(r, g) &\le 3T(r, f) + S(r, g). \end{aligned}$$

Lemma 2.1 shows that $S(r, f) = S(r, g)$ and we denote them by $S(r)$.

Lemma 2.2. *[20] Let* f, g *share* $(0, 0)$, $(1, 0)$, $(\infty, 0)$ *and* $H \equiv 0$ *then* f, g *share* $(0, \infty)$, $(1, \infty)$, (∞, ∞).

Lemma 2.3. *[2] Let f, g share $(0,p)$, (∞,k), $(1,m)$, where $2 \le m < \infty$ and $H \not\equiv 0$. Then*

$$\begin{aligned} T(r,f) \quad \le \quad & \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,0;f,g)+\overline{N}(r,\infty;f) \\ & +\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-m(r,1;g)-\overline{N}(r,1;f\mid=3) \\ & -\ldots-(m-2)\overline{N}(r,1;f\mid=m)-(m-2)\overline{N}_L(r,1;f) \\ & -(m-1)\overline{N}_L(r,1;g)-(m-1)\overline{N}_E^{(m+1}(r,1;f)+S(r). \end{aligned}$$

Lemma 2.4. *If f, g share $(0,0)$ and 0 is not an Picard exceptional value of f and g. Then $\phi_1 \equiv 0$ implies $f \equiv g$.*

Proof. Suppose
$$\phi_1 \equiv 0.$$
Then by integration we obtain
$$f-1 \equiv C(g-1),$$
where $C \neq 0$ is a constant. It is clear that if z_0 is a zero of f then it is a zero of g. So $C=1$ and hence $f \equiv g$. □

Lemma 2.5. *If f, g share $(1,0)$ and 1 is not an Picard exceptional value of f and g. Then $\phi_2 \equiv 0$ implies $f \equiv g$.*

Proof. Suppose
$$\phi_2 \equiv 0.$$
Then by integration we obtain
$$f \equiv Cg,$$
where $C \neq 0$ is a constant. It is clear that if z_0 is an 1 point of f then it is so also of g. So $C=1$ and hence $f \equiv g$. □

Lemma 2.6. *Let f, g share $(0,1)$, (∞,k), $(1,m)$ where $m \ge 2$ is a positive integer or infinity. If $H \not\equiv 0$ then*

$$\begin{aligned} \overline{N}(r,0;f\mid\ge 2) \quad & \le \quad \overline{N}_*(r,1;f,g)+\overline{N}(r,\infty;f\mid\ge k+1)+S(r), \\ \overline{N}_*(r,1;f,g) \quad & \le \quad \frac{2}{m-1}\overline{N}(r,\infty;f\mid\ge k+1)+S(r). \end{aligned}$$

Proof. Suppose 0, 1 are e.v.P. (exceptional value Picard) of f and g then the lemma follows immediately.

Next suppose 0, 1 are not Picard exceptional values of f and g. Since $H \not\equiv 0$, we have $f \not\equiv g$ and so by *Lemmas 2.4-2.5* it follows that $\phi_i \not\equiv 0$ for $i = 1,2$. Now

$$\begin{aligned} \overline{N}(r,0;f\mid\ge 2) \quad & \le \quad N(r,0;\phi_1) \\ & \le \quad T(r,\phi_1)+O(1) \\ & = \quad N(r,\infty;\phi_1)+S(r) \\ & \le \quad \overline{N}_*(r,1;f,g)+\overline{N}(r,\infty;f\mid\ge k+1)+S(r), \end{aligned} \tag{2.1}$$

which is the first part of the lemma.
Again

$$
\begin{aligned}
m\,\overline{N}_*(r,1;f,g) &\leq N(r,0;\phi_2) \\
&\leq N(r,\infty;\phi_2)+S(r) \\
&\leq \overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,\infty;f\mid\geq k+1)+S(r).
\end{aligned} \tag{2.2}
$$

Substituting (2.1) in (2.2) we get the second conclusion of the lemma. □

Lemma 2.7. *[10] Let f, g share $(0,1)$, (∞,k), $(1,m)$ and $f\not\equiv g$, where k and m are positive integers or infinity satisfying $(m-1)(km-1)>(1+m)^2$. Then for $a=0,1,\infty$ we have $\overline{N}(r,a;f\mid\geq 2)=\overline{N}(r,a;g\mid\geq 2)=S(r)$.*

