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SOME INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract

Some inequalities for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1 Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [6, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have
(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;
(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;
(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;
(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.
With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A) \tag{P}$$

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in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [6] and the references therein. For other results, see [13], [7] and [9].

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [11] (see also [6, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [11]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

The following result that provides a reverse of the Mond & Pečarić has been obtained in [3]:

Theorem 2 (Dragomir, 2008, [3]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \tag{1}$$

for any $x \in H$ with $\|x\| = 1$.

Perhaps more convenient reverses of the Mond & Pečarić result are the following inequalities that have been obtained in the same paper [3]:

Theorem 3 (Dragomir, 2008, [3]). *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned} &(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\quad \leq \begin{cases} \frac{1}{2} \cdot (M - m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2} \cdot (f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\ &\quad \leq \frac{1}{4}(M - m)(f'(M) - f'(m)) \end{aligned} \tag{2}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$
\begin{aligned}
(0 \leq) \langle f(A)x,x\rangle - f(\langle Ax,x\rangle) &\leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \\
&- \begin{cases} [\langle Mx-Ax, Ax-mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\
&\leq \frac{1}{4}(M-m)(f'(M)-f'(m)) \qquad (3)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$
\begin{aligned}
(0 \leq) &\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \\
&\leq \begin{cases} \frac{1}{4} \cdot \frac{(M-m)(f'(M)-f'(m))}{\sqrt{Mmf'(M)f'(m)}} \langle Ax,x\rangle \langle f'(A)x,x\rangle, \\ \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{f'(M)}-\sqrt{f'(m)}\right)[\langle Ax,x\rangle \langle f'(A)x,x\rangle]^{\frac{1}{2}}, \end{cases} \qquad (4)
\end{aligned}
$$

for any $x \in H$ with $\|x\| = 1$.

For generalisations to n-tuples of operators as well as for some particular cases of interest, see [3].

The main aim of the present paper is to provide more general vector inequalities for convex functions whose derivatives are continuous.

2 Some Inequalities for Two Operators

The following result holds:

Theorem 4. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$, then*

$$
\begin{aligned}
&\langle f'(A)x,x\rangle \langle By,y\rangle - \langle f'(A)Ax,x\rangle \\
&\leq \langle f(B)y,y\rangle - \langle f(A)x,x\rangle \leq \langle f'(B)By,y\rangle - \langle Ax,x\rangle \langle f'(B)y,y\rangle \qquad (5)
\end{aligned}
$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$
\begin{aligned}
&\langle f'(A)x,x\rangle \langle Ay,y\rangle - \langle f'(A)Ax,x\rangle \\
&\leq \langle f(A)y,y\rangle - \langle f(A)x,x\rangle \leq \langle f'(A)Ay,y\rangle - \langle Ax,x\rangle \langle f'(A)y,y\rangle \qquad (6)
\end{aligned}
$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\begin{aligned}&\langle f'(A)x,x\rangle\langle Bx,x\rangle-\langle f'(A)Ax,x\rangle\\&\quad\le\langle f(B)x,x\rangle-\langle f(A)x,x\rangle\le\langle f'(B)Bx,x\rangle-\langle Ax,x\rangle\langle f'(B)x,x\rangle\end{aligned}\tag{7}$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have that

$$f'(s)\cdot(t-s)\le f(t)-f(s)\le f'(t)\cdot(t-s)\tag{8}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then for any $x \in H$ with $\|x\| = 1$ we have

$$\begin{aligned}&\langle f'(A)\cdot(t\cdot 1_H-A)x,x\rangle\\&\quad\le\langle[f(t)\cdot 1_H-f(A)]x,x\rangle\le\langle f'(t)\cdot(t\cdot 1_H-A)x,x\rangle\end{aligned}\tag{9}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

The inequality (9) is equivalent with

$$t\langle f'(A)x,x\rangle-\langle f'(A)Ax,x\rangle\le f(t)-\langle f(A)x,x\rangle\le f'(t)t-f'(t)\langle Ax,x\rangle\tag{10}$$

for any $t \in [m, M]$ and any $x \in H$ with $\|x\| = 1$.

