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CHAOTIC C-DISTRIBUTION SEMIGROUPS

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Abstract

We introduce and analyze hypercyclic and chaotic properties of C-distribution semigroups and global integrated C-semigroups. The obtained results are incorporated in the study of the dynamics of the backwards L^p heat semigroups ($p > 2$) on symmetric spaces of non-compact type.

1 Introduction

Throughout this paper, we assume that E is a separable infinite-dimensional complex Banach space. The dual space of E and the space of all bounded, linear operators on E, are denoted by E^* and $L(E)$, respectively. Let S be a non-empty closed subset of $\mathbb{C}$ satisfying $\mathrm{S} \setminus \{0\} \neq \emptyset$. For a closed, linear operator A acting on E, we denote by $D(A)$, $\mathrm{R}(A)$, $\mathrm{Kern}(A)$, $\rho(A)$, $\sigma(A)$ and $\sigma_p(A)$ its domain, range, kernel, resolvent set, spectrum and point spectrum, respectively. If F is a closed subspace of E, then A_F denotes the part of A in F; that is, $A_F = \{(x, y) \in A : x,\ y \in F\}$. Set $D_\infty(A) := \bigcap_{n \in \mathbb{N}} D(A^n)$. The solution space for A, denoted by $Z(A)$, is defined to be the set of all $x \in E$ for which there exists a continuous mapping $u(\cdot, x) \in C([0,\infty) : E)$ satisfying $\int_0^t u(s,x)ds \in D(A)$ and $A\int_0^t u(s,x)ds = u(t,x) - x$, $t \geq 0$. In this paper, the notions of fractionally integrated C-semigroups and the (vector-valued) Schwartz spaces are understood in the sense of [26]. We employ the convolution product $*$ and the finite convolution product $*_0$ of measurable functions $\varphi,\ \psi : \mathbb{R} \to \mathbb{C}$:

$$\varphi * \psi(t) =: \int_{-\infty}^{\infty} \varphi(t-s)\psi(s)ds,\ \varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)ds,\ t \in \mathbb{R},$$

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and refer the reader to [26, Section 1.3] for the basic properties of the convolution of vector-valued distributions.

A linear operator T on E is said to be *hypercyclic* if there exists an element $x \in D_\infty(T)$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in E; T is said to be *topologically transitive* if for every pair of open non-empty subsets U, V of E, there exists an element $x \in D_\infty(T)$ and a number $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. A *periodic point* for T is an element $x \in D_\infty(T)$ satisfying that there exists $n \in \mathbb{N}$ with $T^n x = x$. Finally, T is said to be *chaotic* if T is hypercyclic and the set of periodic points of T is dense in E.

Chronologically, the first examples of hypercyclic operators were given on the space $\mathrm{H}(\mathbb{C})$ of entire functions equipped with the topology of uniform convergence on compact subsets of $\mathbb{C}$. More precisely, G. D. Birkhoff proved in 1929 that the translation operator $f \mapsto f(\cdot + a)$, $f \in \mathrm{H}(\mathbb{C})$, $a \in \mathbb{C} \setminus \{0\}$ is hypercyclic in $\mathrm{H}(\mathbb{C})$, and G. R. MacLane proved in 1952 the hypercyclicity of the derivative operator $f \mapsto f'$, $f \in \mathrm{H}(\mathbb{C})$. The first systematic investigation into the hypercyclicity and chaos of strongly continuous semigroups was obtained by W. Desch, W. Schappacher and G. F. Webb [19] in 1997; the references [4], [7]-[13], [23]-[24], [31] and [39] are crucially important in the study of chaotic properties of strongly continuous semigroups. The notion of hypercyclicity and chaos of distribution semigroups as well as unbounded semigroups of linear operators was introduced by R. deLaubenfels, H. Emamirad and K.-G. Grosse-Erdmann in [17]. The main objective in this paper is to enquire into the hypercyclic and chaotic properties of C-distribution semigroups and integrated C-semigroups.

2 Chaos for C-distribution semigroups and integrated C-semigroups

We recall the next definition of a C-distribution semigroup.

Definition 1. *([25]) Let $\mathcal{G} \in \mathcal{D}_0'(L(E))$ satisfy $C\mathcal{G} = \mathcal{G}C$. If*

$$\text{(C.D.S.1)} \quad \mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi), \ \varphi, \ \psi \in \mathcal{D},$$

then $\mathcal{G}$ is called a pre-(C-DS) and if additionally

$$\text{(C.D.S.2)} \ \ \mathcal{N}(\mathcal{G}) := \bigcap_{\varphi \in \mathcal{D}_0} Kern(\mathcal{G}(\varphi)) = \{0\},$$

then $\mathcal{G}$ is called a C-distribution semigroup, (C-DS) in short. A pre-(C-DS) $\mathcal{G}$ is called dense if

$$\text{(C.D.S.3)} \quad \mathcal{R}(\mathcal{G}) := \bigcup_{\varphi \in \mathcal{D}_0} R(\mathcal{G}(\varphi)) \text{ is dense in } E.$$

Let $\mathcal{G}$ be a (C-DS) and $T \in \mathcal{E}_0'(\mathbb{C})$. Then we define $G(T)$ on a subset of E by

$$y = G(T)x \text{ iff } \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0.$$

By (C.D.S.2), $G(T)$ is a function. Moreover, $G(T)$ is a closed linear operator and $G(\delta) = I$. The infinitesimal generator of a (C-DS) $\mathcal{G}$ is defined by $A := G(-\delta')$. Let us also define the operator $G_C(T)$ $(T \in \mathcal{E}_0'(\mathbb{C}))$ by

$$G_C(T) = \{(x, y) \in E \oplus E \ : \mathcal{G}(T * \varphi)Cx = \mathcal{G}(\varphi)y, \ \varphi \in \mathcal{D}_0\}\,.$$

It can be easily seen that $G_C(T)$ is a closed linear operator and that $G(T)C = G_C(T)$, $T \in \mathcal{E}_0'(\mathbb{C})$. Furthermore, for every $T \in \mathcal{E}_0'(\mathbb{C})$ and $\varphi \in \mathcal{D}$, $G(\varphi)G(T) \subseteq G(T)\mathcal{G}(\varphi)$, $CG(T) \subseteq G(T)C$ and $\mathcal{R}(\mathcal{G}) \subseteq D(G(T))$. In the case $C = I$, there is no risk for confusion and we also write G for $\mathcal{G}$. Arguing as in the proofs of [25, Theorem 4.2, Theorem 4.4], we have that a closed linear operator A generates a (C-DS) iff for every $\tau > 0$ there exists $n_\tau \in \mathbb{N}$ such that A is the integral generator of a local n_τ-times integrated C-semigroup on $[0, \tau)$.
Let $\mathcal{G}$ be a (C-DS), $S,\ T \in \mathcal{E}_0'(\mathbb{C})$, $\varphi \in \mathcal{D}_0$, $\psi \in \mathcal{D}$ and $x \in E$. Then

(P_1) $G(S)G(T) \subseteq G(S * T)$ with $D(G(S)G(T)) = D(G(S * T)) \cap D(G(T))$, $G(S) + G(T) \subseteq G(S + T)$ and

(P_2) If $\mathcal{G}$ is dense, then its generator is densely defined.

