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DYNAMICS OF GILPIN-AYALA COMPETITION MODEL WITH RANDOM PERTURBATION

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Abstract

In this paper we study the Gilpin-Ayala competition system with random perturbation which is more general and more realistic than the classical Lotka-Volterra competition model. We verify that the positive solution of the system does not explode in a finite time. Furthermore, it is stochastically ultimately bounded and continuous a.s. We also obtain certain results about asymptotic behavior of the stochastic Gilpin-Ayala competition model.

1 Introduction

One of the most common phenomena considering ecological population is that many species which grow in the same environment compete for the limited resources or in some way inhibit others' growth. A particularly interesting type of this kind of interaction is facultative mutualism, in which the interacting species derive benefit from each other but, not being fully dependent, each can survive without the symbiotic partner (e.g. plants producing fruits are eaten by birds and on the other hand the birds help on dispersing the seeds of the fruit when they excrete them on places far from the parent plant).

One of the famous models that regards dynamics of population systems is the classical Lotka-Volterra competitive system. It was suggested independently by Lotka and Volterra in the 1920s and was described by the following differential equation

$$\frac{dN_i}{dt} = r_i N_i \left(1 - \frac{N_i}{k_i} - \alpha_{ij}\frac{N_j}{k_i}\right), \quad i,j = 1,2, \quad i \neq j,$$

where N_i and r_i are the population size and exponential rate of the growth of the ith species, respectively, k_i is the carrying capacity of the ith species in the absence

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of its competitor - the jth species, and α_{ij} is the linear reduction (in terms of k_i) of the ith species' rate of growth by its competitor - the jth species.

Since that time, many different forms of the Lotka-Volterra competition model have been studied (see, for example, [1], [2]). However, the Lotka-Volterra competition model is linear (i.e. the rate of change in the size of each species is a linear function of sizes of the interacting species) and this property is considered as a disadvantage of this model. In 1973, Gilpin and Ayala [3] claimed that a little more complicated model was needed in order to obtain more realistic solutions, so they proposed a few competition models, for example,

$$\frac{dN_i}{dt} = r_i N_i \left(1 - \left(\frac{N_i}{k_i} \right)^{\theta_i} - \alpha_{ij} \frac{N_j}{k_i} \right), \quad i, j = 1, \ldots, d,$$

where θ_i are the parameters which modify the classical Lotka-Volterra model and they represent a nonlinear measure of interspecific interference ($i = 1, \ldots, d$). It was noticed that the Gilpin-Ayala model has even some properties which do not exist in the Lotka-Volterra model [4].

The mentioned examples are deterministic cases, but the population systems are often subject to the environmental noise, i.e. they are exposed to the impact of a large number of random unpredictable factors. So, it is natural to consider what happens if the noise is included in the model. There are many papers which consider stochastic Lotka-Volterra population models (see [5],[6],[7]), and just a few papers deal with Gilpin-Ayala competition systems ([8],[9],[10]).

In a deterministic case, without further hypothesis on the matrix $(\alpha_{ij})_{d\times d}$, the solutions of these models may not exist on $[0, \infty)$ and may explode at a finite time. Since the solutions of the equations represent the size of the populations, they should be both positive and finite. The main purpose of this paper is to show that, under certain conditions for (α_{ij}), adding the environmental noise to the model will not make any change in that property, i.e. the solution of the considered equation will be positive and will not explode at a finite moment of time.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Moreover, let $w(t) = (w_1(t), \ldots, w_d(t))^T$ be a d-dimensional Brownian motion defined on a filtered space and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i > 0 \text{ for all } 1 \leq i \leq d\}$. Then, let $|A| = \sqrt{\text{trace}(A^T A)}$ be the trace norm of a matrix A (where A^T denotes the transpose of a vector or matrix A) and its operator norm by $\|A\| = \sup\{|Ax| : |x| = 1\}$.