3 Proofs of the theorems.

Proof of Theorem 1.1. Suppose $H\not\equiv 0$. Then using *Lemma 2.3* for $p=1$ and *Lemma 2.6* we get

$$
\begin{aligned}
T(r,f) &\leq 2N(r,0;f\mid=1)+3\overline{N}(r,0;f\mid\geq 2)+2\overline{N}(r,\infty;f) \\
&\quad +\overline{N}(r,\infty;f\mid\geq k+1)-(m-2)\overline{N}_*(r,1;f,g) \\
&\quad -\overline{N}_L(r,1;g)-m(r,1;g)+S(r) \\
&\leq 2N(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+3\overline{N}_*(r,1;f,g) \\
&\quad +4\overline{N}(r,\infty;f\mid\geq k+1)-(m-2)\overline{N}_*(r,1;f,g) \\
&\quad -\overline{N}_L(r,1;g)-m(r,1;g)+S(r) \\
&\leq 2N(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+4\overline{N}(r,\infty;f\mid\geq k+1) \\
&\quad +(5-m)\overline{N}_*(r,1;f,g)-\overline{N}_L(r,1;g)-m(r,1;g)+S(r) \\
&\leq 2N(r,0;f\mid=1)+2\overline{N}(r,\infty;f)+\frac{2(m+3)}{m-1}\overline{N}(r,\infty;f\mid\geq k+1) \\
&\quad -\overline{N}_L(r,1;g)-m(r,1;g)+S(r),
\end{aligned} \tag{3.1}
$$

which contradicts (1.7). So $H\equiv 0$. Hence by *Lemma 2.2* f and g share $(0,\infty)$, $(1,\infty)$, (∞,∞). So $\overline{N}_L(r,1;g)\equiv 0$. Now by *Theorem B* the theorem follows. □

Proof of Theorem K. Suppose $f\not\equiv g$, we shall show that $f\cdot g\equiv 1$. Noting that here both k and m are ≥ 2, using *Lemma 2.7* we get $\overline{N}(r,\infty;f\mid\geq 2)=\overline{N}(r,\infty;f\mid\geq k+1)=\overline{N}_L(r,1;g)=S(r)$ and hence (1.7) reduces to (1.6). So by *Theorem 1.1* we can obtain the conclusion of *Theorem K*. □

4 Some remarks.

In 2003 Yi proved the following theorems.

Theorem L. *[21] Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(\infty,1)$, $(1,5)$. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r\in I}} \frac{3\overline{N}(r,0;f)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2}$$

then either $f\equiv g$ or $f.g\equiv 1$.

Theorem M. *[21] Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(\infty,1)$, $(1,3)$. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r\in I}} \frac{4\overline{N}(r,0;f)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2}$$

then either $f\equiv g$ or $f.g\equiv 1$.

Theorem N. *[21] Let f and g be two nonconstant meromorphic functions sharing $(0,2)$, $(\infty,1)$, $(1,6)$. If*

$$\limsup_{\substack{r\longrightarrow\infty \\ r\in I}} \frac{N(r,0;f\mid=1)+N(r,\infty;f\mid=1)-\frac{1}{2}m(r,1;g)}{T(r,f)}<\frac{1}{2}$$

then either $f\equiv g$ or $f.g\equiv 1$.

Recently the present author [2] have improved *Theorems L* and *M* and Chen, Shen, Lin [4] improved *Theorems L-N* in two different directions. As a consequence of *Theorem 1.1* and *Theorem K* we will improve *Theorems L-N* which will also improve the results of Banerjee [2] as well as Chen, Shen, Lin [4] in this respect.

Theorem 4.1. *Let f and g be two nonconstant meromorphic functions sharing $(0,k)$, $(\infty,1)$, $(1,m)$ where $m\geq 2$. If*

$$\limsup_{\substack{r\to\infty \\ r\in I}} \frac{\left\{\begin{array}{c} N(r,\infty;f\mid=1)+\overline{N}(r,0;f)+\frac{m+3}{m-1}\overline{N}(r,0;f\mid\geq k+1) \\ -\frac{1}{2}m(r,1;g)-\frac{1}{2}\overline{N}_L(r,1;g) \end{array}\right\}}{T(r,f)}<\frac{1}{2} \tag{4.1}$$

then $f\equiv g$ or $f.g\equiv 1$

Proof. Let

$$F=\frac{1}{f},\quad G=\frac{1}{g} \tag{4.2}$$

It is clear that

$$T(r,f)=T(r,F)+O(1) \tag{4.3}$$

and

$$m(r,1;g)=m(r,1;G)+O(1) \tag{4.4}$$

From the given condition and by (4.2)-(4.4) we get

$$\limsup_{\substack{r\to\infty\\ r\in I}} \frac{\left\{\begin{array}{c} N(r,0;F\mid=1)+\overline{N}(r,\infty;F)+\frac{(m+3)}{m-1}\overline{N}(r,\infty;F\mid\geq k+1)\\ -\frac{1}{2}m(r,1;G)-\frac{1}{2}\overline{N}_L(r,1;G)\end{array}\right\}}{T(r,F)}<\frac{1}{2}$$