If we fix $x \in H$ with $\|x\| = 1$ in (10) and apply the property (P) for the operator B, then we get

$$\begin{aligned}&\langle[\langle f'(A)x,x\rangle B-\langle f'(A)Ax,x\rangle 1_H]y,y\rangle\\&\quad\le\langle[f(B)-\langle f(A)x,x\rangle 1_H]y,y\rangle\le\langle[f'(B)B-\langle Ax,x\rangle f'(B)]y,y\rangle\end{aligned}\tag{11}$$

for each $y \in H$ with $\|y\| = 1$, which is clearly equivalent to the desired inequality (5). ∎

Remark 1. *If we fix* $x \in H$ *with* $\|x\| = 1$ *and choose* $B = \langle Ax, x\rangle \cdot 1_H$, *then we obtain from the first inequality in (5) the reverse of the Mond-Pečarić inequality obtained by the author in [3]. The second inequality will provide the inequality (MP) for convex functions whose derivatives are continuous.*

The following corollary is of interest:

Corollary 1. *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *whose derivative* f' *is continuous on* $\mathring{I}$. *Also, suppose that* A *is a selfadjoint operator on the Hilbert space* H *with* $Sp(A) \subseteq [m, M] \subset \mathring{I}$. *If* g *is nonincreasing and continuous on* $[m, M]$ *and*

$$f'(A)[g(A)-A]\ge 0\tag{12}$$

in the operator order of $B(H)$, *then*

$$(f\circ g)(A)\ge f(A)\tag{13}$$

in the operator order of $B(H)$.

Proof. If we apply the first inequality from (7) for $B = g(A)$ we have

$$\langle f'(A)x,x\rangle \langle g(A)x,x\rangle - \langle f'(A)Ax,x\rangle \leq \langle f(g(A))x,x\rangle - \langle f(A)x,x\rangle \tag{14}$$

any $x \in H$ with $\|x\| = 1$.

We use the following Čebyšev type inequality for functions of operators established by the author in [4]:

Let A be a selfadjoint operator with $Sp(A) \subseteq [m,M]$ for some real numbers $m < M$. If $h,g : [m,M] \longrightarrow \mathbb{R}$ are continuous and *synchronous (asynchronous)* on $[m,M]$, then

$$\langle h(A)g(A)x,x\rangle \geq (\leq) \langle h(A)x,x\rangle \cdot \langle g(A)x,x\rangle \tag{15}$$

for any $x \in H$ with $\|x\| = 1$.

Now, since f' and g are continuous and are asynchronous on $[m,M]$, then by (15) we have the inequality

$$\langle f'(A)g(A)x,x\rangle \leq \langle f'(A)x,x\rangle \cdot \langle g(A)x,x\rangle \tag{16}$$

for any $x \in H$ with $\|x\| = 1$.

Subtracting in both sides of (16) the quantity $\langle f'(A)Ax,x\rangle$ and taking into account, by (12), that $\langle f'(A)[g(A)-A]x,x\rangle \geq 0$ for any $x \in H$ with $\|x\| = 1$, we then have

$$\begin{aligned} 0 &\leq \langle f'(A)[g(A)-A]x,x\rangle = \langle f'(A)g(A)x,x\rangle - \langle f'(A)Ax,x\rangle \\ &\leq \langle f'(A)x,x\rangle \cdot \langle g(A)x,x\rangle - \langle f'(A)Ax,x\rangle \end{aligned}$$

which together with (14) will produce the desired result (13). ∎

We provide now some particular inequalities of interest that can be derived from Theorem 4:

Example 1. ***a.*** *Let A,B two positive definite operators on H. Then we have the inequalities*

$$1 - \left\langle A^{-1}x,x\right\rangle \langle By,y\rangle \leq \langle \ln Ax,x\rangle - \langle \ln By,y\rangle \leq \langle Ax,x\rangle \left\langle B^{-1}y,y\right\rangle - 1 \tag{17}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$1 - \left\langle A^{-1}x,x\right\rangle \langle Ay,y\rangle \leq \langle \ln Ax,x\rangle - \langle \ln Ay,y\rangle \leq \langle Ax,x\rangle \left\langle A^{-1}y,y\right\rangle - 1 \tag{18}$$

for any $x,y \in H$ with $\|x\| = \|y\| = 1$ and

$$1 - \left\langle A^{-1}x,x\right\rangle \langle Bx,x\rangle \leq \langle \ln Ax,x\rangle - \langle \ln Bx,x\rangle \leq \langle Ax,x\rangle \left\langle B^{-1}x,x\right\rangle - 1 \tag{19}$$

for any $x \in H$ with $\|x\| = 1$.