Consider the Gilpin-Ayala competition model for a system with d interacting species

$$dx_i(t) = x_i(t) \left[r_i - \left(\frac{x_i(t)}{k_i} \right)^{\theta_i} - \sum_{i\neq j}^{d} \frac{a_{ij} x_j(t)}{k_j} \right] dt \tag{1}$$

for every $1 \leq i \leq d$, where $x_i(t)$, r_i and k_i are the population size at a time t, the intrinsic exponential growth rate and the carrying capacity in the absence of competition, respectively, for the i-th species; then, $a_{ij}, i \neq j = 1, \ldots, d$ represent the effect of interspecific interaction and $\theta_i \geq 0, i = 1, \ldots, d$ are the parameters that modify the classical Lotka-Volterra model. In this paper we observe the situation of the parameter perturbation. In practice the intrinsic growth rate r_i of the species i is estimated by an average value plus an error term. In other words, we can replace the rate r_i with an average growth rate plus a random fluctuation term $r_i + \sigma_i \dot{w}_i(t)$ where σ_i^2 is the intensity of the noise and $\dot{w}_i(t)$ is a white noise $(i = 1, \ldots, d)$. As a result, Eq. (1) becomes the stochastic Gilpin-Ayala competition model:

$$dx_i(t) = x_i(t)\left\{\left[r_i - \left(\frac{x_i(t)}{k_i}\right)^{\theta_i} - \sum_{i\neq j}^{d} \frac{a_{ij}x_j(t)}{k_j}\right]dt + \sigma_i dw_i(t)\right\}. \tag{2}$$

We assume that the competitions among the different spaces are non-negative, i.e.

$$a_{ij} \geq 0, \ i \neq j. \tag{3}$$

The paper is organized as follows: In the next section we prove that the solution of Eq. (2) is global and positive, which is a logical requirement since x_i represents the size of the ith species. Then, we prove that the solution is stochastically ultimately bounded and continuous a.s. in Section 3. Since the explicit solution of Eq. (2) does not exist, we analyze the asymptotic moment behavior of the solution in Section 4 and, also, the pathwise behavior in Section 5.

2 Positive and Global Solutions

In the considered Eq. (2) the coefficients are locally Lipschitz continuous but do not satisfy the linear growth condition. However, according to the existence-and-uniqueness theorem, the coefficients of stochastic differential equations should satisfy both mentioned conditions in order to have a unique global solution (i.e. no explosion in finite time) for any given initial value. To overcome this problem, we need to impose the simple hypotheses (3) to get a solution which is not only positive but also will not explode in any finite time and that is proved in the following theorem.

Theorem 1. *Let us assume that condition* (3) *holds. Then, for any given initial value $x_0 \in \mathbb{R}_+^d$, there is a unique solution $x(t)$ to Eq. (2) on $t \geq 0$. Moreover, this solution remains in $\mathbb{R}_+^d$ with probability 1.*

Proof. Since the coefficients of Eq. (2) are locally Lipschitz continuous, for any given initial data $x_0 \in \mathbb{R}_+^d$ there is a unique maximal local positive solution $x(t)$ defined on $t \in [0, \rho)$, where ρ is an explosion time. To show this solution is global, we need to show that $\rho = \infty$ a.s. Let n_0 be sufficiently large for every component of x_0 lying within the interval $[n_0^{-1}, n_0]$. For each integer $n \geq n_0$, define the stopping time

$$\tau_n = \inf\{t \in [0, \rho) | x_i(t) \notin (n^{-1}, n), \text{ for some } i = 1, \ldots, d\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Because τ_n is increasing as $n \to \infty$, set $\tau_\infty = \lim_{n\to\infty} \tau_n$. If we can show that

$\tau_\infty = \infty$ a.s., then $\rho = \infty$ a.s. and $x(t) \in \mathbb{R}^d_+$ a.s. for all $t \geq 0$. To complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. or for all $T > 0$, we have $P(\tau_n \leq T) \to 0$ when $n \to \infty$. To show this statement, we will define the Lyapunov function $V : \mathbb{R}^d_+ \to \mathbb{R}_+$

$$V(x) = \sum_{i=1}^{d} \big(x_i^\gamma - 1 - \gamma \ln x_i\big), \ \gamma \geq \max_i \theta_i \vee 1.$$

The non-negativity of this function is obvious because $t^\gamma - 1 - \gamma \ln t \geq 0$ for every $t > 0$ and $\gamma > 0$. We use the Itô formula to $V(x(t))$ and obtain:

