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ON EXTENDABILITY OF CAYLEY GRAPHS

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Abstract

A connected graph Γ of even order is *n-extendable*, if it contains a matching of size n and if every such matching is contained in a perfect matching of Γ. Furthermore, a connected graph Γ of odd order is *$n\frac{1}{2}$-extendable*, if for every vertex v of Γ the graph $\Gamma - v$ is n-extendable.

It is proved that every connected Cayley graph of an abelian group of odd order which is not a cycle is $1\frac{1}{2}$-extendable. This result is then used to classify 2-extendable connected Cayley graphs of generalized dihedral groups.

1 Introductory remarks

Throughout this paper graphs are assumed to be finite and simple.

A connected graph Γ of even order is *n-extendable*, if it contains a matching of size n and if every such matching is contained in a perfect matching of Γ. The concept of n-extendable graphs was introduced by Plummer [8] in 1980. Since then a number of papers on this topic have appeared (see [2, 10, 11, 12] and the references therein). In 1993 Yu [11] introduced an analogous concept for graphs of odd order. A connected graph Γ of odd order is *$n\frac{1}{2}$-extendable*, if for every vertex v of Γ the graph $\Gamma - v$ is n-extendable.

The problem of n-extendability of Cayley graphs was first considered in [3] where a classification of 2-extendable Cayley graphs of dihedral groups was obtained. (For a definition of a Cayley graph see Section 2.) A few years later a classification of 2-extendable Cayley graphs of abelian groups was obtained in [2]. In this paper we generalize these results in two different ways. First, we consider $n\frac{1}{2}$-extendability for Cayley graphs of abelian groups of odd order. In particular, we prove the following theorem.

Theorem 1 *Let Γ be a connected Cayley graph on an abelian group of odd order $n \geq 3$. Then either Γ is a cycle, or Γ is $1\frac{1}{2}$-extendable.*

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Second, using Theorem 1 we generalize the result of [3] to generalized dihedral groups as follows.

Theorem 2 *Let Γ be a connected Cayley graph on a generalized dihedral group which is not a cycle. Then Γ is 2-extendable unless it is isomorphic to one of the following Cayley graphs on cyclic groups, also called circulants: $Circ(2n;\{\pm1,\pm2\})$ $(n \geq 3)$, $Circ(4n;\{\pm1,2n\})$ $(n \geq 2)$, $Circ(4n+2;\{\pm2,2n+1\})$ and $Circ(4n+2;\{\pm1,\pm2n\})$.*

2 Preliminaries

In this section we introduce the notation and some results needed in the rest of the paper.

A *Cayley graph* $Cay(G;S)$ of a group G with respect to the *connection set* $S \subseteq G \setminus \{1\}$, where $S^{-1} = S$, is a graph with vertex-set G in which $g \sim gs$ for all $g \in G$, $s \in S$. In the case that $G = \mathbb{Z}_n$ the graph $Cay(G;S)$ is called a circulant and is denoted by $Circ(n;S)$. Let M be a subset of edges of $Cay(G;S)$ and let $g \in G$. Then Mg denotes the set of all edges of the form $\{ug, vg\}$, where $\{u,v\} \in M$.

A *Hamilton path* of a graph is a path visiting all of its vertices. The question of existence of Hamilton paths in vertex-transitive graphs and in particular Cayley graphs has been extensively studied over the last forty years (see for instance [1, 5, 6, 7] and the references therein). The following result on this topic is of particular interest to us.

Proposition 3 *[4] Let Γ be a connected Cayley graph of an abelian group and of valency at least three. If Γ is not bipartite then for any pair of its vertices u and v there exists a Hamilton path of Γ from u to v. If Γ is bipartite then for any pair of vertices u and v from different parts of bipartition of Γ there exists a Hamilton path of Γ from u to v.*

Note that it follows from this proposition that every connected Cayley graph of an abelian group is 1-extendable if the order of the group is even and is $0\frac{1}{2}$-extendable otherwise. However, as the following proposition (which will be used in the proofs of our main results) shows, not all Cayley graphs on abelian groups of even order are 2-extendable.