b. *With the same assumption for A and B we have the inequalities*

$$\langle By,y\rangle - \langle Ax,x\rangle \leq \langle B\ln By,y\rangle - \langle \ln Ax,x\rangle \langle By,y\rangle \tag{20}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have

$$\langle Ay, y\rangle - \langle Ax, x\rangle \leq \langle A \ln Ay, y\rangle - \langle \ln Ax, x\rangle \langle Ay, y\rangle \tag{21}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\langle Bx, x\rangle - \langle Ax, x\rangle \leq \langle B \ln Bx, x\rangle - \langle \ln Ax, x\rangle \langle Bx, x\rangle \tag{22}$$

for any $x \in H$ *with* $\|x\| = 1$.

The proof of Example **a** follows from Theorem 4 for the convex function $f(x) = -\ln x$ while the proof of the second example follows by the same theorem applied for the convex function $f(x) = x \ln x$ and performing the required calculations. The details are omitted.

The following result may be stated as well:

Theorem 5. *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$*. If* A *and* B *are selfadjoint operators on the Hilbert space* H *with* $Sp(A), Sp(B) \subseteq [m, M] \subset \mathring{I}$*, then*

$$\begin{aligned} f'(\langle Ax, x\rangle)(\langle By, y\rangle - \langle Ax, x\rangle) &\leq \langle f(B) y, y\rangle - f(\langle Ax, x\rangle) \\ &\leq \langle f'(B) By, y\rangle - \langle Ax, x\rangle \langle f'(B) y, y\rangle \end{aligned} \tag{23}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have

$$\begin{aligned} f'(\langle Ax, x\rangle)(\langle Ay, y\rangle - \langle Ax, x\rangle) &\leq \langle f(A) y, y\rangle - f(\langle Ax, x\rangle) \\ &\leq \langle f'(A) Ay, y\rangle - \langle Ax, x\rangle \langle f'(A) y, y\rangle \end{aligned} \tag{24}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\begin{aligned} f'(\langle Ax, x\rangle)(\langle Bx, x\rangle - \langle Ax, x\rangle) &\leq \langle f(B) x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \langle f'(B) Bx, x\rangle - \langle Ax, x\rangle \langle f'(B) x, x\rangle \end{aligned} \tag{25}$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Since f is convex and differentiable on $\mathring{\mathrm{I}}$, then we have that

$$f'(s) \cdot (t - s) \leq f(t) - f(s) \leq f'(t) \cdot (t - s) \tag{26}$$

for any $t, s \in [m, M]$.

If we choose $s = \langle Ax, x\rangle \in [m, M]$, with a fix $x \in H$ with $\|x\| = 1$, then we have

$$f'(\langle Ax, x\rangle) \cdot (t - \langle Ax, x\rangle) \leq f(t) - f(\langle Ax, x\rangle) \leq f'(t) \cdot (t - \langle Ax, x\rangle) \tag{27}$$

for any $t \in [m, M]$.

Now, if we apply the property (P) to the inequality (27) and the operator B, then we get

$$\begin{aligned}&\langle f'(\langle Ax,x\rangle)\cdot(B-\langle Ax,x\rangle\cdot 1_H)\,y,y\rangle\\&\qquad\le\langle[f(B)-f(\langle Ax,x\rangle)\cdot 1_H]\,y,y\rangle\le\langle f'(B)\cdot(B-\langle Ax,x\rangle\cdot 1_H)\,y,y\rangle\end{aligned}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$, which is equivalent with the desired result (23). ∎

Remark 2. *We observe that if we choose $B=A$ in (25) or $y=x$ in (24) then we recapture the Mond-Pečarić inequality and its reverse from (1).*

The following particular case of interest follows from Theorem 5

Corollary 2. *Assume that f,A and B are as in Theorem 5. If, either f is increasing on $[m,M]$ and $B\ge A$ in the operator order of $B(H)$ or f is decreasing and $B\le A$, then we have the Jensen's type inequality*

$$\langle f(B)\,x,x\rangle\ge f(\langle Ax,x\rangle) \tag{28}$$

for any $x\in H$ with $\|x\|=1$.