Since f, g share $(0,k)$, $(\infty,1)$ and $(1,m)$ from (4.2) it follows that F, G share $(0,1),(\infty,k)$ and $(1,m)$. So by *Theorem 1.1* we get either $F\equiv G$ or $FG\equiv 1$ from which the theorem follows. □

Considering the substitution (4.2), from *Theorem K* we can easily prove the following.

Corollary 4.1. *Let f and g be two nonconstant meromorphic functions sharing $(0,k)$, $(\infty,1)$, $(1,m)$, where $(m-1)(km-1)>(1+m)^2$. If (1.6) holds then either $f\equiv g$ or $f.g\equiv 1$*

5 Applications of the main results

Definition 5.1. *[6] Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $\bigcup_{a\in S} E_k(a;f)$.*

In this section denote by S_i $i=1,2,3$ the following three sets unless otherwise stated.
$S_1=\{a+b,a+b\omega,\ldots,a+b\omega^{n-1}\}$, $S_2=\{a\}$, $S_3=\{\infty\}$ where $\omega=cos\frac{2\pi}{n}+isin\frac{2\pi}{n}$ and n is a positive integer, a and b $(\neq 0)$ are constants.

In 2003 improving the result of Lahiri [6], Yi [21] proved the following result.

Theorem O. *If $n\geq 2$ and $E_f(S_1,6)=E_g(S_1,6)$, $E_f(S_2,0)=E_g(S_2,0)$, $E_f(S_3,1)=E_g(S_3,1)$, then $f-a\equiv t(g-a)$, where $t^n=1$ or $(f-a)(g-a)\equiv s$ where $s^n=b^{2n}$.*

From *Theorem K* we can prove the following.

Theorem 5.1. *If $n\geq 2$ and $E_f(S_1,m)=E_g(S_1,m)$, $E_f(S_2,0)=E_g(S_2,0)$, $E_f(S_3,k)=E_g(S_3,k)$, where $(m-1)(nk+n-2)>4$ then $f-a\equiv t(g-a)$, where $t^n=1$ or $(f-a)(g-a)\equiv s$ where $s^n=b^{2n}$.*

Proof. Let $F=\left(\frac{f-a}{b}\right)^n$ and $G=\left(\frac{g-a}{b}\right)^n$
We note that

$$\begin{aligned} F-1 &= \left(\frac{f-a}{b}\right)^n-1=\left(\frac{f-a}{b}-1\right)\left(\frac{f-a}{b}-\omega\right)\ldots\left(\frac{f-a}{b}-\omega^{n-1}\right)\\ &= \frac{1}{b^n}\left(f-(a+b)\right)\left(f-(a+b\omega\right)\ldots\left(f-(a+b\omega^{n-1})\right) \end{aligned}$$

and

$$G-1=\frac{1}{b^n}\left(g-(a+b)\right)\left(g-(a+b\omega)\ldots\left(g-(a+b\omega^{n-1})\right)\right).$$

Also since $E_f(S_1,m)=E_g(S_1,m)$ implies that the totality of zeros of $f-(a+b\omega^i)$ and $g-(a+b\omega^i)$, $i=0,1,\ldots,n-1$ coincides in locations as well as multiplicities up to order m, it follows that F and G share $(1,m)$. It is also clear that F and G share $(0,n-1)$, $(\infty,nk+n-1)$. As $n\geq 2$, we note that F and G always share $(0,1)$. Again from $(m-1)(nk+n-2)>4$ we get $(m-1)\left((nk+n-1)m-1\right)>(1+m)^2$. Combining *Theorem K* and $N(r,0;F\mid=1)=N(r,1;F\mid=1)=0$, we get either $F\equiv G$ or $FG\equiv 1$. From this we get the conclusion of the theorem. □

Corollary 5.1. *When* $n\geq 2$ Theorem 5.1 *holds for the following pairs of least values of* m *and* k*: (i) m=2, k=3; (ii) m=3, k=2; (iii) m=4, k=1;*