Proposition 4 *[2] Let Γ be a connected Cayley graph of an abelian group of even order and valency at least three. Then Γ is 2-extendable if and only if it is not isomorphic to any of $Circ(2n;\{\pm1,\pm2\})$ $(n \geq 3)$, $Circ(4n;\{\pm1,2n\})$ $(n \geq 2)$, $Circ(4n+2;\{\pm2,2n+1\})$ and $Circ(4n+2;\{\pm1,\pm2n\})$.*

We remark that none of the exceptional graphs from Proposition 4 is bipartite. This fact will be used in the proof of Theorem 2.

3 Cayley graphs of abelian groups

In this section we prove Theorem 1. To do this, we first need the following result.

Proposition 5 *Let G be an abelian group of odd order with identity* 1*, and let $S \subseteq G \setminus \{1\}$ be a nonempty set such that $S = S^{-1}$. Then $\Gamma = Cay(\langle S \cup \{g, g^{-1}\}\rangle; S \cup \{g, g^{-1}\})$ is $1\frac{1}{2}$-extendable for every $g \in G \setminus \langle S\rangle$.*

Proof. Recall first that $Cay(\langle S\rangle; S)$ is $0\frac{1}{2}$-extendable by the comment following Proposition 3. Let m be the smallest positive integer such that $g^m \in \langle S\rangle$. Note that the subgraphs of Γ induced on cosets $g^i\langle S\rangle$, $i \in \{0, 1, \ldots, m-1\}$ are all isomorphic to $Cay(\langle S\rangle; S)$. Furthermore, for every $i \in \{0, 1, \ldots, m-1\}$ and $s \in \langle S\rangle$, the vertex $g^i s$ of Γ is adjacent to the vertex $g^{i+1}s$. Finally, since the order of G is odd, both $|\langle S\rangle|$ and m are also odd and $m \geq 3$.

Pick an edge $e = \{g^i s_1, g^j s_2\}$, $i, j \in \{0, 1, \ldots, m-1\}$, $s_1, s_2 \in \langle S\rangle$, and a vertex x of Γ. We show that there exists a perfect matching of $\Gamma - x$ containing e. Since Γ is vertex-transitive, we can assume that $x = 1$. The proof is split into four cases depending on the numbers i and j. Note that we can assume $i \leq j$. Observe also that if $i \neq j$, then $s_1 = s_2$ and either $j - i = 1$ or $i = 0$, $j = m - 1$.

Case 1: $i = j = 0$. Since the subgraph of Γ induced on the coset $g^2\langle S\rangle$ is isomorphic to $Cay(\langle S\rangle; S)$, which is $0\frac{1}{2}$-extendable, there exists an almost perfect matching M of this subgraph missing the vertex g^2. But then

$$\{\{s_1, s_2\}, \{gs_1, gs_2\}, \{g, g^2\}\} \cup \{\{s, gs\} : s \in \langle S\rangle \setminus \{1, s_1, s_2\}\} \cup M \cup$$
$$\{\{g^k s, g^{k+1}s\} : k \in \{3, 5, \ldots, m-2\},\ s \in \langle S\rangle\}$$

is an almost perfect matching of Γ missing 1 and containing e.

Case 2: $i = j \neq 0$. Since $Cay(\langle S\rangle; S)$ is $0\frac{1}{2}$-extendable, there exists an almost perfect matching M of $Cay(\langle S\rangle; S)$ missing 1. If i is odd, then

$$M \cup \{\{g^k s, g^{k+1}s\} : k \in \{1, 3, \ldots, i-2, i+2, \ldots, m-2\},\ s \in \langle S\rangle\} \cup$$
$$\{\{g^i s, g^{i+1}s\} : s \in \langle S\rangle \setminus \{s_1, s_2\}\} \cup \{\{g^i s_1, g^i s_2\}, \{g^{i+1}s_1, g^{i+1}s_2\}\}$$