$$\begin{aligned}
dV(x(t)) = \gamma \sum_{i=1}^{d} \Bigg\{ &(x_i^\gamma(t) - 1)\left[r_i - \left(\frac{x_i(t)}{k_i}\right)^{\theta_i} - \sum_{i\neq j}^{d} \frac{a_{ij}x_j(t)}{k_j}\right] \\
&+ \frac{(\gamma-1)\sigma_i^2}{2} x_i^\gamma(t) + \frac{\sigma_i^2}{2}\Bigg\} dt + \gamma \sum_{i=1}^{d} \sigma_i \big(x_i^\gamma(t) - 1\big) dw_i(t) \\
\leq \gamma \sum_{i=1}^{d} \Bigg\{ &r_i\big(x_i^\gamma(t) - 1\big) + \left(\frac{x_i(t)}{k_i}\right)^{\theta_i} + \sum_{i\neq j}^{d} \frac{a_{ij}x_j(t)}{k_j} \\
&+ \frac{(\gamma-1)\sigma_i^2}{2} x_i^\gamma(t) + \frac{\sigma_i^2}{2}\Bigg\} dt + \gamma \sum_{i=1}^{d} \sigma_i \big(x_i^\gamma(t) - 1\big) dw_i(t) \\
\leq \gamma \sum_{i=1}^{d} \Bigg\{ &\left[r_i + \frac{(\gamma-1)\sigma_i^2}{2}\right](1 + x_i(t))^\gamma + \frac{(1 + x_i(t))^{\theta_i}}{k_i^{\theta_i}} \\
&+ \sum_{i\neq j}^{d} \frac{a_{ij}(1 + x_j(t))}{k_j} + \frac{\sigma_i^2}{2} - r_i \Bigg\} dt + \gamma \sum_{i=1}^{d} \sigma_i \big(x_i^\gamma(t) - 1\big) dw_i(t).
\end{aligned}$$

Because $\gamma \geq \max_i \theta_i \vee 1$ it follows that

$$\begin{aligned}
dV(x(t)) &\leq \gamma \sum_{i=1}^{d} \left\{ (1 + x_i(t))^\gamma \left[r_i + \frac{(\gamma-1)\sigma_i^2}{2} + \frac{1}{k_i^{\theta_i}} + \sum_{i\neq j}^{d} \frac{a_{ji}}{k_i}\right] + \frac{\sigma_i^2}{2} - r_i \right\} dt \\
&\quad + \gamma \sum_{i=1}^{d} \sigma_i \big(x_i^\gamma(t) - 1\big) dw_i(t) \\
&= F(x(t))dt + \gamma \sum_{i=1}^{d} \sigma_i \big(x_i^\gamma(t) - 1\big) dw_i(t),
\end{aligned}$$

where

$$F(x(t)) = \gamma\left(A_1 \sum_{i=1}^{d} x_i^\gamma(t)dt + dA_2\right),$$
$$A_1 = (2^{\gamma-1}\vee 1)\left[r + \frac{(\gamma-1)\sigma^2}{2} + \frac{1}{k^\theta} + \frac{(d-1)a}{k}\right],$$
$$A_2 = \big[(2^{\gamma-1}\vee 1) - 1\big]r + \frac{\sigma^2}{2} + (2^{\gamma-1}\vee 1)\left[\frac{(\gamma-1)\sigma^2}{2} + \frac{1}{k^\theta} + \frac{(d-1)a}{k}\right]$$

and $r = \max_i r_i$, $\sigma^2 = \max_i \sigma_i^2$, $a = \max_{i,j} a_{ij}$, $k = \min_i k_i$, $\theta = \min_i \theta_i$. Since $\gamma \geq 1$ and knowing that $x^\gamma \leq 2[x^\gamma - 1 - \gamma \ln x] + 2, \gamma \geq 0$, we obtain

$$\begin{aligned} F(x(t)) &\leq 2\gamma A_1 V(x(t)) + d\gamma(2A_1 + A_2) \\ &= K_1 V(x(t)) + K_2. \end{aligned}$$

Then

$$dV(x(t)) \leq \big(K_1 V(x(t)) + K_2\big)dt + \gamma \sum_{i=1}^{d} \sigma_i(x_i^\gamma(t) - 1)dw_i(t).$$

Integrating the last inequality from 0 to $\tau_n \wedge T$, yields

$$\begin{aligned} V(x(\tau_n \wedge T)) &\leq V(x(0)) + K_1 \int_0^{\tau_n\wedge T} V(x(s))ds + K_2 \cdot (\tau_n \wedge T) \\ &\quad + \gamma \int_0^{\tau_n\wedge T} \sum_{i=1}^{d} \sigma_i(x_i^\gamma(t) - 1)dw_i(t). \end{aligned}$$

By taking expectations we get

$$EV(x(\tau_n \wedge T)) \leq V(x(0)) + K_2 T + K_1 \int_0^T EV(x(\tau_n \wedge t))dt.$$