Corollary 5.2. *When* $n\geq 3$ Theorem 5.1 *holds for the following pairs of least values of* m *and* k*: (i) m=2, k=3; (ii) m=3, k=1; (iii) m=6, k=0;*

Example 5.1. *Let* $f(z)=\frac{(e^{2z}+1)^2}{2e^z(e^{2z}-1)}$ *and* $g(z)=\frac{2ie^z(e^{2z}+1)}{(e^{2z}-1)^2}$ *and* $S_1=\{-1,1\}$, $S_2=\{0\}$, $S_3=\{\infty\}$. *Clearly here* $E_f(S_1,\infty)=E_g(S_1,\infty)$, $E_f(S_2,0)=E_g(S_2,0)$, $E_f(S_3,0)=E_g(S_3,0)$.

Remark 5.1. *When* $n\geq 3$, Theorem 8.3 *in [22] is a better result than (ii) and (iii) in* Corollary 5.2 *obtained in this paper.*

From *Example 5.1* we see that the condition $k\geq 1$ can not be relaxed to $k=0$.

Example 5.2. *Let* $f(z)=a-3b(e^z+e^{2z})$ *and* $g(z)=a+b(1+e^{-z})^3$ *and let* $S_1=\{a+b\}$, $S_2=\{a\}$ *and* $S_3=\{\infty\}$, *where* a *and* $b(\neq 0)$ *are constants.*

We verify that $E_f(S_1,1)=E_g(S_1,1)$, $E_f(S_2,0)=E_g(S_2,0)$, $E_f(S_3,1)=E_g(S_3,1)$, but $f\not\equiv g$ and $fg\not\equiv 1$ in *Example 5.2*. This example shows that the assumption $n\geq 2$ in *Theorem 5.1* is best possible.

In 2003 Yi [21] proved the following theorem.

Theorem P. *If* $n\geq 2$ *and* $E_f(S_1,6)=E_g(S_1,6)$, $E_f(S_2,1)=E_g(S_2,1)$, $E_f(S_3,0)=E_g(S_3,0)$, *then* $f-a\equiv t(g-a)$, *where* $t^n=1$ *or* $(f-a)(g-a)\equiv s$ *where* $s^n=b^{2n}$.

By replacing f and g with $a+\frac{b^2}{f-a}$ and $a+\frac{b^2}{g-a}$ respectively in *Theorem 5.1* we get the following theorem which improves *Theorem P*.

Theorem 5.2. *If* $n\geq 2$ *and* $E_f(S_1,m)=E_g(S_1,m)$, $E_f(S_2,k)=E_g(S_2,k)$, $E_f(S_3,0)=E_g(S_3,0)$, *where* $(m-1)(nk+n-2)>4$ *then* $f-a\equiv t(g-a)$, *where* $t^n=1$ *or* $(f-a)(g-a)\equiv s$ *where* $s^n=b^{2n}$.

Remark 5.2. Theorem 5.1 *and* Theorem 5.2 *of this paper correct* Theorems 5.1-Theorem 5.4 *in [4]. In fact, from the assumptions of* Theorem 5.1 *in [4] we see that* $F=\left(\frac{f-a}{b}\right)^n$ $(n\geq 2)$ *and* $G=\left(\frac{g-a}{b}\right)^n$ $(n\geq 2)$ *share* $(1,m)$, $(0,1)$ *and* (∞,k), *where* $m = 2$ *and* $k = 3$, *which do not satisfies* $(m-1)(km-1) > (1+m)^2$, *and so we can not use* Corollary 1.5 *in [4] to get* Theorem 5.1 *in [4]. Similarly, from the assumptions of* Theorem 5.2 *in [4] we see that* $\frac{1}{F} = \left(\frac{b}{f-a}\right)^n$ $(n \geq 2)$ *and* $\frac{1}{G} = \left(\frac{b}{g-a}\right)^n$ $(n \geq 2)$ *share* $(1, m)$, $(0, 1)$ *and* (∞, k), *where* $m = 2$ *and* $k = 3$, *which do not satisfies* $(m-1)(km-1) > (1+m)^2$ *and so we can not use* Corollary 1.5 *in [4] to get* Theorem 5.2 *in [4], from the assumptions of* Theorem 5.3 *in [4] we see that* $F = \left(\frac{f-a}{b}\right)^n$ $(n \geq 3)$ *and* $G = \left(\frac{g-a}{b}\right)^n$ $(n \geq 3)$ *share* $(0, k_1)$, (∞, k_2) *and* $(1, k_3)$, *where* $k_1 = k_2 = k_3 = 2$, *which do not satisfies* $k_1k_2k_3 > k_1 + k_2 + k_3 + 2$, *and so we can not use* Corollary 1.5 *in [4] and* Theorem 1.1 *in [22] to get* Theorem 5.3 *in [4], from the assumptions of* Theorem 5.4 *in [4] we see that* $1 - F = 1 - \left(\frac{f-a}{b}\right)^n$ $(n \geq 3)$ *and* $1 - G = 1 - \left(\frac{g-a}{b}\right)^n$ $(n \geq 3)$ *share* $(1, m)$, $(0, 1)$ *and* (∞, k), *where* $m = 2$ *and* $k = 5$, *which do not satisfies* $(m-1)(km-1) > (1+m)^2$, *and so we can not use* Corollary 1.5 *in [4] to get* Theorem 5.4 *in [4].*