By (P_1), we have

$$D(G(\delta_s)G(\delta_t)) = D(G(\delta_s * \delta_t)) \cap D(G(\delta_t)) = D(G(\delta_{t+s})) \cap D(G(\delta_t)),\ t,\ s \geq 0, \quad (1)$$

where $\delta_t(\varphi) = \varphi(t)$, $t \in \mathbb{R}$, $\varphi \in \mathcal{D}$. Denote by $D(\mathcal{G})$ the set of all $x \in \bigcap_{t \geq 0} D(G(\delta_t))$ satisfying that the mapping $t \mapsto G(\delta_t)x,\ t \geq 0$ is continuous. Then (1) implies $G(\delta_t)(D(\mathcal{G})) \subseteq D(\mathcal{G})$, $t \geq 0$. A closed linear subspace $\tilde{E}$ of E is said to be $\mathcal{G}$-*admissible* iff $G(\delta_t)(D(\mathcal{G}) \ \cap \ \tilde{E}) \subseteq D(\mathcal{G}) \ \cap \ \tilde{E}$, $t \geq 0$. Define $\mathbf{G}(\varphi)\binom{x}{y} := \binom{\mathcal{G}(\varphi)x}{\mathcal{G}(\varphi)y}$ and $\mathcal{C}\binom{x}{y} := \binom{Cx}{Cy}$, $x,\ y \in E$, $\varphi \in \mathcal{D}$. Then $\mathbf{G}$ is a ($\mathcal{C}$-DS) in $E \oplus E$, and $\tilde{E} \oplus \tilde{E}$ is $\mathbf{G}$-admissible provided that $\tilde{E}$ is $\mathcal{G}$-admissible.

Definition 2. *Let $\mathcal{G}$ be a (C-DS) and let $\tilde{E}$ be $\mathcal{G}$-admissible. Then it is said that $\mathcal{G}$ is:*

(i) $\tilde{E}$-hypercyclic, if there exists $x \in D(\mathcal{G}) \ \cap \ \tilde{E}$ such that the set $\{G(\delta_t)x : t \geq 0\}$ is dense in $\tilde{E}$,

(ii) $\tilde{E}$-chaotic, if $\mathcal{G}$ is $\tilde{E}$-hypercyclic and the set of $\tilde{E}$-periodic points of $\mathcal{G}$, $\mathcal{G}_{\tilde{E},per}$, defined by $\{x \in D(\mathcal{G}) \ \cap \ \tilde{E} : G(\delta_{t_0})x = x$ for some $t_0 > 0\}$, is dense in $\tilde{E}$,

(iii) $\tilde{E}$-topologically transitive, if for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in D(\mathcal{G}) \ \cap \ \tilde{E}$ and $t \geq 0$ such that $||y - v|| < \varepsilon$ and that $||z - G(\delta_t)v|| < \varepsilon$,

(iv) $\tilde{E}$-topologically mixing, if for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in D(\mathcal{G}) \ \cap \ \tilde{E}$ and $t_0 \geq 0$ such that $||y - v|| < \varepsilon$ and that $||z - G(\delta_t)v|| < \varepsilon$, $t \geq t_0$,

(v) $\tilde{E}$-weakly mixing, if $\mathbf{G}$ is $(\tilde{E} \oplus \tilde{E})$-hypercylic in $E \oplus E$,

(vi) *$\tilde{E}$-supercyclic, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that its projective orbit $\{cG(\delta_t)x : c \in \mathbb{C},\ t \geq 0\}$ is dense in $\tilde{E}$,*

(vii) *$\tilde{E}$-positively supercyclic, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that its positive projective orbit $\{cG(\delta_t)x : c \geq 0,\ t \geq 0\}$ is dense in $\tilde{E}$,*

(viii) *$\tilde{E}_S$-hypercyclic, if there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that its S-projective orbit $\{cG(\delta_t)x : c \in S,\ t \geq 0\}$ is dense in $\tilde{E}$,*

(ix) *$\tilde{E}_S$-topologically transitive, if for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in D(\mathcal{G}) \cap \tilde{E}$, $t \geq 0$ and $c \in S$ such that $||y - v|| < \varepsilon$ and that $||z - cG(\delta_t)v|| < \varepsilon$,*

(x) *sub-chaotic, if there exists a $\mathcal{G}$-admissible subset $\hat{E}$ such that $\mathcal{G}$ is $\hat{E}$-chaotic.*

Let $\alpha \in (0, \infty)$, $\alpha \notin \mathbb{N}$ and $f \in \mathcal{S}$. Put $n = \lceil \alpha \rceil := \inf\{k \in \mathbb{Z} : k \geq \alpha\}$. Recall ([33]), the Weyl fractional derivatives W_+^α and W_-^α of order α are defined by:

$$W_+^\alpha f(t) := \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_t^\infty (s-t)^{n-\alpha-1} f(s)\,ds,\ t \in \mathbb{R} \text{ and}$$

$$W_-^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int\limits_{-\infty}^t (t-s)^{n-\alpha-1} f(s)\,ds,\ t \in \mathbb{R}.$$

If $\alpha = n \in \mathbb{N}$, put $W_+^n := (-1)^n \frac{d^n}{dt^n}$ and $W_-^n := \frac{d^n}{dt^n}$. Then $W_\pm^{\alpha+\beta} = W_\pm^\alpha W_\pm^\beta$, $\alpha,\ \beta > 0$. Assume that A is the integral generator of an α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ for some $\alpha \geq 0$. Set $\mathcal{G}_\alpha(\varphi)x := \int\limits_0^\infty W_+^\alpha \varphi(t) S_\alpha(t)x\,dt,\ x \in E,\ \varphi \in \mathcal{D}$. Then $\mathcal{G}_\alpha$ is a (C-DS) generated by A ([25], [33]).