The Gronwall-Bellman inequality implies that

$$EV(x(\tau_n \wedge T)) \leq \big[V(x(0)) + K_2 T\big]e^{K_1 T}.$$

For every $\omega \in \{\tau_n \leq T\}$, there is some i such that $x_i(\tau_n, \omega) \notin (n^{-1}, n)$. Hence,

$$V(x(\tau_n)) \geq (x_i(\tau_n))^\gamma - 1 - \gamma \ln x_i(\tau_n) = \left[\left(\frac{1}{n^\gamma} - 1 - \gamma \ln\frac{1}{n}\right) \wedge \left(n^\gamma - 1 - \gamma \ln n\right)\right]$$

and then it follows that

$$\begin{aligned} \infty &> [V(x(0)) + K_2 T\big]e^{K_1 T} \geq EV(x(\tau_n \wedge T)) \\ &= P(\tau_n \leq T)V(x(\tau_n)) + P(\tau_n > T)V(x(T)) \geq P(\tau_n \leq T)V(x(\tau_n)) \\ &\geq P(\tau_n \leq T)\left[\left(\frac{1}{n^\gamma} - 1 - \gamma \ln\frac{1}{n}\right) \wedge \left(n^\gamma - 1 - \gamma \ln n\right)\right]. \end{aligned}$$

Because $\left(\frac{1}{n^\gamma} - 1 - \gamma \ln \frac{1}{n}\right) \wedge \left(n^\gamma - 1 - \gamma \ln n\right)$ tends to infinity when $n \to \infty$, this implies that $\lim_{n\to\infty} P(\tau_n \leq T) = 0$ and, therefore, $P(\tau_\infty \leq T) = 0$. Since $T > 0$ is arbitrary, we deduce that $P(\tau_\infty = \infty) = 1$. Thus, the theorem is proved. □

3 Properties of the solution

In this section, firstly we deduce that the p-th moment of the solution of Eq. (2) is finite for every $p > 0$. Then, we prove that the solution is stochastically ultimately bounded and continuous.

Theorem 2. *Let condition (3) holds. Then for any $p > 0$,*

$$\sup_{t\geq 0} E\left[\sum_{i=1}^{d} x_i^p(t)\right] \leq K < \infty. \tag{4}$$

Proof. Define the function $V : \mathbb{R}_+^d \to \mathbb{R}_+$ by

$$V(x) = \sum_{i=1}^{d} x_i^p, \;\; p > 0.$$

By applying the Itô formula to $e^t V(x(t))$, we obtain

$$\begin{aligned} d[e^t V(x(t))] &= e^t \sum_{i=1}^{d} x_i^p(t)dt + pe^t \sum_{i=1}^{d} x_i^p(t) \Bigg[r_i - \Big(\frac{x_i(t)}{k_i}\Big)^{\theta_i} \\ &\quad - \sum_{i\neq j}^{d} \frac{a_{ij}x_j(t)}{k_j} + \frac{(p-1)\sigma_i^2}{2} \Bigg] dt + pe^t \sum_{i=1}^{d} \sigma_i x_i^p(t) dw_i(t) \\ &\leq e^t F(x(t))dt + pe^t \sum_{i=1}^{d} \sigma_i x_i^p(t) dw_i(t), \end{aligned}$$

where

$$F(x(t)) = p \sum_{i=1}^{d} x_i^p(t) \left[\frac{1}{p} + r_i - \Big(\frac{x_i(t)}{k_i}\Big)^{\theta_i} + \frac{(p-1)\sigma_i^2}{2} \right].$$

It is easy to conclude that there exists a positive constant K such that $F(x(t)) \leq K$. Therefore,

$$d[e^t V(x(t))] \leq K e^t dt + pe^t \sum_{i=1}^{d} \sigma_i x_i^p(t) dw_i(t).$$

Integrating the last inequality from 0 to $\tau_n \wedge t$ and then taking expectations, yields

$$e^t EV(x(\tau_n \wedge t)) \leq V(x(0)) + KE \int_0^{\tau_n \wedge t} e^s ds \leq V(x(0)) + K\big(e^t - 1\big).$$

We proved in Theorem 1 that $\lim_{n\to\infty} \tau_n = \infty$, a.s. Then, by virtue of Fatou's lemma and by letting $n \to \infty$ the required assertion (4) follows. □

Corollary 1. *Under the condition of Theorem 2, for any $p > 0$,*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T E|x(t)|^p dt < \infty. \tag{5}$$