is an almost perfect matching of Γ missing 1 and containing e. If i is even, then

$$M \cup \{\{g^k s, g^{k+1}s\} : k \in \{1, 3, \ldots, i-3, i+1, \ldots, m-2\},\ s \in \langle S\rangle\} \cup$$
$$\{\{g^{i-1}s, g^i s\} : s \in \langle S\rangle \setminus \{s_1, s_2\}\} \cup \{\{g^{i-1}s_1, g^{i-1}s_2\}, \{g^i s_1, g^i s_2\}\}$$

is an almost perfect matching of Γ missing 1 and containing e.

Case 3: $i = j - 1 \neq 0$. Recall that in this case $s_1 = s_2$. Since $Cay(\langle S\rangle; S)$ is $0\frac{1}{2}$-extendable, there exists an almost perfect matching M of $Cay(\langle S\rangle; S)$ missing 1. If i is odd, then

$$M \cup \{\{g^k s, g^{k+1}s\} : k \in \{1, 3, \ldots, m-2\},\ s \in \langle S\rangle\}$$

is an almost perfect matching of Γ missing 1 and containing e.

Assume now that i is even. Pick an edge $\{s, s'\}$ of M. Since subgraphs of Γ induced on the cosets $g\langle S\rangle$ and $g^{m-1}\langle S\rangle$ are $0\frac{1}{2}$-extendable, there exist almost perfect matchings M_1 and M_{m-1} of these subgraphs, which miss vertices gs and $g^{-1}s'$, respectively. But now

$$(M \setminus \{\{s, s'\}\}) \cup \{\{s, sg\}, \{s', g^{-1}s'\}\} \cup M_1 \cup M_{m-1} \cup \{\{g^k s, g^{k+1}s\} : \\ k \in \{2, 4, \ldots, m-3\},\ s \in \langle S\rangle\}$$

is an almost perfect matching of Γ missing 1 and containing e.

Case 4: $i = 0$, $j \in \{1, m-1\}$. Without loss of generality we can assume $j = 1$ (otherwise replace g by g^{-1}). Since a subgraph of Γ induced on the coset $g^2\langle S\rangle$ is isomorphic to $Cay(\langle S\rangle; S)$, there exist an almost perfect matching M of this subgraph missing the vertex g^2. But then

$$\{\{s, gs\} : s \in \langle S\rangle \setminus \{0\}\} \cup \{\{g, g^2\}\} \cup M \cup \{\{g^k s, g^{k+1}s\} : k \in \{3, 5, \ldots, m-2\}, s \in \langle S\rangle\}$$

is an almost perfect matching of Γ missing 1 and containing e. ■

Proof. [Of Theorem 1] Assume that $\Gamma = Cay(G; S)$ is not a cycle and note that this implies $|S| \geq 4$. We show that Γ is $1\frac{1}{2}$-extendable using induction on $|S|$.

Suppose first that $|S| = 4$. If for some $s \in S$ we have that $\langle s\rangle \neq G$, then $Cay(G; S)$ is $1\frac{1}{2}$-extendable by Proposition 5. We are left with the possibility that $S = \{s, s^{-1}, t, t^{-1}\}$ where $\langle s\rangle = \langle t\rangle = G$. Pick a vertex x and an edge e of $Cay(G; S)$. Let n denote the order of G. Without loss of generality we can assume that $x = 1$, that $s = t^\ell$ for some $\ell \in \{2, 3, \ldots, n-2\}$, and that $e = \{t^i, t^i s\}$ for some $i \in \{1, 2, \ldots, n-\ell-1, n-\ell+1, \ldots, n-1\}$. We now construct an almost perfect matching M of Γ containing e and missing x depending on the parity of i and ℓ. If i and ℓ are both odd, then