Definition 3. *Let $\tilde{E}$ be a closed linear subspace of E. Then it is said that $\tilde{E}$ is $(S_\alpha(t))_{t\geq 0}$-admissible iff $\tilde{E}$ is $\mathcal{G}_\alpha$-admissible, and that $(S_\alpha(t))_{t\geq 0}$ is $\tilde{E}$-hypercyclic iff $\mathcal{G}_\alpha$ is; all other dynamical properties of $(S_\alpha(t))_{t\geq 0}$ are understood in the same sense. A point $x \in \tilde{E}$ is said to be a $\tilde{E}$-periodic point of $(S_\alpha(t))_{t\geq 0}$ iff x is a $\tilde{E}$-periodic point of $\mathcal{G}_\alpha$.*

It is clear that the notion of $\tilde{E}_S$-hypercyclicity generalizes the notions of (positive) $\tilde{E}$-supercyclicity and $\tilde{E}$-hypercyclicity. In the case $\tilde{E} = E$, it is also said that $\mathcal{G}$ ($(S_\alpha(t))_{t\geq 0}$) is hypercyclic, chaotic, ..., S-hypercyclic, S-topologically transitive, and we write $\mathcal{G}_{per}$ instead of $\mathcal{G}_{\tilde{E},per}$.

Let $\beta > \alpha$ and $S_\beta(t)x = \int\limits_0^t \frac{(t-s)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} S_\alpha(s)x\,ds,\ t \geq 0,\ x \in E$. Then $\mathcal{G}_\alpha(\varphi)x = \int\limits_0^\infty W_+^\beta \varphi(t) S_\beta(t)x\,dt = \mathcal{G}_\beta(\varphi)x,\ x \in E,\ \varphi \in \mathcal{D}$, and this implies that a closed linear subspace $\tilde{E}$ of E is $(S_\alpha(t))_{t\geq 0}$-admissible iff $\tilde{E}$ is $(S_\beta(t))_{t\geq 0}$-admissible, and that $(S_\alpha(t))_{t\geq 0}$ is $\tilde{E}$-hypercyclic ($\tilde{E}$-chaotic, ..., sub-chaotic) iff $(S_\beta(t))_{t\geq 0}$ is; because of this, we assume in the sequel that $\alpha = n \in \mathbb{N}_0$.

Example 4. *Let* $\rho(x) := \frac{1}{x^2+1}$, $x \in \mathbb{R}$, $E := C_{0,\rho}(\mathbb{R})$, $F := E \oplus E \oplus E$ *and* $E_0 = \{f \in E : f$ *is continuously differentiable* , $f' \in E\}$ *(see [19]). Put*

$$S_1(t)\binom{f}{g} := \begin{pmatrix} \int_0^t f(\cdot+s)ds + tg(\cdot+t) - \int_0^t g(\cdot+s)ds \\ \int_0^t g(\cdot+s)ds \end{pmatrix}, \ t \geq 0, \ f, \ g \in E.$$

Then $(S_1(t))_{t\geq 0}$ *is a polynomially bounded once integrated semigroup in* $E \oplus E$, *and the integral generator of* $(S_1(t))_{t\geq 0}$ *is the operator* A, *given by* $D(A) := E_0 \oplus E_0$ *and* $A\binom{f}{g} := \binom{f'+g'}{g'}$, $\binom{f}{g} \in D(A)$. *It can be straightforwardly proved that* A *does not generate a strongly continuous semigroup in* $E \oplus E$ *and that* A *generates a global* $R(1:A)$*-regularized semigroup* $(T_1(t))_{t\geq 0}$ *satisfying* $S_1(t)\binom{f}{g} = (1-A)\int_0^t T_1(s)\binom{f}{g}ds$, $t \geq 0$, f, $g \in E$. *Denote by* G *a distribution semigroup generated by* A. *Then* $G(\delta_t)\binom{f}{g} = \frac{d}{dt}S_1(t)\binom{f}{g} = (1-A)T_1(t)\binom{f}{g} = \binom{f(\cdot+t)+tg'(\cdot+t)}{g(\cdot+t)}$, $t \geq 0$, f, $g \in \mathcal{D}$. *Applying these equalities and* [17, *Theorem 3.4*] *with* $Y_1 = Y_2 = \mathcal{D} \oplus \mathcal{D}$ *and* $S\binom{f}{g} = \binom{f(\cdot-1)-g'(\cdot-1)}{g(\cdot-1)}$, f, $g \in \mathcal{D}$, *we get that the operator* $G(\delta_1)$ *is hypercyclic, which implies that* $(S_1(t))_{t\geq 0}$ *is hypercyclic; by the proof of Theorem 9 given below, it also follows that* $(S_1(t))_{t\geq 0}$ *is weakly mixing. Finally, one can use the preceding construction in order to see that there exist an injective operator* $C \in L(F)$ *and a closed densely defined operator* B *acting on* F *satisfying that* B *is the integral generator of a hypercyclic once integrated* C*-semigroup and that* B *is not a subgenerator of any (local)* C*-regularized semigroup in* F *(cf. [29, Examples 8.1, 8.2] and [36, Proposition 2.4] for further information).*

Proposition 5. *Assume that* $\mathcal{G}$ *is a (C-DS) and that* $\tilde{E}$ *is* $\mathcal{G}$*-admissible. If* $\mathcal{G}$ *is* $\tilde{E}$*-weakly mixing, then* $\mathcal{G}$ *is both* $\tilde{E}$*-topologically transitive and* $\tilde{E}$*-hypercyclic.*

Proof. We will only prove that $\mathcal{G}$ is $\tilde{E}$-topologically transitive provided that $\mathcal{G}$ is $\tilde{E}$-weakly mixing. Assume y, $z \in \tilde{E}$ and $\varepsilon > 0$. By $\tilde{E}$-hypercyclicity of $\mathbf{G}$, it follows that there exist $t_0 \geq 0$, $t_1 \geq 0$, $x_1 \in D(\mathcal{G}) \cap \tilde{E}$ and $x_2 \in D(\mathcal{G}) \cap \tilde{E}$ such that:

$$||y - G(\delta_{t_0})x_1|| < \varepsilon, \ ||z - G(\delta_{t_0})x_2|| < \varepsilon \text{ and}$$

$$||z - G(\delta_{t_1})x_1|| < \varepsilon, \ ||y - G(\delta_{t_1})x_2|| < \varepsilon.$$

Put $t = |t_1 - t_0|$, and $v = G(\delta_{t_1})x_2$, resp. $v = G(\delta_{t_0})x_1$, if $t_0 \geq t_1$, resp. $t_1 > t_0$. Then $v \in D(\mathcal{G}) \cap \tilde{E}$, $||y - v|| < \varepsilon$ and $||z - G(\delta_t)v|| < \varepsilon$.

Assume that the semigroup $(e^{tA})_{t\geq 0}$ is hypercyclic (chaotic) in the sense of [17, Definition 3.2] and let $L(E) \ni C$ be an injective operator satisfying that $(W(t) := e^{tA}C)_{t\geq 0}$ is a C-regularized semigroup generated by A. Then it can be simply proved that $(W(t))_{t\geq 0}$ is hypercyclic (chaotic) in the sense of Definition 3. Hence, examples given in [17, Sections 5, 6] can be used for the construction of chaotic C-regularized semigroups. For more examples of chaotic and various kinds of S-hypercyclic strongly continuous semigroups, we refer to [3]-[4], [11]-[12], [19]-[20], [31] and [35].