Proof. Because

$$E|x(t)|^p \leq d^{\frac{p-2}{2}} E\left(\sum_{i=1}^{d} x_i^p(t)\right), \tag{6}$$

from the previous theorem (5) follows directly. □

Definition 1. *The solution of Eq.* (2) *is said to be stochastically ultimately bounded if for any $\varepsilon \in (0,1)$, there is a positive constant $H = H(\varepsilon)$ such that for any initial value $x(0) \in \mathbb{R}_+^d$, the solution $x(t)$ of Eq.* (2) *satisfies*

$$\limsup_{t\to\infty} \mathbb{P}\left\{|x(t)| \leq H\right\} \geq 1 - \varepsilon. \tag{7}$$

Theorem 3. *Let condition (3) holds. The solution of Eq.* (2) *is stochastically ultimately bounded.*

Proof. By virtue of Tchebychev's inequality and (6) we have that for $H > 0$

$$\mathbb{P}\Big\{|x(t)| > H\Big\} \leq \frac{E|x(t)|^2}{H^2} \leq \frac{K}{H^2}.$$

Therefore, by choosing H sufficiently large, (7) follows. □

In order to prove that the solution of the Eq. (2) is continuous a.s. it is necessary to apply the following lemma.

Lemma 1. *(Kolmogorov-Čentsov theorem on the continuity of stochastic process) [11] Suppose that a d-dimensional stochastic process $\{x(t), t \geq 0\}$ satisfies the condition*

$$E|x(t) - x(s)|^\alpha \leq C|t-s|^{1+\beta}, \quad 0 < s,t < \infty,$$

for some positive constants α, β and C. Then, there exists a continuous modification $\tilde{x}(t)$ of $x(t)$, which has the property that for every $\gamma \in \left(0, \frac{\beta}{\alpha}\right)$, there is a positive random variable $h(\omega)$ such that

$$P\left\{\omega : \sup_{0<|t-s|<h(\omega);0\leq s,t<\infty} \frac{|\tilde{x}(t)-\tilde{x}(s)|}{|t-s|^\gamma} \leq \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every simple path of $\tilde{x}(t)$ is locally but uniformly Hölder-continuous with exponent γ.

Theorem 4. *The solution $x(t)$ of Eq.* (2) *is continuous a.s.*

Proof. Denote for $i = 1, \dots, d$

$$f_i(x(t)) = x_i(t)\left[r_i - \Big(\frac{x_i(t)}{k_i}\Big)^{\theta_i} - \sum_{i\neq j}^{d} \frac{a_{ij}x_j(t)}{k_j}\right],$$
$$g_i(x(t)) = \sigma_i x_i(t).$$

Let $0 < s < t < \infty$, $t - s \leq 1$, $p \geq 2$. Since $dx_i(t) = f_i(x(t))dt + g_i(x(t))dw_i(t)$, it follows that

$$x_i(t) - x_i(s) = \int_s^t f_i(x(u))du + \int_s^t g_i(x(u))dw_i(u),$$

and hence

$$|x_i(t) - x_i(s)|^p \leq 2^{p-1}\left\{\left|\int_s^t f_i(x(u))du\right|^p + \left|\int_s^t g_i(x(u))dw_i(u)\right|^p\right\}.$$

Taking expectations on both sides of the last inequality and using the Hölder's inequality and moment inequalities for the Itô integral, yields

$$E|x_i(t) - x_i(s)|^p \leq 2^{p-1}\left\{(t-s)^{p-1}E\int_s^t \big|f_i(x(u))\big|^p du + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(t-s)^{\frac{p-2}{2}}E\int_s^t \big|g_i(x(u))\big|^p du\right\}. \tag{8}$$

By using the inequality $\left(\sum_{i=1}^d x_i\right)^p \leq d^{p-1}\sum_{i=1}^d x_i^p$, Fubini theorem and Hölder's inequality we get

$$\begin{aligned}E\int_s^t \big|f_i(x(u))\big|^p du &\leq (d+1)^{p-1}E\int_s^t \left\{r_i^p\big|x_i(u)\big|^p + \frac{1}{k_i^{p\theta_i}}\big|x_i(u)\big|^{p(1+\theta_i)} + a^p\sum_{i\neq j}^{d}\left|x_i(u)\frac{x_j(u)}{k_j}\right|^p\right\}du\\
&\leq (d+1)^{p-1}\int_s^t\left\{r_i^p E\big|x_i(u)\big|^p + \frac{1}{k_i^{p\theta_i}}E\big|x_i(u)\big|^{p(1+\theta_i)} + a^p\sum_{i\neq j}^{d}\frac{1}{k_j^p}\Big(E\big|x_i(u)\big|^{2p}\Big)^{1/2}\Big(E\big|x_j(u)\big|^{2p}\Big)^{1/2}\right\}du\end{aligned}$$