The proof is obvious by the first inequality in (25) and the details are omitted.

We provide now some particular inequalities of interest that can be derived from Theorem 5:

Example 2. ***a.*** *Let A,B be two positive definite operators on H. Then we have the inequalities*

$$1-\langle Ax,x\rangle^{-1}\langle By,y\rangle\le\ln(\langle Ax,x\rangle)-\langle\ln By,y\rangle\le\langle Ax,x\rangle\left\langle B^{-1}y,y\right\rangle-1 \tag{29}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

In particular, we have

$$1-\langle Ax,x\rangle^{-1}\langle Ay,y\rangle\le\ln(\langle Ax,x\rangle)-\langle\ln Ay,y\rangle\le\langle Ax,x\rangle\left\langle A^{-1}y,y\right\rangle-1 \tag{30}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and

$$1-\langle Ax,x\rangle^{-1}\langle Bx,x\rangle\le\ln(\langle Ax,x\rangle)-\langle\ln Bx,x\rangle\le\langle Ax,x\rangle\left\langle B^{-1}x,x\right\rangle-1 \tag{31}$$

for any $x\in H$ with $\|x\|=1$.

b. *With the same assumption for A and B, we have the inequalities*

$$\langle By,y\rangle-\langle Ax,x\rangle\le\langle B\ln By,y\rangle-\langle By,y\rangle\ln(\langle Ax,x\rangle) \tag{32}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

In particular, we have

$$\langle Ay,y\rangle-\langle Ax,x\rangle\le\langle A\ln Ay,y\rangle-\langle Ay,y\rangle\ln(\langle Ax,x\rangle) \tag{33}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$ and

$$\langle Bx,x\rangle-\langle Ax,x\rangle\le\langle B\ln Bx,x\rangle-\langle Bx,x\rangle\ln(\langle Ax,x\rangle) \tag{34}$$

for any $x\in H$ with $\|x\|=1$.

3 Inequalities for Two Sequences of Operators

The following result may be stated:

Theorem 6. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, ..., n\}$, then*

$$\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle \leq \sum_{j=1}^{n} \langle f(B_j) y_j, y_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \leq \sum_{j=1}^{n} \langle f'(B_j) B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(B_j) y_j, y_j \rangle \quad (35)$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

In particular, we have

$$\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle A_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle \leq \sum_{j=1}^{n} \langle f(A_j) y_j, y_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \leq \sum_{j=1}^{n} \langle f'(A_j) A_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(A_j) y_j, y_j \rangle \quad (36)$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$ and

$$\sum_{j=1}^{n} \langle f'(A_j) x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle f'(A_j) A_j x_j, x_j \rangle \leq \sum_{j=1}^{n} \langle f(B_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \leq \sum_{j=1}^{n} \langle f'(B_j) B_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle f'(B_j) x_j, x_j \rangle \quad (37)$$

for any $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [6, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix}, \ \widetilde{B} := \begin{pmatrix} B_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & B_n \end{pmatrix}$$

and

$$\widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}, \widetilde{y} = \begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right), Sp\left(\widetilde{B}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = \|\widetilde{y}\| = 1$,

$$\left\langle f'\left(\widetilde{A}\right)\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \left\langle f'\left(A_j\right)x_j, x_j\right\rangle, \left\langle B\widetilde{y}, \widetilde{y}\right\rangle = \sum_{j=1}^{n} \left\langle By_j, y_j\right\rangle$$

and so on.