The proof of the next extension of [17, Lemma 4.3] is provided for the sake of completeness.

Lemma 6. *(i) Assume A generates a (C-DS) $\mathcal{G}$. Then $Z(A) = D(\mathcal{G})$. If $x \in Z(A)$, then $u(t,x) = G(\delta_t)x$, $t \geq 0$ and*

$$\mathcal{G}(\psi)x = \int_0^\infty \psi(t)Cu(t,x)dt = \int_0^\infty \psi(t)G_C(\delta_t)xdt, \ \psi \in \mathcal{D}. \tag{2}$$

(ii) Assume that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that A is a subgenerator of a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$. Then the solution space $Z(A)$ is the space which consists of all elements $x \in E$ such that, for every $\tau > 0$, $S_{n_\tau}(t)x \in R(C)$ and that the mapping $t \mapsto C^{-1}S_{n_\tau}(t)x$, $t \in [0,\tau)$ is n_τ-times continuously differentiable. If $x \in Z(A)$ and $t \in [0,\tau)$, then $u(t,x) = \frac{d^{n_\tau}}{dt^{n_\tau}}C^{-1}S_{n_\tau}(t)x$.

Proof. To prove (i), notice that, for every $\tau > 0$, A is the integral generator of a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$. Assume first that $x \in D(\mathcal{G})$ and that $0 \leq t$ is fixed. It suffices to show that $\mathcal{G}(-\varphi')\int_0^t G(\delta_s)xds = \mathcal{G}(\varphi)(G(\delta_t)x - x)$, $\varphi \in \mathcal{D}_0$. Let $\varphi \in \mathcal{D}_0$ and $\mathrm{supp}\varphi \cup \mathrm{supp}\varphi(\cdot - t) \cup \{t\} \subseteq (-\infty,\tau)$ for some $\tau > 0$. We must prove that

$$\int_0^t \int_0^\infty \varphi^{(n+1)}(v)S_{n_\tau}(v)G(\delta_s)xdvds = -\int_0^\infty \varphi^{(n)}(v)S_{n_\tau}(v)(G(\delta_t)x - x)dv.$$

This follows from Fubini's theorem, the equality $\mathcal{G}(\varphi(\cdot - s))x = \mathcal{G}(\varphi)G(\delta_s)x$, $s \in [0,t]$ and an elementary computation. In order to prove the converse statement, let us assume $x \in Z(A)$, $t \geq 0$, $\varphi \in \mathcal{D}_0$ and $\mathrm{supp}\varphi \cup \mathrm{supp}\varphi(\cdot - t) \cup \{t\} \subseteq (-\infty,\tau)$ for some $\tau > 0$. Set $u_k(s,x) := \int_0^s \frac{(s-r)^{k-1}}{(k-1)!}u(r,x)dr$, $s \in [0,t]$, $k \in \mathbb{N}$. Then one can inductively prove that $A\int_0^s u_k(r,x)dr = u_k(s,x) - \frac{s^k}{k!}x$, $s \in [0,t]$, $k \in \mathbb{N}$. Using the proof of [34, Proposition 2.6], one gets that $Cu_{n_\tau}(s,x) = S_{n_\tau}(s)x$, $s \in [0,t]$. This implies $u(t,x) = (\frac{d^{n_\tau}}{ds^{n_\tau}}C^{-1}S_{n_\tau}(s)x)_{s=t}$. On the other hand, the proof of [26, Theorem 3.2.1.15] gives that $x \in D(G(\delta_t))$ and that $G(\delta_t)x = (\frac{d^{n_\tau}}{ds^{n_\tau}}C^{-1}S_{n_\tau}(s)x)_{s=t} = u(t,x)$. The proof of (2) follows from an application of the partial integration and this completes the proof of (i). The proof of (ii) is a consequence of the proof of (i) and a similar argumentation.

Assume A generates a (C-DS) $\mathcal{G}$ and $x \in Z(A)$. By Lemma 6, we have that $C(Z(A)) \subseteq \overline{\mathcal{R}(\mathcal{G})}$ and that $\mathcal{G}(\psi)x \in \mathrm{R}(C)$, $\psi \in \mathcal{D}$. Further on, $\mathcal{R}(\mathcal{G}) \subseteq Z(A)$, $G(\delta_t)(Z(A)) \subseteq Z(A) \subseteq \overline{D(A)}$, $t \geq 0$ and $\tilde{E}_{\mathrm{S}}$-hypercyclicity ($\tilde{E}_{\mathrm{S}}$-topological transitivity) of $\mathcal{G}$ implies $\overline{\tilde{E} \cap Z(A)} = \tilde{E}$ and $\tilde{E} \subseteq \overline{D(A)}$. Given $t > 0$ and $\sigma > 0$, set

$$\Phi_{t,\sigma} := \{\varphi \in \mathcal{D}_0 : \text{supp}\varphi \subseteq (t-\sigma, t+\sigma),\ \varphi \geq 0,\ \int \varphi(s)ds = 1\}.$$

Keeping in mind Lemma 6 and the proofs of [17, Proposition 3.3, Theorem 4.6], we have the following theorem.

Theorem 7. *(i) Assume $n \in \mathbb{N}_0$, A is the integral generator of an n-times integrated C-semigroup $(S_n(t))_{t\geq 0}$, $\overline{C(\tilde{E})} = \tilde{E}$ and $\tilde{E}$ is $\mathcal{G}_n$-admissible. Then the following holds.*

(i.1) $(S_n(t))_{t\geq 0}$ is $\tilde{E}_S$-hypercyclic iff there exists $x \in \tilde{E}$ such that the mapping $t \mapsto S_n(t)x$, $t \geq 0$ is n-times continuously differentiable and that the set $\{c\frac{d^n}{dt^n}S_n(t)x : c \in S,\ t \geq 0\}$ is dense in $\tilde{E}$.

(i.2) $(S_n(t))_{t\geq 0}$ is $\tilde{E}_S$-topologically transitive iff for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $v \in \tilde{E}$, $t_0 > 0$ and $c \in S$ such that the mapping $t \mapsto S_n(t)v$, $t \geq 0$ is n-times continuously differentiable and that $||y - v|| < \varepsilon$ as well as $||z - c(\frac{d^n}{dt^n}S_n(t)v)_{t=t_0}|| < \varepsilon$.

(i.3) $(S_n(t))_{t\geq 0}$ is $\tilde{E}$-chaotic iff $(S_n(t))_{t\geq 0}$ is $\tilde{E}$-hypercyclic and there exists a dense subset of $\tilde{E}$ consisting of those vectors $x \in \tilde{E}$ for which there exists $t_0 > 0$ such that the mapping $t \mapsto S_n(t)x$, $t \geq 0$ is n-times continuously differentiable and that $(\frac{d^n}{dt^n}S_n(t)x)_{t=t_0} = Cx$.

