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MOMENT DECAY RATES OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH TIME-VARYING DELAY

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Abstract

One of the most important questions, especially in applications, is how to choose a decay function in the study of stability for a concrete equation. Motivated by the fact that the coefficients of the considered equation mainly suggest the choice of the decay function, the point of analysis in the paper is to carry out the Lyapunov function approach and to state coercivity conditions dependent on decay in the study of the pth moment stability with a general decay rate for a certain stochastic differential equation with variable time delay. Some criteria including the usual exponential decay are discussed, particularly the ones with a concave decay, special cases of which are polynomial and logarithmic decays. Some consequences and examples are given to illustrate the theory.

1 Introduction

Stochastic differential delay equations (SDDE), as a special case of stochastic functional differential equations, represent a mathematical formulation of dynamical systems in science and engineering where some of the past states are included to determine the present state. The significance of SDDEs has also become more evident in recent years due to their applications in modelling of real-life phenomena in population dynamics, for instance. However, researchers hit upon enormous difficulties in their attempt to solve and study even simple such equations. For example, general criteria for the stability of the following SDDE with constant coefficients $dx(t) = [ax(t) + bx(t-\tau)]\,dt + [cx(t) + dx(t-\tau)]\,dw(t)$ have not been derived yet. Moreover, SDDEs used in many applications are becoming more and more complex and unsolvable in almost all cases. For that reason, the main interest in the field

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has been often referred to the existence and uniqueness of the solutions with a special emphasis on stability problems. We refer the reader to monographs [2] by R.Z. Has'minskii, [5] by G.S. Ladde and V. Lakshmikantham, [12, 14] by X. Mao and [15] by S. Mohamed, among others, and the literature cited therein.

It is evident that there is numerous literature ion exponential stability in the sense of pth mean or almost sure for SDDEs. We mention here [1, 7, 7] by K. Liu *et al.*, [3, 4] by S. Jankovic *et al.*, [10] by J. Luo and [9, 11, 13, 17] by X. Mao, for instance. However, it should be pointed out that some SDDEs are in fact stable but with respect to a certain lower decay rate which is different from exponential decay, for instance, polynomial or logarithmic one. For example, some illustrative examples are given in [1, 6, 7, 8] justifying the investigation of mean square, pth mean or almost sure stability with respect to an arbitrary decay rate which includes the exponential rate as a special case. These examples gave a new impetus motivation to formulate conditions dependent on decay functions guaranteeing the pth moment stability for SDDEs considered in the paper.

It should also be pointed out that there are not many papers about SDDEs referring to mean square, pth moment and almost sure stability with decay. We call readers' attention to [7, 8, 11, 18], for instance. It seems important, from the theoretical point of view and even more for applications, to find some other more applicable criteria to verify the required type of stability and thus to describe the behavior of the solution with respect to the desired decay rate. The topic of the paper are just some new criteria, based on the Lyapunov functions approach, for the pth moment stability with decay of the solutions to a class of SDDEs with time-varying lags. Note that in [7] similar problems are discussed under conditions that are not comparable with the ones in the present paper.

The paper is organized as follows: In the remainder of this section, we introduce some notations and notions needed in our investigation and we present SDDE which will be the topic of our study. In Section 2, we state the main results – conditions guaranteeing that the considered equation is pth moment stable with respect to an arbitrary decay rate. Some criteria with a concave decay including the polynomial and logarithmic decays as special cases are particularly discussed. The paper is closed with Section 3, where some consequences and examples are given to illustrate the usefulness of the theoretical considerations.

Throughout the paper, we assume that all random variables and processes are given on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration is right-continuous and $\mathcal{F}_0$ contains all P-null sets). As usual, let $|\cdot|$ denote the Euclidean norm in R^d and $||\cdot||$ the matrix trace-norm, that is, $||A|| = \sqrt{trace(A^T A)}$ for a matrix A, where A^T is the transpose of A. For a fixed $\tau > 0$, denote that $C([-\tau, 0]; R^d)$ is the family of R^d-valued continuous functions φ defined on $[-\tau, 0]$, equipped with the norm $||\varphi|| = \sup_{s\in[-\tau,0]} |\varphi(s)|$. For $p > 0$, let $L^p(\Omega, \mathcal{F}_0, C([-\tau, 0]; R^d))$ be the family of all $C([-\tau, 0]; R^d)$-valued $\mathcal{F}_0$-adapted random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ satisfying $E||\xi||^p < \infty$.

Suppose that there exist continuously differentiable functions $\rho_i : R_+ \to R$,

$i = 1, \dots, n$, satisfying the following conditions,

$$\rho_i(t) \le t, \quad \rho_i'(t) \ge 1, \quad t \ge 0. \tag{1}$$

From here,

$$t + \rho_i(0) \le \rho_i(t), \quad \rho_i^{-1}(t) \le t - \rho_i(0), \quad t \ge 0, \tag{2}$$

where $\rho_i^{-1}(\cdot)$ is the inverse function of $\rho_i(\cdot)$. Likewise, let us just denote that $\tau = \max\{-\rho_i(0), i = 1, \dots, n\}$.