where $a = \max_{i,j} a_{ij}$. According to Theorem 2 we get

$$E\int_s^t \big|f_i(x(u))\big|^p du \leq C_1(t-s) \tag{9}$$

where C_1 is a generic constant. Furthermore,

$$E\int_s^t \left|g_i(x(u))\right|^p du \leq |\sigma|^p E\int_s^t \left|x_i(u)\right|^p du \leq C_2(t-s) \tag{10}$$

where $|\sigma| = \max_i |\sigma_i|$ and C_2 is a generic constant. From (8)-(10) it follows

$$E|x_i(t) - x_i(s)|^p \leq \tilde{K}(t-s)^{\frac{p}{2}}.$$

where $\tilde{K}$ is some constant. By applying Lemma 1, we conclude that almost every simple path of $x_i(t)$ is locally but uniformly Hölder-continuous with exponent $\gamma \in \left(0, \frac{p-2}{2p}\right)$ and, therefore, almost every simple path of $x_i(t)$ must be uniformly continuous on $t \geq 0$, i.e. the solution $x(t) = (x_1(t), \ldots, x_d(t))$ of Eq. (2) is continuous a.s. □

4 Asymptotic behavior of the solution

Since the considered Eq. (2) does not have an explicit solution, it is reasonable to study an asymptotic moment estimation.

Theorem 5. *For any $\theta > 0$, there exists a positive constant K such that, for any initial value $x_0 \in \mathbb{R}_+^d$, the solution of Eq.* (2) *satisfies the property*

$$\limsup_{t\to\infty} \frac{1}{t} E\int_0^t \sum_{i=1}^d \frac{1}{2k_i^{\theta_i}} x_i(s)^{\theta+\theta_i}\; ds \leq K.$$

Proof. Define the function $V : \mathbb{R}_+^d \to \mathbb{R}_+$ by

$$V(x) = \sum_{i=1}^d x_i^\theta.$$

Now we use the Itô formula to $V(x(t))$:

$$dV(x(t)) \leq \left[F(x(t)) - \frac{\theta}{2}\sum_{i=1}^d \frac{x_i^{\theta+\theta_i}(t)}{k_i^{\theta_i}}\right]dt + \sum_{i=1}^d \theta\sigma_i x_i^\theta(t) dw_i(t),$$

where

$$F(x(t)) = \theta\sum_{i=1}^d x_i^\theta(t)\left[r_i - \frac{x_i^{\theta_i}(t)}{2k_i^{\theta_i}} - \frac{(1-\theta)\sigma_i^2}{2}\right].$$

Furthermore, by taking into consideration the fact that polynomial $F(x(t))$ has an upper positive bound K, the last equality becomes

$$dV(x(t)) \leq Kdt - \frac{\theta}{2}\sum_{i=1}^d \frac{x_i^{\theta+\theta_i}(t)}{k_i^{\theta_i}} dt + \sum_{i=1}^d \theta\sigma_i x_i^\theta(t) dw_i(t).$$

Integrating from 0 to t gives

$$V(x(t)) + \frac{\theta}{2}\int_0^t \sum_{i=1}^d \frac{x_i^{\theta+\theta_i}(s)}{k_i^{\theta_i}}ds \le V(x(0)) + \bar{K}t + M_t, \tag{11}$$

where $M_t = \theta\int_0^t \sum_{i=1}^d \sigma_i x_i^\theta(s)dw_i(t)$ is a real-valued continuous local martingale vanishing at $t=0$. Taking expectations on both sides of (11) results in

$$E\int_0^t \sum_{i=1}^d \frac{x_i^{\theta+\theta_i}(s)}{k_i^{\theta_i}}ds \le \frac{2}{\theta}\big[V(x(0)) + \bar{K}t\big].$$

The required assertion follows immediately from this for $K = \frac{2}{\theta}\bar{K}$. □

The conclusion of Theorem 4 is very powerful since it is universal in the sense that it is independent of the system parameters a_{ij}, and of the initial value $x_0 \in \mathbb{R}_+^d$. It is also independent of the noise intensity σ_i^2, $i = 1, \ldots, d$.