$$M = \{e\} \cup \{\{t^j, t^{j+1}\} \ :\ j \in J\},$$

where $J = \{1, 3, \ldots, i-2, i+1, i+3, \ldots, i+\ell-2, i+\ell+1, i+\ell+3, \ldots, n-2\}$. If i is odd and ℓ is even, then

$$M = \{e, \{t^{i+1}, t^{i+\ell+1}\}\} \cup \{\{t^j, t^{j+1}\} \ :\ j \in J\},$$

where $J = \{1, 3, \ldots, i-2, i+2, i+4, \ldots, i+\ell-2, i+\ell+2, i+\ell+4, \ldots, n-2\}$. If i and ℓ are both even, then

$$M = \{e, \{t^{i-1}, t^{i+\ell-1}\}\} \cup \{\{t^j, t^{j+1}\} \ :\ j \in J\},$$

where $J = \{1, 3, \ldots, i-3, i+1, i+3, \ldots, i+\ell-3, i+\ell+1, i+\ell+3, \ldots, n-2\}$. Finally, if i is even and ℓ is odd, then

$$M = \{e, \{t^{i-1}, t^{i+\ell-1}\}, \{t^{i+1}, t^{i+\ell+1}\}\} \cup \{\{t^j, t^{j+1}\} \ :\ j \in J\},$$

where $J = \{1, 3, \ldots, i-3, i+2, i+4, \ldots, i+\ell-3, i+\ell+2, i+\ell+4, \ldots, n-2\}$.

Now suppose $|S| \geq 6$ and pick a vertex x and an edge $e = \{u, us\}$, $s \in S$, of $Cay(G;S)$. We will show that there exists an almost perfect matching of $Cay(G;S)$ which contains e and misses x. Let $t \in S \setminus \{s, s^{-1}\}$, let $S' = S \setminus \{t, t^{-1}\}$ and consider the subgraph $\Gamma' = Cay(\langle S' \rangle; S')$, which, by induction, is $1\frac{1}{2}$-extendable. If $\langle S' \rangle = G$, then an almost perfect matching of Γ', containing e and missing x, is also an almost perfect matching of Γ containing e and missing x. If however $\langle S' \rangle \neq G$, then Γ is $1\frac{1}{2}$-extendable by Proposition 5. ■

4 Cayley graphs of generalized dihedral groups

A group G containing an abelian subgroup H of index 2 and an involution $t \notin H$ such that $tht = h^{-1}$ for each $h \in H$ is called a *generalized dihedral group.* In this case we denote G by D_H. Observe that if $\Gamma = Cay(D_H; S)$ is a Cayley graph of a generalized dihedral group D_H and $h, th' \in S$, then for any vertex x of Γ, $(x, xh, xth^{-1}h', xth')$ is a 4-cycle of Γ. Note also that for each $ta \in S$ and for each subgroup $H' \leq H$ the edges corresponding to ta introduce perfect matchings between components of the subgraph $Cay(D_H; S \cap H')$.

Proof. [Of Theorem 2] Let $\Gamma = Cay(D_H; S)$ and let $S_1 = H \cap S$ and $S_2 = S \setminus S_1$. Let Γ_1 be the subgraph of Γ induced by Γ on H and let Γ_2 be the subgraph of Γ induced on tH. Furthermore pick any two disjoint edges e_1 and e_2 of Γ. We distinguish four cases depending on whether the edges e_i belong to Γ_1 or Γ_2 or neither of them.

CASE 1: $e_1 \in \Gamma_1$ and $e_2 \notin \Gamma_1 \cup \Gamma_2$. (The case $e_1 \in \Gamma_2$, $e_2 \notin \Gamma_1 \cup \Gamma_2$ is done analogously.)
Let $ta \in S_2$ be the unique element such that $e_2 = \{x, tx^{-1}a\}$ for some $x \in H$ and let $h \in S_1$ be such that $e_1 = \{y, yh\}$ for some $y \in H$. Then a perfect matching of Γ containing e_1 and e_2 is

$$\{e_1, e_2\} \cup \{\{z, tz^{-1}a\} : z \in H \setminus \{y, yh, x\}\} \cup \{\{ty^{-1}a, ty^{-1}h^{-1}a\}\}.$$