The topic of our investigation is the following SDDE with time-varying lags,

$$\begin{aligned} dx(t) = {} & F(x(t), x(\rho_1(t)), \dots, x(\rho_n(t)), t)\, dt \\ & + G(x(t), x(\rho_1(t)), \dots, x(\rho_n(t)), t)\, dw(t), \quad t \ge 0, \end{aligned} \tag{3}$$

with the initial condition $x_0 = \xi \in L^p(\Omega, \mathcal{F}_0, C([-\tau, 0], R^d))$. The functions $F : R^d \times R^{d\times n} \times R_+ \to R^d$ and $G : R^d \times R^{d\times n} \times R_+ \to R^{d\times m}$ are Borel measurable, $w = \{w(t), \mathcal{F}_t, t \ge 0\}$ is an m-dimensional Brownian motion and $x(t)$ is an unknown stochastic process.

An $\mathcal{F}_t$-adapted process $x = \{x(t), -\tau \le t \le \infty\}$ is said to be the solution to Eq. (3) if it satisfies a.s. the initial condition for $t \in [-\tau, 0]$ and the corresponding integral equation for every $t \ge 0$.

Since our investigation is focused on stability problems, we assume with no emphasis on conditions that there exists a unique global solution $x(t;\xi), t \in [-\tau, \infty)$ to Eq. (3) satisfying $E \sup_{t\in[-\tau,\infty)} |x(t;\xi)|^p < \infty$ for $p \ge 0$ (for more details see [14], for instance), as well as that all the Lebesgue and Ito integrals employed further are well defined.

For the stability purpose, we usually assume that $F(0, 0, \dots, 0, t) \equiv 0$ and $G(0, 0, \dots, 0, t) \equiv 0$, so that Eq. (3) admits a trivial solution $x(t; 0) \equiv 0$.

Before formulating our stability criteria, let us give the general definition of the pth moment stability with a certain decay function.

Definition 1. *Let the function $\lambda \in C(R_+; R_+)$ be increasing and let $\lambda(t) \uparrow \infty$ as $t \to \infty$. Eq. (3) (or the trivial solution) is said to be pth moment stable with decay $\lambda(t)$ of order γ if there exists a pair of constants $\gamma > 0$ and $c(\xi) > 0$ such that*

$$E|x(t;\xi)|^p \le c(\xi) \cdot \lambda^{-\gamma}(t), \quad t \ge 0$$

holds for any $\xi \in L^p(\Omega, \mathcal{F}_0, C([-\tau, 0], R^d))$, or equivalently,

$$\limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{\ln \lambda(t)} \le -\gamma.$$

Obviously, replacing the decay function $\lambda(t)$ by $e^t, 1 + t$ and $\ln(1 + t)$ leads to the usual moment stability with exponential, polynomial and logarithmic decays, respectively.

Since our investigation is based on the Lyapunov functions approach, we usually introduce the following differential operator associated to Eq. (3).

Let $C^{2,1}(R^d \times R_+; R_+)$ denote the family of all functions $V(x,t): R^d \times R_+ \to R_+$ with continuous second-order and first-order partial derivatives in x and t, respectively. For each $V(x,t) \in C^{2,1}(R^d \times R_+; R_+)$, define an operator $L\tilde{V}$ from $R^d \times R^{d\times n} \times R_+$ to R by

$$\begin{aligned} L\tilde{V}&(x, y_1, y_2, \ldots, y_n, t) \\ &:= V_t(x,t) + V_x(x,t)F(x, y_1, y_2, \ldots, y_n, t) \\ &\quad + \frac{1}{2}\, trace[G^T(x, y_1, y_2, \ldots, y_n, t)V_{xx}(x,t)G(x, y_1, y_2, \ldots, y_n, t)], \end{aligned} \tag{4}$$

where we denote that $(y_1, y_2, \ldots, y_n) \in R^{d\times n}$, $V_t = \frac{\partial V}{\partial t}$, $V_x = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d}\right)$ and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{d\times d}$.

2 Main results

An important question is how to choose a decay function in the study of stability for a certain equation. Clearly, the coefficients of the considered equation mainly suggest our choice of the decay rate. For instance, if $L\tilde{V}$ would be contracted by terms dependent on a function $\lambda(t)$, it seems to be sometimes appropriate to use $\lambda(t)$ as the decay rate. The above mentioned illustrative examples in [1, 6, 7, 8] just explain a motivation to state all the assertions in this section.

Before stating our main results, let us first introduce some general assumptions about decay rates, coercivity terms and Lyapunov functions. Precisely, we suppose that there exist functions λ, θ and V satisfying the following conditions:

(H_1) The decay function $\lambda \in C^1(R_+; R_+)$ is strictly increasing ($\lambda'(t) > 0$) and $\lambda(t) \uparrow \infty$ as $t \to \infty$ ($\lambda(0) = 1$ for the simplicity reason).

(H_2) The function $\theta \in C(R_+; R_+)$ is such that $\theta(t) = o\big(\lambda^\delta(t)\big)$ as $t \to \infty$ for an arbitrary $\delta > 0$.

(H_3) For some constants $p > 0$ and $c_1, c_2 > 0$, the function $V \in C^{2,1}(R^d \times R_+; R_+)$ is such that

$$c_1|x|^p \le V(x,t) \le c_2|x|^p, \quad (x,t) \in R^d \times R_+. \tag{5}$$

In addition to the general assumption (H_1), we require some particular conditions for the decay rates $\lambda(t)$ in the forthcoming assertions. In other words, we state coercivity conditions which will play a key role in our stability results.