5 Pathwise estimation

In this section, we consider some limit inequalities for growth rates of the population size.

Theorem 6. *Let us assume that condition (3) holds. Then, there exists a positive constant K such that, for any initial value $x_0 \in \mathbb{R}_+^d$, the solution of Eq.* (2) *has the property that*

$$\limsup_{t\to\infty} \frac{\ln\Big(\prod_{i=1}^d x_i(t)\Big)}{t} \le K \quad a.s.$$

Proof. For each $1 \le i \le d$, we apply Itô's formula to $\ln x_i(t)$ and obtain

$$d\big(\ln x_i(t)\big) = \left[r_i - \left(\frac{x_i(t)}{k_i}\right)^{\theta_i} - \sum_{i\neq j}^d \frac{a_{ij}x_j(t)}{k_j} - \frac{\sigma_i^2}{2}\right]dt + \sigma_i dw_i(t).$$

Integrating both sides of this equality from 0 to t yields

$$\ln x_i(t) = \ln x_i(0) + M_i(t) + \int_0^t \left[r_i - \left(\frac{x_i(s)}{k_i}\right)^{\theta_i} - \sum_{i\neq j}^d \frac{a_{ij}x_j(s)}{k_j} - \frac{\sigma_i^2}{2}\right]ds, \tag{12}$$

where $M_i(t) = \sigma_i w_i(t)$ is the real-valued continuous local martingale vanishing at $t=0$, with the quadratic variation $\langle M_i(t), M_i(t)\rangle = \sigma_i^2 t$. From (12), it follows that

$$\sum_{i=1}^d \ln x_i(t) \le \sum_{i=1}^d \ln x_i(0) + \sum_{i=1}^d M_i(t) + \int_0^t \sum_{i=1}^d \left[r_i - \left(\frac{x_i(s)}{k_i}\right)^{\theta_i} - \frac{\sigma_i^2}{2}\right]ds.$$

Thus, by the strong law of large numbers for martingales, we have that $\lim_{t\to 0} \frac{M_i(t)}{t} = 0$, for $i = 1, \ldots, d$. Since

$$\sum_{i=1}^{d} \left[r_i - \left(\frac{x_i(s)}{k_i} \right)^{\theta_i} - \frac{\sigma_i^2}{2} \right] \leq K$$

for some positive constant K, it follows that

$$\limsup_{t\to\infty} \frac{1}{t} \left[\ln \prod_{i=1}^{d} x_i(t) \right] \leq \limsup_{t\to\infty} \left[\frac{\ln \prod_{i=1}^{d} x_i(0)}{t} + K \right] = K, \; s.i.$$

which is the required assertion □

Theorem 7. *Under the condition (2), there exists a positive constant K such that, for any initial value $x_0 \in \mathbb{R}_+^d$, for the solution of Eq.* (2) *it holds that*

$$\limsup_{t\to\infty} \frac{\ln \left(\prod_{i=1}^{d} x_i(t) \right)}{\ln t} \leq d \quad a.s.$$

Proof. For each $1 \leq i \leq d$, if we apply Itô's formula to $e^{\delta t} \ln x_i(t)$ for $\delta > 0$ and integrate from 0 to t, we have

$$\begin{aligned} e^{\delta t} \ln x_i(t) &= \ln x_i(0) + M_i(t) \\ &\quad + \int_0^t e^{\delta s} \left\{ \delta \ln x_i(s) + r_i - \left(\frac{x_i(s)}{k_i} \right)^{\theta_i} - \sum_{i\neq j}^{d} \frac{a_{ij} x_j(s)}{k_j} - \frac{\sigma_i^2}{2} \right\} dt \end{aligned} \tag{13}$$

where $M_i(t) = \int_0^t e^{\delta s} \sigma_i dw_i(s)$ is a real-valued continuous local martingale vanishing at $t = 0$ with quadratic variation

$$\langle M_i(t), M_i(t) \rangle = \frac{\sigma_i^2 (e^{2\delta t} - 1)}{2\delta}.$$