CASE 2: $e_1, e_2 \in \Gamma_1$. (The case $e_1, e_2 \in \Gamma_2$ is done analogously.)
Since Γ is connected, S_2 is nonempty. With no loss of generality we can assume $t \in S_2$. Letting $h, h' \in S_1$ be such that $e_1 = \{x, xh\}$ and $e_2 = \{y, yh'\}$ for some $x, y \in H$ a perfect matching of Γ containing e_1 and e_2 is

$$\{e_1, e_2\} \cup \{\{z, tz^{-1}\} : z \in H \setminus \{x, y, xh, yh'\}\} \cup \{\{tx^{-1}, tx^{-1}h^{-1}\}, \{ty^{-1}, ty^{-1}h'^{-1}\}\}.$$

CASE 3: $e_1 \in \Gamma_1$, $e_2 \in \Gamma_2$.
If $H' = \langle S_1 \rangle$ is of even order, then each of the $[H : H']$ components of Γ_1 (and Γ_2), and thus Γ_1 (and Γ_2) itself, is 1-extendable by the remark following Proposition 3. Thus in this case Γ clearly contains a desired perfect matching. We can therefore assume that H' is of odd order. Moreover, we can also assume that $e_1 = \{1, h\}$ for some $h \in S_1$. Let $x, h' \in H$ be such that $e_2 = \{tx, txh'\}$. If there exists an element

$ta \in S_2$ such that $\{ta, th^{-1}a\} \cap e_2 = \emptyset$, then $\{x^{-1}a, x^{-1}h'^{-1}a\} \cap e_1 = \emptyset$, and so a perfect matching of Γ containing e_1 and e_2 is

$$\{e_1, e_2\} \cup \{\{z, tz^{-1}a\} : z \in H \backslash \{1, h, x^{-1}a, x^{-1}h'^{-1}a\}\} \cup \{\{ta, th^{-1}a\}, \{x^{-1}a, x^{-1}h'^{-1}a\}\}.$$

Similarly, if for some $ta \in S_2$ we have that $e_2 = \{ta, th^{-1}a\}$, then a desired perfect matching of Γ is

$$\{e_1, e_2\} \cup \{\{z, tz^{-1}a\} : z \in H \setminus \{1, h\}\}.$$

We are left with the possibility that for each $ta \in S_2$ we have $|\{ta, th^{-1}a\} \cap e_2| = 1$. In view of the connectedness of Γ this implies $H' = H$. Suppose first that $|S_1| > 2$. Then $|H| > 4$, and so there exists an edge $e = \{y, ty^{-1}a\}$ such that $e \cap (e_1 \cup e_2) = \emptyset$. By Theorem 1 both Γ_1 and Γ_2 are $1\frac{1}{2}$-extendable, and so a desired perfect matching of Γ clearly exists. Suppose now that $S_1 = \{h, h^{-1}\}$. In this case each of Γ_1 and Γ_2 is isomorphic to a cycle of odd length, say $2n + 1$. Using the remarks from the beginning of this section it is easy to see that the above assumptions imply $|S_2| \leq 2$ and $\Gamma \cong Cay(\mathbb{Z}_{2n+1} \times \mathbb{Z}_2; \{(\pm 1, 0), (0, 1)\}) \cong Circ(4n + 2; \{\pm 2, 2n + 1\})$ in the case of $|S_2| = 1$, and $\Gamma \cong Cay(\mathbb{Z}_{2n+1} \times \mathbb{Z}_2; \{(\pm 1, 0), (\pm 1, 1)\}) \cong Circ(4n + 2; \{\pm 1, \pm 2n\})$ in the case of $|S_2| = 2$. Hence, in either case Γ is a Cayley graph of an abelian group, so that Proposition 4 applies.