Theorem 1. *Let the assumptions (H_1), (H_2) and (H_3) hold for the functions λ, θ and V, respectively, and let $0 < \lambda'(t) \le \lambda(t)$ for $t \ge 0$. Also, let there exist constants $\mu > 0$, $\nu_1, \ldots, \nu_n > 0$ and $\lambda_0, \lambda_1, \ldots, \lambda_n \ge 0$, where $0 \le \frac{c_2}{c_1}\sum_{i=1}^n \lambda_i < \lambda_0$, such that*

$$L\tilde{V}(x, y_1, \ldots, y_n, t) \le -\lambda_0 V(x,t) + \sum_{i=1}^n \lambda_i \lambda^{-\nu_i}(t) V(y_i, t) + \theta(t)\cdot\lambda^{-\mu}(t) \tag{6}$$

for all $x, y_1, \ldots, y_n \in R^d$ *and* $t \geq 0$. *Then, Eq. (3) is pth moment stable in the sense that, for all* $\xi \in L^p(\Omega, \mathcal{F}_0, C([-\tau, 0], R^d))$,

$$\limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{\ln \lambda(t)} \leq -\left[\mu \wedge \nu_1 \wedge \ldots \wedge \nu_n \wedge \left(\frac{c_1}{c_2}\lambda_0 - \sum_{i=1}^{n} \lambda_i\right)\right]. \tag{7}$$

Proof. For simplicity, we will further use the notation $x(t)$ instead of $x(t;\xi)$ to denote the solution to Eq. (3) for a given initial condition ξ.

First, we extend the decay function $\lambda(t)$ so that $\lambda(t) = 1$ for $-\tau \leq t \leq 0$. If we put $u(t) = \ln \lambda(t)$, then $u(t) = 0$ for $-\tau \leq t \leq 0$ and $0 < u'(t) \leq 1$ for $t \geq 0$. Note that although $\lambda'(t)$ and $u'(t)$ are discontinuous in $t = 0$, we will understand without special emphasis that $\lambda'(0) = \lambda'(0+)$ and $u'(0) = u'(0+)$ in the sequel.

Let us denote that $\gamma = \mu \wedge \nu_1 \wedge \ldots \wedge \nu_n \wedge \left(\frac{c_1}{c_2}\lambda_0 - \sum_{i=1}^{n} \lambda_i\right)$ and let $\varepsilon \in (0, \gamma)$ be arbitrary. The application of the Itô formula to $e^{(\gamma-\varepsilon)u(t)}V(x(t), t)$ yields

$$\begin{aligned}
&e^{(\gamma-\varepsilon)u(t)}V(x(t), t)\\
&\quad= V(x(0), 0) + E\int_0^t (\gamma - \varepsilon)e^{(\gamma-\varepsilon)u(s)}u'(s)V(x(s), s)\, ds\\
&\qquad+ \int_0^t e^{(\gamma-\varepsilon)u(s)} L\tilde{V}(x(s), x(\rho_1(s)), \ldots, x(\rho_n(s)), s)\, ds\\
&\qquad+ \int_0^t e^{(\gamma-\varepsilon)u(s)} V_x(x(s), s) G(x(s), x(\rho_1(s)), \ldots, x(\rho_n(s)), s)\, dw(s).
\end{aligned}$$

By virtue of condition (6) and the fact that $u'(t) \leq 1$, we come to the following relation,

$$\begin{aligned}
&Ee^{(\gamma-\varepsilon)u(t)}V(x(t), t)\\
&\quad\leq EV(x(0), 0) + (\gamma - \varepsilon - \lambda_0)E\int_0^t e^{(\gamma-\varepsilon)u(s)}V(x(s), s)\, ds\\
&\qquad+\sum_{i=1}^{n} \lambda_i E\int_0^t e^{(\gamma-\varepsilon-\nu_i)u(s)}V(x(\rho_i(s)), s)\, ds + \int_0^t \theta(s)\lambda^{\gamma-\varepsilon-\mu}(s)\, ds.
\end{aligned}$$

Since $\gamma - \varepsilon - \mu < 0$, the assumption (H_2) yields that there exists δ small enough such that $\theta(t) = o\big(\lambda^{\delta}(t)\big)$ and $\int_0^\infty \theta(s)\lambda^{\gamma-\varepsilon-\mu}(s)\, ds < \infty$. Moreover, by applying condition (5), we see that

$$\begin{aligned}
&c_1 Ee^{(\gamma-\varepsilon)u(t)}|x(t)|^p \\
&\quad\leq Ee^{(\gamma-\varepsilon)u(t)}V(x(t), t)\\
&\quad\leq c_2 E||\xi||^p + [c_2(\gamma - \varepsilon) - c_1\lambda_0)]E\int_0^t e^{(\gamma-\varepsilon)u(s)}|x(s)|^p\, ds\\
&\qquad+c_2\sum_{i=1}^{n} \lambda_i E\int_0^t e^{(\gamma-\varepsilon-\nu_i)u(s)}|x(\rho_i(s))|^p\, ds + \int_0^\infty \theta(s)\lambda^{\gamma-\varepsilon-\mu}(s)\, ds.
\end{aligned} \tag{8}$$