Fix $\varepsilon \in (0, \frac{1}{2})$ arbitrarily and $\theta > 1$. For every integer $n \geq 1$, using the exponential martingale inequality we have

$$P \left\{ \sup_{0\leq t\leq n} \left[M_i(t) - \frac{\varepsilon}{2e^{\delta n}} \langle M_i(t), M_i(t) \rangle \right] \geq \frac{\theta e^{\delta n} \ln n}{\varepsilon} \right\} \leq \frac{1}{n^\theta}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^\theta}$ converges, the application of the Borel-Cantelli lemma yields that there exists an $\Omega_i \subset \Omega$ with $P(\Omega_i) = 1$ such that for any $\omega \in \Omega_i$ an integer $n_i = n_i(\omega)$ can be found such that

$$M_i(t) \leq \frac{\varepsilon}{2e^{\delta n}} \langle M_i(t), M_i(t) \rangle + \frac{\theta e^{\delta n} \ln n}{\varepsilon},$$

for $0 \le t \le n$ and $n \ge n_i(\omega)$, $i = 1, \dots, d$. Thus (13) results in

$$e^{\delta t} \ln x_i(t) \le \ln x_i(0) + \frac{\theta e^{\delta n} \ln n}{\varepsilon} + \frac{\varepsilon}{2} \int_0^t \sigma_i^2 e^{2\delta s - \delta n} ds + \int_0^t e^{\delta s} \left\{ \delta \ln x_i(s) + r_i - \left(\frac{x_i(s)}{k_i} \right)^{\theta_i} - \sum_{i \ne j}^{d} \frac{a_{ij} x_j(s)}{k_j} - \frac{\sigma_i^2}{2} \right\} ds$$

for $0 \le t \le n_i(\omega)$ and $n \ge n_i(\omega)$ whenever $\omega \in \Omega_i$. Now let $\Omega_0 = \bigcap_{i=1}^d \Omega_i$. Clearly, $P(\Omega_0) = 1$. Moreover, for all $\omega \in \Omega_0$, let $n_0 = \max\{n_i(\omega) : 1 \le i \le d\}$. Then, for all $\omega \in \Omega_0$, it follows from the last inequality that

$$e^{\delta t} \sum_{i=1}^d \ln x_i(t) \le \sum_{i=1}^d \ln x_i(0) + \frac{\theta d e^{\delta n} \ln n}{\varepsilon} + \int_0^t e^{\delta s} \sum_{i=1}^d \left[\delta \ln x_i(s) + r_i - \left(\frac{x_i(s)}{k_i} \right)^{\theta_i} - \frac{\sigma_i^2 (1 - \varepsilon e^{\delta(s-n)})}{2} \right] ds$$

for $0 \le t \le n$ and $n \ge n_0(\omega)$. Since

$$\sum_{i=1}^d \left[\delta \ln x_i(s) + r_i - \left(\frac{x_i(s)}{k_i} \right)^{\theta_i} - \frac{\sigma_i^2 (1 - \varepsilon e^{\delta(s-n)})}{2} \right] \le K,$$

for all $x \in \mathbb{R}_+^d$ for some positive constant $K = K(\theta_i)$, we have

$$e^{\delta t} \ln \prod_{i=1}^d x_i(t) \le \ln \prod_{i=1}^d x_i(0) + \frac{\theta d e^{\delta n} \ln n}{\varepsilon} + K \frac{e^{\delta t} - 1}{\delta}$$

for $0 \le t \le n$ and $n \ge n_0(\omega)$. Consequently, for all $\omega \in \Omega_0$, if $n - 1 \le t \le n$ and $n \ge n_0(\omega)$,

$$\frac{\ln \prod_{i=1}^d x_i(t)}{\ln t} \le \frac{e^{-\delta(n-1)} \ln \prod_{i=1}^d x_i(0) + \frac{\theta d e^{\delta} \ln n}{\varepsilon} + \frac{K}{\delta} - \frac{K}{\delta} e^{-\delta(n-1)}}{\ln(n-1)}$$

which implies that

$$\limsup_{t \to \infty} \frac{\ln \prod_{i=1}^d x_i(t)}{\ln t} \le \frac{\theta d e^{\delta}}{\varepsilon} \quad s.i.$$

In the end, by letting $\theta \to 1$, $\delta \to 0$ and $\varepsilon \to 1$, it follows that

$$\limsup_{t \to \infty} \frac{\ln \prod_{i=1}^d x_i(t)}{\ln t} \le d \quad s.i.$$

and the proof is completed. □

The conclusion of Theorem 7 is very powerful since it is universal in the sense that it is independent both of the noise intensity σ_i^2, $i = 1, \dots, d$, and of the initial value $x_0 \in \mathbb{R}_+^d$. It is also independent of the system parameters a_{ij} as long as it exists in the sense of hypothesis (3).

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