(ii) Let A be the generator of a (C-DS) $\mathcal{G}$ and let $\tilde{E}$ be $\mathcal{G}$-admissible. Then:

(ii.1) $\mathcal{G}$ is $\tilde{E}_S$-hypercyclic iff there exists $x_0 \in D(\mathcal{G}) \cap \tilde{E}$ such that, for every $x \in \tilde{E}$ and $\varepsilon > 0$, there exist $t_0 > 0$, $c \in S$ and $\sigma > 0$ such that

$$||cC^{-1}\mathcal{G}(\varphi)x_0 - x|| < \varepsilon,\ \varphi \in \Phi_{t_0,\sigma}.$$

(ii.2) $\mathcal{G}$ is $\tilde{E}_S$-topologically transitive iff for every $y,\ z \in \tilde{E}$ and $\varepsilon > 0$, there exist $t_0 > 0$, $c \in S$, $\sigma > 0$ and $v \in D(\mathcal{G}) \cap \tilde{E}$ such that, for every $\varphi \in \Phi_{t_0,\sigma}$,

$$||y - v|| < \varepsilon \text{ and } ||z - cC^{-1}\mathcal{G}(\varphi)v|| < \varepsilon.$$

(ii.3) $\mathcal{G}$ is $\tilde{E}$-chaotic iff $\mathcal{G}$ is $\tilde{E}$-hypercyclic and there exists a dense set in $\tilde{E}$ of vectors $x \in D(\mathcal{G}) \cap \tilde{E}$ for which there exists $\tau > 0$ such that, for every $\varepsilon > 0$, there exists $\sigma > 0$ satisfying

$$||C^{-1}\mathcal{G}(\varphi)x - x|| < \varepsilon,\ \varphi \in \Phi_{\tau,\sigma}.$$

It is not known whether the converse of the assertion (P_2) holds unless $\overline{\mathrm{R}(C)} = E$ and $\rho(A) \neq \emptyset$. Nevertheless, Theorem 7 enables one to prove the following

Corollary 8. *Let A be the generator of a (C-DS) $\mathcal{G}$. Assume that $\tilde{E}$ is $\mathcal{G}$-admissible and that $\mathcal{G}$ is $\tilde{E}_S$-hypercyclic ($\tilde{E}_S$-topologically transitive). Then $\overline{C(\tilde{E})} \subseteq \overline{\mathcal{R}(\mathcal{G})}$.*

The Hypercyclicity Criterion for C-distribution semigroups reads as follows.

Theorem 9. *Let A be the generator of a (C-DS) $\mathcal{G}$ and let $\tilde{E}$ be $\mathcal{G}$-admissible. Assume that there exist subsets $\overline{Y_1},\ \overline{Y_2} \subseteq Z(A) \cap\ \tilde{E}$, both dense in $\tilde{E}$, a mapping $\overline{S} : \overline{Y_1} \to \overline{Y_1}$ and a bounded linear operator D in $\tilde{E}$ such that:*

(i) $G(\delta_1)\overline{S}y = y,\ y \in \overline{Y_1}$,

(ii) $\lim\limits_{n\to\infty} \overline{S}^n y = 0,\ y \in \overline{Y_1}$,

(iii) $\lim\limits_{n\to\infty} G(\delta_n)\omega = 0,\ \omega \in \overline{Y_2}$,

(iv) $R(D)$ *is dense in* $\tilde{E}$,

(v) $R(D) \subseteq Z(A)\ \cap\ \tilde{E},\ G(\delta_n)D \in L(\tilde{E}),\ n \in \mathbb{N}$ *and*

(vi) $DG(\delta_1)x = G(\delta_1)Dx,\ x \in Z(A)\ \cap\ \tilde{E}$.

Then $\mathbf{G}$ *is both* $(\tilde{E} \oplus \tilde{E})$*-hypercyclic and* $(\tilde{E} \oplus \tilde{E})$*-topologically transitive; in particular,* $\mathcal{G}$ *is* $\tilde{E}$*-weakly mixing.*

Proof. Let T_1 be the restriction of the operator $G(\delta_1)$ to $Z(A)\ \cap\ \tilde{E}$, $T_1 = G(\delta_1)_{Z(A)\ \cap\ \tilde{E}}$. Put $T := T_1 \oplus T_1$, $Y_1 := \overline{Y_1} \oplus \overline{Y_1}$, $Y_2 := \overline{Y_2} \oplus \overline{Y_2}$, $\tilde{D} := D \oplus D$ and define $S : Y_1 \to Y_1$ by $S(x, y) := (\overline{S}x, \overline{S}y)$, $x,\ y \in \overline{Y_1}$. Since $G(\delta_1)(Z(A) \cap \tilde{E}) \subseteq Z(A) \cap \tilde{E}$, $D_\infty(T) = Z(A)\ \cap\ \tilde{E}$ and $G(\delta_1)^n x = G(\delta_n)x$, $x \in Z(A)\ \cap\ \tilde{E}$, one can apply [17, Theorem 2.3] in order to see that the operator T is hypercyclic in $\tilde{E} \oplus \tilde{E}$. Under the aegis of the proofs of [17, Theorem 2.3] and [21, Theorem 2], one yields that T is also topologically transitive. The proof of theorem completes a routine argument.

Assume $\overline{\mathrm{R}(C)} = E$, $\tilde{E} = E$, A is the integral generator of a C-regularized semigroup $(T(t))_{t\geq 0}$ and $\mathcal{G}(\varphi)x = \int\limits_0^\infty \varphi(t)T(t)x\,dt$, $x \in E$, $\varphi \in \mathcal{D}$. Then the conditions (iv)-(vi) quoted in the formulation of Theorem 9 hold with $D = C$ and, in this case, Theorem 9 reduces to the Hypercyclicity Criterion for C-regularized semigroups (cf. [17, Theorem 3.4]). Using [8, Remark 2.6] and the proofs of [6, Lemma 3.1] and [17, Theorem 2.3], we are in a position to clarify S-Hypercyclicity Criterion for C-distribution semigroups (C-regularized semigroups).