Case 4: $e_1, e_2 \notin \Gamma_1 \cup \Gamma_2$.
With no loss of generality we can assume that $e_1 = \{1, t\}$ and $e_2 = \{x, tx^{-1}a\}$ for some $x, a \in H$. If $a = 1$ then a perfect matching of Γ containing e_1 and e_2 is $\{\{z, tz^{-1}\} : z \in H\}$. We can thus assume $a \neq 1$ (implying that $|S_2| \geq 2$). We distinguish two subcases depending on whether $|S_2| = 2$ or not.

Subcase 4.1: $|S_2| \geq 3$.
We show that in this case a desired perfect matching of Γ can be constructed using just some of the edges corresponding to elements of S_2. Now, if $x \notin \langle a \rangle$ then a desired perfect matching of Γ is given by

$$\{\{z, tz^{-1}\} : z \in H \setminus \langle a \rangle x\} \cup \{\{z, tz^{-1}a\} : z \in \langle a \rangle x\},$$

so that we can assume $x \in \langle a \rangle$. Let $tb \in S_2 \backslash \{t, ta\}$, let $H' = \langle a, b \rangle \leq H$ and consider the subgraph Γ' of Γ induced on $H' \cup H't$ by the edges corresponding to t, ta and tb. Note that it suffices to prove that Γ' is 2-extendable. To prove this we use a result of [9] that a bipartite graph with bipartition $A \cup B$, where $|A| = |B|$, is 2-extendable if and only if for each subset $X \subset A$ with $|X| \leq |A| - 2$ we have that $|N(X)| \geq |X| + 2$ (here $N(X)$ denotes the set of neighbours of vertices from X). Suppose there exists a subset X of H' of cardinality at most $|H'| - 2$ for which $|N(X)| \leq |X| + 1$. Since $tx^{-1} \in N(X)$ for each $x \in X$, there cannot exist distinct $x_1, x_2 \in X$ with $x_1a^{-1}, x_2a^{-1} \notin X$ (in this case $\{tx^{-1} : x \in X\} \cup \{tx_1^{-1}a, tx_2^{-1}a\} \subseteq N(X)$ would contradict $|N(X)| \leq |X| + 1$). Hence, except possibly with one exception, for each $x \in X$ we have that $xa^{-1} \in X$. Similarly, except possibly with one exception, for each $x \in X$ we have that $xb^{-1} \in X$. It is easy to see that these two conditions imply that $|X| \geq |H'| - 1$, a contradiction, showing that Γ' and thus Γ is 2-extendable.

Subcase 4.2: $|S_2| = 2$, that is $S_2 = \{t, ta\}$. Note that since, by assumption, Γ is not a cycle, this forces S_1 to be nonempty.
Subsubcase 4.2.1: $\langle S_1 \rangle$ is of even order.
Suppose first that $x \notin \langle S_1 \rangle$ and $tx^{-1}a \notin t\langle S_1 \rangle$. Then a desired perfect matching of Γ is obtained by taking $\{\{z, tz^{-1}\} : z \in \langle S_1 \rangle\} \cup \{\{xz, tx^{-1}z^{-1}a\} : z \in \langle S_1 \rangle\}$ together with perfect matchings of the remaining $2([H : \langle S_1 \rangle] - 2)$ components of $\Gamma_1 \cup \Gamma_2$ (which exist as they are Cayley graphs of an abelian group of even order). Next, suppose $x \in \langle S_1 \rangle$ but $tx^{-1}a \notin t\langle S_1 \rangle$ (the case $x \notin \langle S_1 \rangle$, $tx^{-1}a \in t\langle S_1 \rangle$ is dealt with analogously). If $|S_1| = 1$ (that is, S_1 consists of a single involution), then either $\langle a \rangle = H$ or $[H : \langle a \rangle] = 2$. Hence either $\Gamma \cong Circ(4n; \{\pm 1, 2n\})$ (the cycle of length $4n$ corresponding to ± 1 is given by the edges corresponding to t and ta) or $\Gamma \cong Cay(\mathbb{Z}_{2n} \times \mathbb{Z}_2; \{(\pm 1, 0), (0, 1)\})$, depending on whether $\langle a \rangle = H$ or not, respectively. This shows that Γ is a Cayley graph of an abelian group of even order and valency three, so Proposition 4 applies. We can thus assume that $|S_1| \geq 2$ implying that there exists some $h \in S_1 \setminus \{x\}$. Since $\Gamma' = Cay(\langle S_1 \rangle; S_1)$ is 1-extendable, there exists a perfect matching M of Γ' containing $\{1, h\}$. Let $h' \in H$ be such that $\{x, xh'\} \in M$. Taking $M_1 = Mt \setminus \{\{t, th^{-1}\}\}$ and $M_2 = Mta \setminus \{\{tx^{-1}a, tx^{-1}h'^{-1}a\}\}$ a desired perfect matching of Γ is obtained by taking