To estimate the integrals $E\int_0^t e^{(\gamma-\varepsilon-\nu_i)u(s)}|x(\rho_i(s))|^p\,ds$, we use the properties (1) and (2) of the functions $\rho_i(t)$. If we put $\rho_i(s)=v$, then $v=\rho_i(s)\le s\le t$ and $dv=\rho_i'(s)\,ds\ge ds$. On the other hand, since $0\le s=\rho_i^{-1}(v)\le v-\rho_i(0)\le v+\tau$, we see that $-\tau\le v\le t$. Moreover, since $\gamma-\varepsilon-\nu_i<0$, we derive

$$\begin{aligned}E\int_0^t e^{(\gamma-\varepsilon-\nu_i)u(s)}|x(\rho_i(s))|^p ds &= E\int_0^t \lambda^{\gamma-\varepsilon-\nu_i}(s)|x(\rho_i(s))|^p\,ds \\ &\le E\int_{-\tau}^t \lambda^{\gamma-\varepsilon-\nu_i}(v)|x(v)|^p\,dv \\ &\le E||\xi||^p\tau + E\int_0^t e^{(\gamma-\varepsilon)u(v)}|x(v)|^p\,dv.\end{aligned} \tag{9}$$

This estimate and (8) yield

$$\begin{aligned}&Ee^{(\gamma-\varepsilon)u(t)}|x(t)|^p \\ &\quad\le c(\xi,\varepsilon)+\left[\frac{c_2}{c_1}\Big(\gamma-\varepsilon+\sum_{i=1}^n\lambda_i\Big)-\lambda_0\right]E\int_0^t e^{(\gamma-\varepsilon)u(s)}|x(s)|^p\,ds,\end{aligned}$$

where $c(\xi,\varepsilon)=\frac{c_2}{c_1}\big(1+\tau\sum_{i=1}^n\lambda_i\big)E||\xi||^p+\frac{1}{c_1}\int_0^\infty\theta(s)\lambda^{\gamma-\varepsilon-\mu}(s)\,ds<\infty$. Having in mind that $\frac{c_2}{c_1}\big(\gamma-\varepsilon+\sum_{i=1}^n\lambda_i\big)-\lambda_0<0$, we get

$$Ee^{(\gamma-\varepsilon)u(t)}|x(t)|^p\le c(\xi,\varepsilon),\quad t\ge 0,$$

that is,

$$\limsup_{t\to\infty}\frac{\ln E|x(t)|^p}{u(t)}\le-(\gamma-\varepsilon).$$

The required result (7) follows straightforwardly letting $\varepsilon\to 0$. $\diamondsuit$

Note that although $\lambda^{-\nu_i}(t)\le 1$ for $t\ge 0$, the presence of $\lambda^{-\nu_i}(t)$ on the right-hand side in (6) is of the basic interest for Theorem 1; on the contrary, the relation (9) could not be proved. Similar reasoning holds for all the assertions in the sequel.

Theorem 2. *Let the assumptions (H_1), (H_2) and (H_3) hold for the functions λ, θ and V, respectively, and let $0<\lambda'(t)\le\lambda(t)$ and $\lambda(t+s)\le\lambda(t)\cdot\lambda(s)$ for $t,s\ge 0$. Also, let there exist constants $\mu>0$, $\nu_1,\dots,\nu_n\ge 0$ and $\lambda_1,\dots,\lambda_n\ge 0$, where $\mu>(\nu_1\vee\ldots\vee\nu_n)\ge 0$ and $0\le\frac{c_2}{c_1}\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)<\lambda_0$, such that condition (6) holds for all $x,y_1,\dots,y_n\in R^d$ and $t\ge 0$. Then,*

$$\limsup_{t\to\infty}\frac{\ln E|x(t;\xi)|^p}{\ln\lambda(t)}\le-(\mu\wedge\alpha^*) \tag{10}$$

for all $\xi\in L^p(\Omega,\mathcal{F}_0,C([-\tau,0],R^d))$, where $\alpha^*\in\Big(0,\frac{c_1}{c_2}\lambda_0-\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\Big)$ is the unique root of the equation

$$c_2\Big(\alpha+\lambda^\alpha(\tau)\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\Big)-c_1\lambda_0=0. \tag{11}$$

Proof. As above, let us extend $\lambda(t)$ so that $\lambda(t) = 1$ for $-\tau \le t \le 0$. If we take $u(t) = \ln \lambda(t)$, then $0 < u'(t) \le 1$ and $u(t+s) \le u(t) + u(s)$ for $t, s \ge 0$.

Let us denote that

$$h(\alpha) = c_2\Big(\alpha + \lambda^{\alpha}(\tau)\sum_{i=1}^{n} \lambda_i \lambda^{-\nu_i}(\tau)\Big) - c_1\lambda_0, \quad \alpha \ge 0.$$

We see that $h(0) = c_2 \sum_{i=1}^{n} \lambda_i \lambda^{-\nu_i}(\tau) - c_1\lambda_0 < 0$, $h\left(\frac{c_1}{c_2}\lambda_0 - \sum_{i=1}^{n} \lambda_i\lambda^{-\nu_i}(\tau)\right) = c_2 \sum_{i=1}^{n} \lambda_i\lambda^{-\nu_i}(\tau) \cdot \left(\lambda^{\frac{c_1}{c_2}\lambda_0 - \sum_{i=1}^{n}\lambda_i\lambda^{-\nu_i}(\tau)}(\tau) - 1\right) > 0$ and $h'(\alpha) > 0$. Therefore, there exists a unique root $\alpha^* \in \left(0, \frac{c_1}{c_2}\lambda_0 - \sum_{i=1}^{n} \lambda_i\lambda^{-\nu_i}(\tau)\right)$ of the equation $h(\alpha) = 0$, that is, of Eq. (11).