Example 10. *Assume* $n \in \mathbb{N}$, $\Omega = (0, \infty)^n$, $\alpha_i > 0$, $1 \leq i \leq n$ *and* $\alpha := \min\{\alpha_i : 1 \leq i \leq n\}$. *Set* $\rho(x) := e^{-(x_1^\alpha + \cdots + x_n^\alpha)}$ *and*

$$\varphi(t, x) := ((t + x_1^{\alpha_1})^{1/\alpha_1}, \cdots, (t + x_n^{\alpha_n})^{1/\alpha_n}),\ t \geq 0,\ x = (x_1, \cdots, x_n) \in \Omega.$$

Let us remind that the space $C_{0,\rho}(\Omega, \mathbb{C})$ *consists of all continuous functions* $f : \Omega \to \mathbb{C}$ *satisfying that, for every* $\varepsilon > 0$, $\{x \in \Omega : |f(x)|\rho(x) \geq \varepsilon\}$ *is a compact subset of* Ω; *equipped with the norm* $||f|| := \sup\limits_{x\in\Omega} |f(x)|\rho(x)$, $C_{0,\rho}(\Omega, \mathbb{C})$ *becomes a Banach space. The space of all continuous functions* $f : \Omega \to \mathbb{C}$ *whose support is a compact subset of* Ω, *denoted by* $C_c(\Omega, \mathbb{C})$, *is dense in* $C_{0,\rho}(\Omega, \mathbb{C})$. *Define*

$(T_\varphi(t)f)(x) := f(\varphi(t,x))$, $t \geq 0$, $x \in \Omega$ and $Cf(x) := e^{-(x_1+\cdots+x_n)}f(x)$, $x \in \Omega$, $f \in C_{0,\rho}(\Omega,\mathbb{C})$. Then one can simply prove that $T_\varphi(t) \notin L(C_{0,\rho}(\Omega,\mathbb{C}))$, $t > 0$ and that $(T_\varphi(t)C)_{t\geq 0}$ is a bounded C-regularized semigroup. Given $f \in C_c(\Omega,\mathbb{C})$, define $\tilde{f} : [0,\infty)^n \to \mathbb{C}$ by $\tilde{f}(x) := f(x)$, $x \in \Omega$ and $\tilde{f}(x) := 0$, $x \in [0,\infty)^n \setminus \Omega$. Applying [17, Theorem 3.4] with $Y_1 = Y_2 = C_c(\Omega,\mathbb{C})$ and $Sf(x_1,\cdots,x_n) = \tilde{f}((x_1^{\alpha_1} - 1)^{1/\alpha_1}\chi_{[(a_1^{\alpha_1}+1)^{1/\alpha_1},(b_1^{\alpha_1}+1)^{1/\alpha_1}]}(x_1),\cdots,(x_n^{\alpha_n}-1)^{1/\alpha_n}\chi_{[(a_n^{\alpha_n}+1)^{1/\alpha_n},(b_n^{\alpha_n}+1)^{1/\alpha_n}]}(x_n))$, $x \in \Omega$, $f \in Y_1$, supp$f \subseteq \prod_{i=1}^n [a_i,b_i] \subseteq \Omega$, we get that $(T_\varphi(t)C)_{t\geq 0}$ is weakly mixing. Furthermore, $(T_\varphi(t)C)_{t\geq 0}$ is topologically mixing and, thanks to the proof of [23, Theorem 5.7], $(T_\varphi(t)C)_{t\geq 0}$ is chaotic.

Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then $(T(t))_{t\geq 0}$ is S-topologically transitive in the sense of Definition 3 iff $(T(t))_{t\geq 0}$ is S-topologically transitive in the sense of the definition introduced on pages 50-51 of [27]. It is well known that S-topological transitivity of $(T(t))_{t\geq 0}$ is equivalent to its S-hypercyclicity and that $(T(t))_{t\geq 0}$ is weakly mixing provided that $(T(t))_{t\geq 0}$ is chaotic ([27]); it is not clear whether the above assertions continue to hold in the case of C-distribution semigroups. In the sequel of the paper, we will use the fact that the notions of $\tilde{E}$-topological transitivity, or more generally $\tilde{E}_{\mathrm{S}}$-topological transitivity, and $\tilde{E}$-periodic points of a (C-DS) $\mathcal{G}$ (or an n-times integrated C-semigroup $(S_n(t))_{t\geq 0}$) can be understood in the sense of Definition 2 even in the case when the set $\tilde{E}$ is not $\mathcal{G}$-admissible.

The next theorem is a strengthening of [19, Theorem 3.1] and [2, Criterion 2.3].

Theorem 11. *(i) Let A be the generator of a (C-DS) $\mathcal{G}$. Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying $f(\lambda) \in Kern(A-\lambda) \cap (Z(A) \setminus \{0\})$, $\lambda \in \Omega$. Assume, further, that the superposition $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, for some $x^* \in E^*$, implies $x^* = 0$. Then $\mathcal{G}$ is topologically transitive and the set $\mathcal{G}_{per}$ is dense in E.*

(ii) Let A be the generator of a (C-DS) $\mathcal{G}$. Assume that there exists an open connected subset Ω of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying $f(\lambda) \in Kern(A-\lambda) \cap (Z(A) \setminus \{0\})$, $\lambda \in \Omega$. Put $E_0 := span\{f(\lambda) : \lambda \in \Omega\}$ and $\tilde{E} := \overline{E_0}$. Then $\mathcal{G}$ is $\tilde{E}$-topologically transitive and the set $\mathcal{G}_{\tilde{E},per}$ is dense in $\tilde{E}$.

Proof. To prove (i), notice that the prescribed assumptions imply that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that A is the integral generator of a local n_τ-times integrated C-semigroup $(S_{n_\tau}(t))_{t\in[0,\tau)}$. Fix temporarily $t \in [0,\tau)$ and $\lambda \in \Omega$. Then $AC\int_0^t \frac{(t-s)^{n_\tau-1}}{(n_\tau-1)!}u(s,f(\lambda))ds = AS_{n_\tau}(t)f(\lambda) = S_{n_\tau}(t)Af(\lambda) = S_{n_\tau}(t)\lambda f(\lambda) = C\int_0^t \frac{(t-s)^{n_\tau-1}}{(n_\tau-1)!}u(s,\lambda f(\lambda))ds = C\int_0^t \frac{(t-s)^{n_\tau-1}}{(n_\tau-1)!}u(s,Af(\lambda))ds$. Since $C^{-1}AC = A$, one gets

$$A\int_0^t \frac{(t-s)^{n_\tau-1}}{(n_\tau-1)!}u(s,f(\lambda))ds = \int_0^t \frac{(t-s)^{n_\tau-1}}{(n_\tau-1)!}u(s,Af(\lambda))ds,\ t\geq 0,\ \lambda\in\Omega.$$

Hence, $u(t,f(\lambda))\in D(A)$, $t\geq 0$, $\lambda\in\Omega$ and

$$Au(t,f(\lambda)) = u(t,Af(\lambda)),\ t\geq 0,\ \lambda\in\Omega. \tag{3}$$

On the other hand, the partial integration implies

$$(A-\lambda)\int_0^t e^{-\lambda s}u(s,x)ds = e^{-\lambda t}u(t,x)-x,\ x\in Z(A),\ t\geq 0,\ \lambda\in\mathbb{C}. \tag{4}$$

Keeping in mind (3)-(4), one gets $(A-\lambda)u(t,f(\lambda))=0$ and

$$G(\delta_t)f(\lambda) = e^{\lambda t}f(\lambda),\ t\geq 0,\ \lambda\in\Omega. \tag{5}$$