$$M \setminus \{\{1, h\}, \{x, xh'\}\} \cup \{e_1, e_2\} \cup \{\{h, th^{-1}\}, \{xh', tx^{-1}h'^{-1}a\}\} \cup M_1 \cup M_2$$

together with perfect matchings of the remaining $2[H : \langle S_1 \rangle] - 3$ components of $\Gamma_1 \cup \Gamma_2$, each of which is isomorphic to Γ'. Finally, suppose $x \in \langle S_1 \rangle$ and $tx^{-1}a \in t\langle S_1 \rangle$ (note that this implies $\langle S_1 \rangle = H$.) We can clearly assume $|S_1| > 1$ (otherwise $\Gamma = K_4$). Now, if $|S_1| = 2$, then each of Γ_1 and Γ_2 is isomorphic to a cycle of length $2n$ for some n. We can thus identify the vertex set of Γ with the set $V = \mathbb{Z}_{2n} \times \mathbb{Z}_2$ in such a way that $(i, j) \sim (i + 1, j)$ for each $i \in \mathbb{Z}_{2n}$ and $j \in \{0, 1\}$, $(i, 0) \sim (i, 1)$ for each $i \in \mathbb{Z}_{2n}$ and $(i, 0) \sim (i + k, 1)$ for each $i \in \mathbb{Z}_{2n}$ and some fixed nonzero $k \in \mathbb{Z}_{2n}$. If $k = 2k_1$ for some k_1, then (relabeling the vertices $(i, 1)$ by $(i - k_1, 1)$ for $i \in \mathbb{Z}_{2n}$) we clearly have that $\Gamma \cong Cay(\mathbb{Z}_{2n} \times \mathbb{Z}_2; \{(\pm 1, 0), (\pm k_1, 1)\})$. If on the other hand $k = 2k_1 - 1$ for some k_1, then the permutation ρ of the vertex set V defined by $\rho((i, 0)) = (i + k_1, 1)$ and $\rho((i, 1)) = (i - k_1 + 1, 0)$ for every $i \in \mathbb{Z}_{2n}$ is easily seen to be an automorphism of Γ of order $4n$, so that Γ is a circulant in this case (in particular, $\Gamma \cong Circ(4n; \{\pm 2, \pm k\})$). In either case Γ is a Cayley graph of an abelian group, so that Proposition 4 applies. We can thus assume $|S_1| \geq 3$, and so Proposition 3 applies to Γ_1 and Γ_2. If Γ_1 is not bipartite, there is a Hamilton path of Γ_1 between 1 and x and there is a Hamilton path of Γ_2 between t and $tx^{-1}a$. Together with e_1 and e_2 this gives a Hamilton cycle of Γ, and so a desired perfect matching of Γ can be obtained by taking every other edge of this cycle, starting with e_1 (recall that Γ_1 is of even order). If Γ_1 is bipartite then it is 2-extendable by Proposition 4. It is easy to see that, since Γ_1 contains no triangles, there exist disjoint edges $e = \{1, h\}$ and $e' = \{x, xh'\}$ such that et and $e'ta$ are also disjoint. As Γ_1 is 2-extendable we can now find a perfect matching of Γ_1 containing e and e' as well as a perfect matching of Γ_2 containing et and $e'ta$. It is now clear how to construct a desired perfect matching of Γ.
Subsubcase 4.2.2: $\langle S_1 \rangle$ is of odd order.