Acknowledgement

The author wish to thank the referee for his/her valuable comments and suggestions towards the improvement of the paper. The author is also grateful to Prof. H.X.Yi for supplying him the electronic file of the paper [21] .

References

[1] A.Banerjee, *Uniqueness of meromorphic functions that share three values*, Austral. J. Math. Anal. Appl. 3 (2) (2006) Art.1, 10pp. [ONLINE *http://ajmaa.org/*]

[2] A.Banerjee, *Meromorphic functions sharing three values with finite weight*, Ann. Polon. Math. 91(1) (2007) 57-69.

[3] G.Brosch, *Eindeutigkeitssätze für meromorphe Funktionen*, Thesis Technical University of Aachen, 1989.

[4] J.Chen, S. Shen and W.C.Lin, *Meromorphic functions sharing three values with weight*, J.Inequal.Pure Appl. Math., 8(1)(2007), Art.19, 11pp. [ONLINE *http://jipam.vu.edu.au/*].

[5] W.K.Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[6] I.Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math.J., 161(2001), 193-206.

[7] I.Lahiri, *Weighted sharing of three values and uniqueness of meromorphic functions*, Kodai Math.J., 24(2001), 421-435.

[8] I.Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., 46(2001), 241-253.

[9] I.Lahiri, *On a results of Ozawa concerning uniqueness of meromorphic function*, J. Math. Anal. Appl., 271(2002), 206-216.

[10] I.Lahiri, *On a results of Ozawa concerning uniqueness of meromorphic functions II*, J. Math. Anal. Appl., 283(2003), 66-76.

[11] I.Lahiri, *Meromorphic functions sharing three values*, Southeast Asian Bull. Math., (2003) 26, 961-966.

[12] P.Li, *Meromorphic functions sharing three values or sets CM*, Kodai Math. J., 21 (1998) , 138-152.

[13] M.Ozawa, *Unicity theorems for entire functions*, J.Anal. Math., 30(1976), 411-420.

[14] N.Terglane, *Identitätssätze in $\mathbb{C}$ meromorpher Funktionen als Ergebnis von Werteteilung*, Diplmarbeit RWTH. Aachen 1989.

[15] H.Ueda, *Unicity theorems for meromorphic or entire function II*, Kodai Math. J., 6 (1983), 26-36.

[16] H.Ueda, *On the zero-one-pole set of a meromorphic function II*, Kodai Math.J., 13 (1990), 134-142.

[17] H.X.Yi, *Meromorphic functions sharing three values*, Chinese Annals of Math., 9A(1988), 433-440.

[18] H.X.Yi, *Meromorphic functions that share two or three values*, Kodai Math.J., 13 (1990), 363-372.

[19] H.X.Yi, *Unicity theorems for meromorphic functions that share three values*, Kodai Math.J., 18 (1995) 300-314.

[20] H.X.Yi, *Meromorphic functions that share three values*, Bull. Hong Kong Math. Soc., 2(1), (1998) 55-64.

[21] H.X.Yi, *On some results of Lahiri*, J. Math. Anal. Appl., 284 (2003), 481-495.

[22] H.X.Yi, *Meromorphic functions with weighted sharing of three values*, Complex Var. Theory Appl., 50 (12) (2005), 923-934.

[23] Q.C.Zhang, *Meromorphic functions sharing three values*, Indian J. Pure Appl. Math., 30(1999), 667-682.

Department of Mathematics, West Bengal State University, Barasat, 24 Pargans (North), West Bengal, Kolkata 700126, India
E-mail: abanerjee_kal@yahoo.co.in abanerjee_kal@rediffmail.com
abanerjeekal@gmail.com