For an arbitrary $\varepsilon \in (0, \mu - (\nu_1 \vee \ldots \vee \nu_n))$, if we apply the Itô formula to $e^{(\mu-\varepsilon)u(s)}V(x(t), t)$, conditions (5) and (6) and the fact that $u(t)' \le 1$, we derive

$$\begin{aligned} & c_1 E e^{(\mu-\varepsilon)u(t)}|x(t)|^p \\ & \le c_2 E||\xi||^p + \big[c_2(\mu-\varepsilon) - c_1\lambda_0\big]\, E\int_0^t e^{(\mu-\varepsilon)u(s)}|x(s)|^p\, ds \\ & \quad + c_2 \sum_{i=1}^{n} \lambda_i E \int_0^t e^{(\mu-\varepsilon-\nu_i)u(s)}|x(\rho_i(s))|^p\, ds + \int_0^t \theta(s)\lambda^{-\varepsilon}(s)\, ds. \end{aligned} \tag{12}$$

As before, if we take $v = \rho_i(s)$ and apply the procedure just as in (8), where we use the fact that $\mu - \varepsilon - \nu_i > 0$, we come to the following estimate,

$$\begin{aligned} & E\int_0^t e^{(\mu-\varepsilon-\nu_i)u(s)}|x(\rho_i(s))|^p\, ds \\ & \le E\int_{-\tau}^t e^{(\mu-\varepsilon-\nu_i)u(v+\tau)}|x(v)|^p\, dv \\ & \le E\int_{-\tau}^t e^{(\mu-\varepsilon-\nu_i)(u(v)+u(\tau))}|x(v)|^p\, dv \\ & \le \lambda^{\mu-\varepsilon-\nu_i}(\tau) E\int_{-\tau}^t e^{(\mu-\varepsilon-\nu_i)u(v)}|x(v)|^p\, dv \\ & \le \lambda^{\mu-\varepsilon-\nu_i}(\tau)\left(E||\xi||^p\,\tau + E\int_0^t e^{(\mu-\varepsilon)u(v)}|x(v)|^p\, dv\right). \end{aligned}$$

This estimate together with (12) yields

$$\begin{aligned} & E e^{(\mu-\varepsilon)u(s)}|x(t)|^p \\ & \le c(\xi, \varepsilon, \mu) \\ & \quad + \left[\frac{c_2}{c_1}\Big(\mu - \varepsilon + \lambda^{\mu-\varepsilon}(\tau)\sum_{i=1}^{n}\lambda_i\lambda^{-\nu_i}(\tau)\Big) - \lambda_0\right] E\int_0^t e^{(\mu-\varepsilon)u(s)}|x(s)|^p\, ds, \end{aligned} \tag{13}$$

where

$$c(\xi,\varepsilon,\mu) \tag{14}$$
$$= \frac{c_2}{c_1} E||\xi||^p \Big(1+\tau\lambda^{\mu-\varepsilon}(\tau)\sum_{i=1}^n \lambda_i\lambda^{-\nu_i}(\tau)\Big) + \frac{1}{c_1}\int_0^\infty \theta(s)\lambda^{-\varepsilon}(s)\,ds < \infty.$$

We can now distinguish two cases. First, let $\mu \le \alpha^*$. Since

$$\frac{c_2}{c_1}\Big(\mu-\varepsilon+\lambda^{\mu-\varepsilon}(\tau)\sum_{i=1}^n \lambda_i\lambda^{-\nu_i}(\tau)\Big)-\lambda_0 = \frac{1}{c_1}h(\mu-\varepsilon) < \frac{1}{c_1}h(\alpha^*) = 0,$$

it follows from (13) that

$$E|x(t)|^p \le c(\xi,\varepsilon,\mu)\cdot e^{-(\mu-\varepsilon)u(t)}, \quad t\ge 0$$

and, therefore, (10) holds letting $\varepsilon \to 0$.

Let $\mu > \alpha^*$. Then, from (6),

$$L\tilde V(x,y_1,\dots,y_n,t) \le -\lambda_0 V(x,t) + \sum_{i=1}^n \lambda_i\lambda^{-\nu_i}(t)V(y_i,t) + \theta(t)\cdot\lambda^{-\alpha^*}(t)$$

for all $x, y_1,\dots,y_n \in R^d$ and $t \ge 0$. By putting α^* instead of μ in (13), we see that

$$\begin{aligned} &Ee^{(\alpha^*-\varepsilon)u(s)}|x(t)|^p \\ &\le c(\xi,\varepsilon,\alpha^*) \\ &\quad + \left[\frac{c_2}{c_1}\Big(\alpha^*-\varepsilon+\lambda^{\alpha^*-\varepsilon}(\tau)\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\Big)-\lambda_0\right] E\int_0^t e^{(\alpha^*-\varepsilon)u(s)}|x(s)|^p\,ds, \end{aligned}$$

where $c(\xi,\varepsilon,\alpha^*)$ is of the form (14) with α^* instead of μ. However, because

$$\frac{c_2}{c_1}\Big(\alpha^*-\varepsilon+\lambda^{\alpha^*-\varepsilon}(\tau)\sum_{i=1}^n \lambda_i\lambda^{-\nu_i}(\tau)\Big)-\lambda_0 = \frac{1}{c_1}h(\alpha^*-\varepsilon) < \frac{1}{c_1}h(\alpha^*) = 0,$$

then

$$E|x(t)|^p \le c(\xi,\varepsilon,\alpha^*)\cdot e^{-(\alpha^*-\varepsilon)u(t)}, \quad t\ge 0.$$