Consider first the case that $x \notin \langle S_1\rangle$ and $tx^{-1}a \notin t\langle S_1\rangle$. Note that this implies $[H : \langle S_1\rangle] \geq 2$, and therefore the connectivity of Γ forces that $a \notin \langle S_1\rangle$. Let $y \in \langle S_1\rangle x$, $y \neq x$, and observe that then $ya^{-1} \notin \langle S_1\rangle$ (otherwise $tx^{-1}a \in t\langle S_1\rangle$). Letting $e = \{y, ty^{-1}\}$ and $e' = \{ya^{-1}, ty^{-1}a\}$, a desired perfect matching of Γ is

$$\{e, e'\} \cup \{\{z, tz^{-1}a\} : z \in \langle S_1\rangle x \setminus \{y\}\} \cup \{\{z, tz^{-1}\} : z \in H \setminus (\langle S_1\rangle x \cup \langle S_1\rangle xa^{-1})\} \cup M_1 \cup M_2,$$

where M_1 is an almost perfect matching of the component of Γ_1 containing xa^{-1} which misses ya^{-1} and M_2 is an almost perfect matching of the component of Γ_2 containing tx^{-1} which misses ty^{-1}. In the case that $x \in \langle S_1\rangle$ and $tx^{-1}a \in t\langle S_1\rangle$ we clearly have $\langle S_1\rangle = H$. The existence of a desired perfect matching of Γ then depends on $|S_1|$. If $|S_1| = 2$, then each of Γ_1 and Γ_2 is just a cycle. Similar argument as in Subsubcase 4.2.1 shows that then Γ is a Cayley graph of an abelian group, so that Proposition 4 applies. If $|S_1| > 2$, then Γ_1 and Γ_2 are $1\frac{1}{2}$-extendable by Theorem 1. Taking $h \in S_1 \setminus \{x, xa^{-1}\}$ (which exists since $|S_1| > 2$) there thus exists an almost perfect matching of Γ_1 which contains $\{1, h\}$ but misses x and there exists an almost perfect matching of Γ_2 which contains $\{t, th^{-1}\}$ but misses $tx^{-1}a$. It is now clear how to obtain a desired perfect matching of Γ. We are left with the possibility that $x \in \langle S_1\rangle$ but $tx^{-1}a \notin t\langle S_1\rangle$ (the case $x \notin \langle S_1\rangle$, $tx^{-1}a \in t\langle S_1\rangle$ is dealt with analogously). Let M_1 be an almost perfect matching of the component of Γ_2 containing t which misses t. By Proposition 3 each component of $\Gamma_1 \cup \Gamma_2$ contains a Hamilton cycle (if $|S_1| = 2$, then each component of $\Gamma_1 \cup \Gamma_2$ consists of a single cycle). Take a Hamilton cycle C of the component containing 1 and let y be the neighbor of x on this cycle, such that the length of the subpath of C from 1 to x not passing through y consists of an even number of vertices. Let M_2 be the unique matching in $Cay(\langle S_1\rangle; S_1)$ consisting of edges of C which misses 1, x and y. Furthermore, let $M_3 = M_2ta$ and let M_4 be an almost perfect matching of the component containing a^{-1} which misses a^{-1}. Then a desired perfect matching of Γ is

$$\{\{z, tz^{-1}\} : z \in H \setminus (\langle S_1\rangle \cup \langle S_1\rangle a^{-1})\} \cup \{e_1, e_2, \{y, ty^{-1}a\}, \{a^{-1}, ta\}\} \cup M_1 \cup M_2 \cup M_3 \cup M_4.$$

■

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