Applying Theorem 4 for $\widetilde{A}$, $\widetilde{B}$, $\widetilde{x}$ and $\widetilde{y}$ we deduce the desired result (35). ∎

The following particular case may be of interest:

Corollary 3. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A_j and B_j are selfadjoint operators on the Hilbert space H with $Sp(A_j), Sp(B_j) \subseteq [m, M] \subset \mathring{I}$ for any $j \in \{1, ..., n\}$, then for any $p_j, q_j \geq 0$ with $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$, we have the inequalities*

$$\begin{aligned} &\left\langle \sum_{j=1}^{n} p_j f'\left(A_j\right)x, x\right\rangle \left\langle \sum_{j=1}^{n} q_j B_j y, y\right\rangle - \left\langle \sum_{j=1}^{n} p_j f'\left(A_j\right)A_j x, x\right\rangle \\ &\qquad \leq \left\langle \sum_{j=1}^{n} q_j f\left(B_j\right)y, y\right\rangle - \left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle \\ &\qquad \leq \left\langle \sum_{j=1}^{n} q_j f'\left(B_j\right)B_j y, y\right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle \left\langle \sum_{j=1}^{n} q_j f'\left(B_j\right)y, y\right\rangle \end{aligned} \tag{38}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j)x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j A_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j) A_j x, x \right\rangle$$
$$\leq \left\langle \sum_{j=1}^{n} q_j f(A_j) y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle$$
$$\leq \left\langle \sum_{j=1}^{n} q_j f'(A_j) B_j y, y \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} q_j f'(A_j) y, y \right\rangle \quad (39)$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\left\langle \sum_{j=1}^{n} p_j f'(A_j)x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f'(A_j) A_j x, x \right\rangle$$
$$\leq \left\langle \sum_{j=1}^{n} p_j f(B_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle$$
$$\leq \left\langle \sum_{j=1}^{n} p_j f'(B_j) B_j x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^{n} p_j f'(B_j) x, x \right\rangle \quad (40)$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Follows from Theorem 6 on choosing $x_j = \sqrt{p_j} \cdot x$, $y_j = \sqrt{q_j} \cdot y$, $j \in \{1, ..., n\}$, where $p_j, q_j \geq 0, j \in \{1, ..., n\}$, $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$ and $x, y \in H$, with $\|x\| = \|y\| = 1$. The details are omitted. ∎

Example 3. ***a.*** *Let* A_j, B_j, $j \in \{1, ..., n\}$, *be two sequences of positive definite operators on* H. *Then we have the inequalities*

$$1 - \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle$$
$$\leq \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle \ln B_j y_j, y_j \rangle \leq \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \left\langle B_j^{-1} y_j, y_j \right\rangle - 1 \quad (41)$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

b. *With the same assumption for* A_j *and* B_j *we have the inequalities*

$$\sum_{j=1}^{n} \langle B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle$$
$$\leq \sum_{j=1}^{n} \langle B_j \ln B_j y_j, y_j \rangle - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle B_j y_j, y_j \rangle \quad (42)$$

for any $x_j, y_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = \sum_{j=1}^{n} \|y_j\|^2 = 1$.

Finally, we have

Example 4. ***a.*** *Let A_j, B_j, $j \in \{1,...,n\}$, be two sequences of positive definite operators on H. Then for any $p_j, q_j \geq 0$ with $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j = 1$, we have the inequalities*

$$1 - \left\langle \sum_{j=1}^n p_j A_j^{-1} x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle \leq \left\langle \sum_{j=1}^n p_j \ln A_j x, x \right\rangle - \left\langle \sum_{j=1}^n q_j \ln B_j y, y \right\rangle \leq \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j^{-1} y, y \right\rangle - 1 \quad (43)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

b. *With the same assumption for A_j, B_j, p_j and q_j, we have the inequalities*

$$\left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j A_j x, x \right\rangle \leq \left\langle \sum_{j=1}^n q_j B_j \ln B_j y, y \right\rangle - \left\langle \sum_{j=1}^n p_j \ln A_j x, x \right\rangle \left\langle \sum_{j=1}^n q_j B_j y, y \right\rangle \quad (44)$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Remark 3. *We observe that all the other inequalities for two operators obtained in Section 2 can be extended for two sequences of operators in a similar way. However, the details are left to the interested reader.*

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