What remains is to let $\varepsilon \to 0$ to obtain the desired result (10). Thus, the proof becomes complete. $\diamondsuit$

Especially, if $\lambda(t) = e^t$, the previous assertions refer to the conditions under which Eq. (3) is pth moment exponentially stable, that is,

$$\limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{t} \le -\gamma.$$

However, the next theorems refer to the pth moment stability with concave decays ($\lambda''(t) \le 0$). Although they are not valid for the pth moment exponential stability, their importance is evident because of polynomial and logarithmic decays, $\lambda(t) = 1+t$, $\lambda(t) = \ln(1+t)$, $\lambda(t) = \ln\ln(1+t)$, for instance.

Theorem 3. *Let the assumptions* (H_1), (H_2) *and* (H_3) *hold for the functions* λ, θ *and* V, *respectively. Moreover, let* $\lambda \in C^2(R_+; R_+)$ *and* $\lambda''(t) \le 0$ *for* $t \ge 0$. *Also, let there exist constants* $\mu > 0$, $\nu_1, \ldots, \nu_n > 0$, $\eta_1, \ldots, \eta_n \ge 0$ *and* $\lambda_0, \lambda_1, \ldots, \lambda_n \ge 0$, *where* $0 \le \frac{c_2}{c_1} \sum_{i=1}^n \lambda_i \lambda'^{\eta_i}(0) < \lambda_0$, *such that*

$$\begin{aligned} & L\tilde{V}(x, y_1, \ldots, y_n, t) \\ & \le -\lambda_0 V(x,t) + \sum_{i=1}^n \lambda_i \lambda^{-\nu_i}(t) \lambda'^{\eta_i}(t) V(y_i, t) + \theta(t) \cdot \lambda^{-\mu}(t) \end{aligned} \tag{15}$$

for all $x, y_1, \ldots, y_n \in R^d$ *and* $t \ge 0$. *Then, Eq. (3) is pth moment stable in the sense that, for all* $\xi \in L^p(\Omega, \mathcal{F}_0, C([-\tau, 0], R^d))$,

$$\begin{aligned} & \limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{\ln \lambda(t)} \\ & \le -\left[\mu \wedge \nu_1 \wedge \ldots \wedge \nu_n \wedge \left(\frac{c_1}{c_2 \lambda'(0)} \lambda_0 - \frac{1}{\lambda'(0)} \sum_{i=1}^n \lambda_i \lambda'^{\eta_i}(0) \right) \right]. \end{aligned} \tag{16}$$

Proof. As above, we extend $\lambda(t)$ so that $\lambda(t) = 1$ for $-\tau \le t \le 0$, and we introduce $u(t) = \ln \lambda(t)$ for $-\tau \le t < \infty$.

Let us take $\gamma = \mu \wedge \nu_1 \wedge \ldots \wedge \nu_n \wedge \left(\frac{c_1}{c_2 \lambda'(0)} \lambda_0 - \frac{1}{\lambda'(0)} \sum_{i=1}^n \lambda_i \lambda'^{\eta_i}(0) \right)$ and apply the Itô formula and condition (15). Then, for an arbitrary $\varepsilon \in (0, \gamma)$,

$$\begin{aligned} & c_1 E e^{(\gamma-\varepsilon)u(t)} |x(t)|^p \\ & \le c_2 E||\xi||^p + c_2(\gamma - \varepsilon) E \int_0^t e^{(\gamma-\varepsilon)u(s)} u'(s) |x(s)|^p \, ds \\ & \quad - c_1 \lambda_0 E \int_0^t e^{(\gamma-\varepsilon)u(s)} |x(s)|^p \, ds \\ & \quad + c_2 \sum_{i=1}^n \lambda_i E \int_0^t e^{(\gamma-\varepsilon-\nu_i)u(s)} \lambda'^{\eta_i}(s) |x(\rho_i(s))|^p \, ds + \int_0^\infty \theta(s) \lambda^{\gamma-\varepsilon-\mu}(s) \, ds. \end{aligned} \tag{17}$$