Noticing that $G(\delta_t)G(\delta_s)f(\lambda) = G(\delta_{t+s})f(\lambda)$, $t,\ s\geq 0$, $\lambda\in\Omega$, one can repeat literally the final part of the proof of [19, Theorem 3.1] to deduce that the set $\mathcal{G}_{per}$ is dense in E. A slight technical modification of the proofs of [19, Theorem 2.3, Theorem 3.1] combined with (5) shows that the sets $X_0 := \text{span}\{f(\lambda) : \lambda\in\Omega,\ \text{Re}(\lambda)<0\}$, $X_\infty := \text{span}\{f(\lambda) : \lambda\in\Omega,\ \text{Re}(\lambda)>0\}$ and $X_{per} := \text{span}\{f(\lambda) : \lambda\in\Omega\ \cap\ i\mathbb{Q}\}$ are dense in E as well as that $\mathcal{G}$ is topologically transitive, finishing the proof of (i). In order to prove (ii), notice first that [2, Lemma 2.2] implies that the sets X_0, X_∞ and X_{per} are dense in $\tilde{E}$. With (5) in view, it follows that, for every $x\in X_0$ and $\varepsilon>0$, resp. $x\in X_\infty$ and $\varepsilon>0$, there exists an arbitrarily large $t>0$ such that $||G(\delta_t)x||<\varepsilon$, resp. there exist an arbitrarily large $t>0$ and $y(t)\in D(\mathcal{G})\ \cap\ E_0$ such that $||y(t)||<\varepsilon$ and that $G(\delta_t)y(t)=y(t)$. By the proof of [2, Criterion 2.3], $\mathcal{G}$ is $\tilde{E}$-topologically transitive. One can similarly prove that $X_{per}\subseteq\mathcal{G}_{\tilde{E},per}$, which completes the proof of (ii).

Remark 12. *(i) It is not clear whether the set $\tilde{E}$, appearing in the formulation of the assertion (ii) of the previous theorem, is $\mathcal{G}$-admissible.*

(ii) Assume A is the integral generator of a C-regularized semigroup $(T(t))_{t\geq 0}$ and $R(C)$ is dense in E. Let Ω and $f(\cdot)$ satisfy the assumptions quoted in the formulation of Theorem 11(i). Then $(T(t))_{t\geq 0}$ is chaotic, weakly mixing and, for every $t>0$, the operator $C^{-1}T(t)$ is chaotic.

Theorem 13. *Let $\theta\in(0,\frac{\pi}{2})$ and let $-A$ generate an analytic strongly continuous semigroup of angle θ. Assume $n\in\mathbb{N}$, $a_n>0$, $a_{n-i}\in\mathbb{C}$, $1\leq i\leq n$, $D(p(A)) = D(A^n)$, $p(A)=\sum_{i=0}^{n}a_iA^i$ and $n(\frac{\pi}{2}-\theta)<\frac{\pi}{2}$.*

(i) *Assume that there exists an open connected subset* Ω *of* $\mathbb{C}$, *which satisfies* $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, *and let* $f : \Omega \to E$ *be an analytic mapping satisfying* $f(\lambda) \in Kern(-A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. *Assume, further, that the superposition* $(x^* \circ f)(\lambda) = 0$, $\lambda \in \Omega$, *for some* $x^* \in E^*$, *implies* $x^* = 0$. *Then, for every* $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, *there exists* $\omega \in \mathbb{R}$ *such that* $p(A)$ *generates an entire* $e^{-(p(A)-\omega)^\alpha}$*-regularized group* $(T(t))_{t\in\mathbb{C}}$. *Furthermore,* $(T(t))_{t\geq 0}$ *is chaotic, weakly mixing and, for every* $t > 0$, *the operator* $C^{-1}T(t)$ *is chaotic.*

(ii) *Assume that there exists an open connected subset* Ω *of* $\mathbb{C}$, *which satisfies* $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, *and let* $f : \Omega \to E$ *be an analytic mapping satisfying* $f(\lambda) \in Kern(-A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. *Let* E_0 *and* $\tilde{E}$ *be as in the formulation of Theorem 11(ii). Then there exists* $\omega \in \mathbb{R}$ *such that, for every* $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, $p(A)$ *generates an entire* $e^{-(p(A)-\omega)^\alpha}$*-regularized group* $(T(t))_{t\in\mathbb{C}}$ *such that* $(T(t))_{t\geq 0}$ *is* $\tilde{E}$*-topologically transitive and that the set of* $\tilde{E}$*-periodic points of* $(T(t))_{t\geq 0}$ *is dense in* $\tilde{E}$.