Since $u''(t) = \big(\lambda'(t)/\lambda(t)\big)' = \big[\lambda''(t)\lambda(t) - \lambda'^2(t)\big]/\lambda^2(t) < 0$ for $t \ge 0$, then $u'(t)$ decreases and, therefore, $u'(t) \le \lambda'(0)$ for $t \ge 0$. Likewise, $\lambda'(t) \le \lambda'(0)\lambda(t)$ for $t \ge 0$ and $\lambda'(t) = 0$ for $t < 0$. If we take $v = \rho_i(s)$, then $\lambda'(s) \le \lambda'(\rho_i(s))$ and $e^{(\gamma-\varepsilon-\nu_i)u(s)} \le e^{(\gamma-\varepsilon-\nu_i)u(\rho_i(s))}$. Therefore,

$$\begin{aligned} & E \int_0^t e^{(\gamma-\varepsilon-\nu_i)u(s)} \lambda'^{\eta_i}(s) |x(\rho_i(s))|^p \, ds \\ & \le E \int_{-\tau}^0 e^{(\gamma-\varepsilon-\nu_i)u(v)} \lambda'^{\eta_i}(v) |x(v)|^p \, dv + E \int_0^t e^{(\gamma-\varepsilon-\nu_i)u(v)} \lambda'^{\eta_i}(v) |x(v)|^p \, dv \\ & \le \lambda'^{\eta_i}(0) E \int_0^t e^{(\gamma-\varepsilon)u(v)} |x(v)|^p \, dv. \end{aligned}$$

In view of (17), we see that

$$\begin{aligned}&Ee^{(\gamma-\varepsilon)u(t)}|x(t)|^p\\&\quad\le c(\xi,\varepsilon)+\left[\frac{c_2\lambda'(0)}{c_1}(\gamma-\varepsilon)+\frac{c_2}{c_1}\sum_{i=1}^n\lambda_i\lambda'^{\eta_i}(0)-\lambda_0\right]E\int_0^t e^{(\gamma-\varepsilon)u(s)}|x(s)|^p\,ds,\end{aligned}$$

where $c(\xi,\varepsilon)=\frac{c_2}{c_1}E||\xi||^p+\frac{1}{c_1}\int_0^\infty\theta(s)\lambda^{\gamma-\varepsilon-\mu}(s)\,ds<\infty$. On the other hand, since $\frac{c_2\lambda'(0)}{c_1}(\gamma-\varepsilon)+\frac{c_2}{c_1}\sum_{i=1}^n\lambda_i\lambda'^{\eta_i}(0)-\lambda_0<0$, then

$$E|x(t)|^p\le c(\xi,\varepsilon)\cdot e^{-(\gamma-\varepsilon)u(t)},\quad t\ge 0$$

and, therefore, the proof becomes complete letting $\varepsilon\to 0$. $\diamondsuit$

Theorem 4. *Let the assumptions* (H_1), (H_2) *and* (H_3) *hold for the functions* λ, θ *and* V, *respectively, and let* $\lambda\in C^2(R_+;R_+)$, $\lambda''(t)\le 0$ *and* $\lambda(t+s)\le\lambda(t)\cdot\lambda(s)$ *for* $t,s\ge 0$. *Also, let there exist constants* $\mu>0$, $\nu_1,\dots,\nu_n\ge 0$, $\eta_1,\dots,\eta_n\ge 0$ *and* $\lambda_0,\lambda_1,\dots,\lambda_n\ge 0$, *where* $\mu>(\nu_1\vee\ldots\vee\nu_n)$, *and* $0\le\frac{c_2}{c_1}\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta_i}(0)<\lambda_0$, *such that condition (15) holds for all* $x,y_1,\dots,y_n\in R^d$ *and* $t\ge 0$. *Then, for all* $\xi\in L^p(\Omega,\mathcal{F}_0,C([-\tau,0],R^d))$,

$$\limsup_{t\to\infty}\frac{\ln E|x(t;\xi)|^p}{\ln\lambda(t)}\le-(\mu\wedge\alpha^*),\tag{18}$$

where $\alpha^*\in\Big(0,\frac{c_1}{c_2\lambda'(0)}\lambda_0-\frac{1}{\lambda'(0)}\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta_i}(0)\Big)$ *is the unique root of the equation*

$$c_2\Big(\lambda'(0)\cdot\alpha+\lambda^{\alpha}(\tau)\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta_i}(0)\Big)-c_1\lambda_0=0.\tag{19}$$

Proof. Let us take

$$h(\alpha)=c_2\Big(\lambda'(0)\cdot\alpha+\lambda^{\alpha}(\tau)\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta_i}(0)\Big)-c_1\lambda_0,\quad\alpha\ge 0.$$

It is easy to check that $h(0)<0$, $h\left(\frac{c_1}{c_2\lambda'(0)}\lambda_0-\frac{1}{\lambda'(0)}\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta_i}(0)\right)>0$ and $h'(\alpha)>0$, so that the equation $h(\alpha)=0$, that is, Eq. (19), has a unique root $\alpha^*\in\Big(0,\frac{c_1}{c_2\lambda'(0)}\lambda_0-\frac{1}{\lambda'(0)}\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta_i}(0)\Big)$.