Proof. The proof of (i) can be obtained as follows. By the arguments given in [15, Section XXIV], we have that the operator $-p(A)$ generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. Let $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$. By [15, Theorem 8.2], one gets that there exists a convenable chosen number $\omega \in \mathbb{R}$ such that $p(A)$ generates an entire $e^{-(p(A)-\omega)^\alpha} \equiv C$-regularized group $(T(t))_{t\in\mathbb{C}}$. Thanks to the proof of [17, Lemma 5.6], $\sigma_p(-p(A)) = -p(-\sigma_p(-A))$ and $f(\lambda) \in \text{Kern}(-p(A) + p(-\lambda))$, $\lambda \in \Omega$. Without loss of generality, one can assume that $p'(z) \neq 0$, $z \in -\Omega$; otherwise, one can replace Ω by $\Omega \setminus \{\gamma_1, \cdots, \gamma_{n-1}\}$, where $\gamma_1, \cdots, \gamma_{n-1}$ are not necessarily distinct zeros of the polynomial $z \mapsto p'(z)$, $z \in \mathbb{C}$. Hence, the mapping $\lambda \mapsto p(-\lambda)$, $\lambda \in \Omega$ and its inverse mapping $z \mapsto -p^{-1}(z)$, $z \in p(-\Omega)$, are analytic and open. The set $-p(-\Omega)$ is open, connected and intersects the imaginary axis. Moreover, the mapping $z \mapsto f(-p^{-1}(-z))$, $z \in -p(-\Omega)$ is analytic, $f(-p^{-1}(-z)) \in \text{Kern}(-p(A) - z)$, $z \in -p(-\Omega)$ and the supposition $x^*(f(-p^{-1}(-z))) = 0$, $z \in -p(-\Omega)$, for some $x^* \in E^*$, implies $x^* = 0$. Therefore, it suffices to prove (i) in the case $p(z) = z$. In order to do that, notice that $-\Omega \subseteq \sigma_p(A)$, $f(-\lambda) \in D_\infty(A)$ and $A^k f(-\lambda) = \lambda^k f(-\lambda)$, $\lambda \in -\Omega$, $k \in \mathbb{N}$. Thereby, the series $\sum_{k\geq 0} \frac{t^k}{k!} A^k f(-\lambda)$ converges for all $\lambda \in -\Omega$, and certainly, $T(t)f(-\lambda) = \sum_{k\geq 0} \frac{t^k}{k!} A^k C f(-\lambda) = C \sum_{k\geq 0} \frac{t^k}{k!} A^k f(-\lambda) \in \text{R}(C)$ and $C^{-1}T(t)f(-\lambda) = \sum_{k\geq 0} \frac{t^k}{k!} A^k f(-\lambda)$ for all $\lambda \in -\Omega$ and $t \in \mathbb{C}$. Therefore, $f(-\lambda) \in Z(A)$, $\lambda \in -\Omega$, $f(-\lambda) \in \text{Kern}(A - \lambda) \cap (Z(A) \setminus \{0\})$, $\lambda \in -\Omega$ and $C^{-1}T(t)f(\lambda) = e^{-\lambda t} f(\lambda)$, $t \geq 0$, $\lambda \in \Omega$. By Theorem 11(i), one has that the set of periodic points of $(T(t))_{t\geq 0}$ is dense in E. Since $\text{R}(C)$ is dense in E ([15]), one can apply [17, Theorem 3.4] with $Y_1 = X_0 \oplus X_0$, $Y_2 = X_\infty \oplus X_\infty$ and $S : Y_1 \to Y_1$, defined by $S(\sum_{i=1}^{k} \alpha_i f(\lambda_i), \sum_{i=1}^{l} \beta_i f(z_i)) := (\sum_{i=1}^{k} \alpha_i e^{\lambda_i} f(\lambda_i), \sum_{i=1}^{l} \beta_i e^{z_i} f(z_i))$, $k, l \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $\text{Re}(\lambda_i) < 0$, $1 \leq i \leq k$, $\beta_i \in \mathbb{C}$, $\text{Re}(z_i) < 0$, $1 \leq i \leq l$, in order to see that, for every $t > 0$, the operator $C^{-1}T(t) \oplus C^{-1}T(t)$ is hypercyclic. This implies that $(T(t))_{t\geq 0}$ is weakly mixing and chaotic. The chaoticity of the operator $C^{-1}T(t)$

$(t > 0)$ can be shown as in the proof of [24, Theorem 4.9] and this completes the proof of (i). The proof of (ii) can be obtained along the same lines.

Remark 14. *(i) Assume that $\mathcal{G}$ is a (C-DS) and that the set $\tilde{E}$ is not $\mathcal{G}$-admissible. Then one can define the notion of $\tilde{E}$-hypercyclicity ($\tilde{E}_S$-hypercyclicity) of $\mathcal{G}$ in several different ways. In the second part of this remark, it will be said that $\mathcal{G}$ is $\tilde{E}$-hypercyclic iff there exists $x \in D(\mathcal{G}) \cap \tilde{E}$ such that the set $\{G(\delta_t)x : t \geq 0\}$ is a dense subset of $\tilde{E}$, and that $\mathcal{G}$ is $\tilde{E}$-chaotic iff $\mathcal{G}$ is $\tilde{E}$-hypercyclic and the set $\mathcal{G}_{\tilde{E},per}$ is dense in $\tilde{E}$.*

(ii) Under the assumptions of Theorem 13(ii), $(T(t))_{t\geq 0}$ is $\tilde{E}$-weakly mixing and $\tilde{E}$-chaotic. We will prove this statement only in the case $p(z) = z$. Clearly, for every $\lambda \in \Omega$, $R(\xi : A)f(\lambda) = \frac{f(\lambda)}{\xi-\lambda}$, $\xi \in \rho(A) \setminus \{\lambda\}$. By the representation formula [15, p. 70, l. 2], one can show that there exists a mapping $g : \Omega \to \mathbb{C} \setminus \{0\}$ such that $Cf(\lambda) = g(\lambda)f(\lambda)$, $\lambda \in \Omega$. This implies that $C(E_0) = E_0$ and that $R(C_{\tilde{E}})$ is dense in $\tilde{E}$. Let $D(T_1) = \{x \in Z(A) \cap \tilde{E} : G(\delta_1)x \in Z(A) \cap \tilde{E}\}$ and $T_1x = G(\delta_1)x$, $x \in D(T_1)$. Using [17, Theorem 2.3] with $T = T_1 \oplus T_1$, $Y_1 = X_0 \oplus X_0$, $Y_2 = X_\infty \oplus X_\infty$, $S(x, y) = (e^\lambda x, e^\lambda y)$, $x,\ y \in X_0$, and $C_{\tilde{E}}$, one yields that the operator T is hypercyclic in $\tilde{E}$. As an outcome, we get that $(T(t))_{t\geq 0}$ is both $\tilde{E}$-weakly mixing and $\tilde{E}$-chaotic.

Example 15. *([19, Example 4.12])*
In what follows, we analyze chaotic properties of a convection-diffusion type equation of the form

$$\begin{cases} u_t = au_{xx} + bu_x + cu := -Au, \\ u(0,t) = 0,\ t \geq 0, \\ u(x,0) = u_0(x),\ x \geq 0. \end{cases}$$

It is well known that the operator $-A$, considered with the domain $D(-A) = \{f \in W^{2,2}([0,\infty)) : f(0) = 0\}$, generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2}$ in the space $E = L^2([0,\infty))$, provided $a,\ b,\ c > 0$ and $c < \frac{b^2}{2a} < 1$. The same conclusion holds if we consider the operator $-A$ with the domain $D(-A) = \{f \in W^{2,1}([0,\infty)) : f(0) = 0\}$ in $E = L^1([0,\infty))$ (cf. [18, Example 2.4]). Let

$$\Omega = \{\lambda \in \mathbb{C} : |\lambda - (c - \frac{b^2}{4a})| \leq \frac{b^2}{4a},\ Im(\lambda) \neq 0 \text{ if } Re(\lambda) \leq c - \frac{b^2}{4a}\}$$

and let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a nonconstant polynomial such that $a_n > 0$ and that $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$. Notice that the last condition holds provided $a_0 \in i\mathbb{R}$. By Theorem 13, it follows that there exists an injective operator $C \in L(E)$ such that $p(A)$ generates an entire C-regularized group $(T(t))_{t\in\mathbb{C}}$ satisfying that $(T(t))_{t\geq 0}$ is chaotic and weakly mixing.

L. Ji and A. Weber [22] have recently investigated the dynamics of L^p heat semigroups $(p > 2)$ on symmetric spaces of non-compact type. We close the paper by observing that Theorem 13 and Remark 14 can be applied to the operators considered in [22, Theorem 3.1(a), Theorem 3.2, Corollary 3.3] and that convenable chosen shifts (polynomials) of the backwards heat operator, acting on such spaces, possess a certain (sub-)chaotical behavior.

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