The requirement $\lambda''(t)<0$ implies that $\lambda'(t)$ and $u'(t)$ decrease and $\lambda'(t)<\lambda'(0)\lambda(t)$, $u'(t)<\lambda'(0)$. This fact and the procedure used in the proof of Theorem 2 yield finally, for $\mu\le\alpha^*$ and arbitrary $\varepsilon\in\big(0,\mu-(\nu_1\vee\ldots\vee\nu_n)\big)$, that

$$\begin{aligned}&Ee^{(\mu-\varepsilon)u(s)}|x(t)|^p\\&\quad\le c(\xi,\varepsilon,\mu)+\left[\frac{c_2}{c_1}\Big(\lambda'(0)(\mu-\varepsilon)+\lambda^{\mu-\varepsilon}(\tau)\sum_{i=1}^n\lambda_i\lambda^{-\nu_i}(\tau)\lambda'^{\eta}(0)\Big)-\lambda_0\right]\\&\qquad\times E\int_0^t e^{(\mu-\varepsilon)u(s)}|x(s)|^p\,ds,\end{aligned}$$

where $c(\xi,\varepsilon,\mu) = \frac{c_2}{c_1} E||\xi||^p + \frac{1}{c_1}\int_0^\infty \theta(s)\lambda^{-\varepsilon}(s)\,ds < \infty$. Since the term multiplying the integral $E\int_0^t e^{(\mu-\varepsilon)u(s)}|x(s)|^p\,ds$ is equal to $\frac{1}{c_1}h(\mu-\varepsilon) < \frac{1}{c_1}h(\alpha^*) = 0$, it follows that $E|x(t)|^p \le c(\xi,\varepsilon,\mu)\cdot e^{-(\mu-\varepsilon)u(t)}$, $t \ge 0$. The proof holds now straightforwardly by putting $\varepsilon \to 0$.

For $\mu > \alpha^*$, we start from (15) with α^* instead of μ and repeat completely the procedure in the second part of the proof of Theorem 2. Finally, we deduce that $Ee^{(\alpha^*-\varepsilon)u(s)}|x(t)|^p \le c(\xi,\varepsilon,\alpha^*)$ for $t \ge 0$, which completes the proof. $\diamondsuit$

Note that Theorem 1 and Theorem 3 contain conditions under which Eq. (3) is absolutely pth moment stable with a certain decay, in the sense that the conditions of these assertions do not depend on τ. Therefore, Eq. (3) is always pth moment stable without regard to how large or small is τ.

3 Some consequences and examples

The application of the previous stability criteria could be endangered in many concrete cases because of the difficulties in finding Lyapunov functions $V(x,t)$ satisfying conditions (6) and (15). Motivated by papers [3, 4, 16], for $p \ge 2$ we can state a more effective and relatively easy way to verify criteria, in fact the consequences of Theorems 1–4, by taking particularly $V(x,t) \equiv |x|^p$. Then, in view of (4),

$$\begin{aligned} &L\tilde{V}(x,y_1,y_2,\dots,y_n,t) \\ &\quad\le p|x|^{p-2}|x^T F(x,y_1,\dots,y_n,t)| + \frac{p(p-1)}{2}\,|x|^{p-2}||G(x,y_1,y_2,\dots,y_n,t)||^2 \end{aligned} \tag{20}$$

for all $x, y_1,\dots,y_n \in R^d$ and $t \ge 0$, so that conditions (6) and (15) become, respectively,

$$L\tilde{V}(x,y_1,y_2,\dots,y_n,t) \le -\lambda_0|x|^p + \sum_{i=1}^n \lambda_i\lambda^{-\nu_i}(t)|y_i|^p + \theta(t)\cdot\lambda^{-\mu}(t) \tag{21}$$

$$L\tilde{V}(x,y_1,y_2,\dots,y_n,t) \le -\lambda_0|x|^p + \sum_{i=1}^n \lambda_i\lambda^{-\nu_i}(t)\lambda'^{\eta_i}(t)|y_i|^p + \theta(t)\cdot\lambda^{-\mu}(t). \tag{22}$$

In fact, Eq. (3) could be regarded as a stochastically perturbed system with delay,

$$\dot{x}(t) = F(x(t),x(\rho_1(t)),\dots,x(\rho_n(t)),t)\,dt \quad t \ge 0.$$

Moreover, if $F(x,y_1,\dots,y_n,t) \equiv f(x,t) + h(y_1,\dots,y_n,t)$, the equation

$$\begin{aligned} dx(t) = [&f(x(t),t) + h(x(\rho_1(t)),\dots,x(\rho_n(t)),t) \\ &+G(x(t),x(\rho_1(t)),\dots,x(\rho_n(t)),t)\,dw(t), \quad t \ge 0, \end{aligned} \tag{23}$$

could be understood as a stochastic perturbation of the deterministic system

$$\dot{x}(t) = f(x(t),t), \quad t \ge 0, \tag{24}$$

where the perturbation depends on several states of the past with variable lags. Note that the mean square exponential stability for Eq. (23) was earlier studied in [11], for instance, under some other conditions.

The basic task of robustness analysis in stochastic control theory, among other things, is focused on conditions guaranteeing that the stable system (24) would remain stable in spite of random excitations transforming it in the stochastic system (23). In connection with our considerations in the paper, the main question is: If Eq. (24) is pth moment stable, what amount of stochastic perturbation can be tolerated by this equation without losing the property of stability, that is, under what assumptions would Eq. (23) remain stable? The following assertions refer to such a problem. To prove them, we need the elementary inequalities: For $p \geq 2$ and $a, b \geq 0$,

$$a^{p-1}b \leq \frac{p-1}{p}\, a^p + \frac{1}{p}\, b^p, \qquad a^{p-2}b^2 \leq \frac{p-2}{p}\, a^p + \frac{2}{p}\, b^p. \tag{25}$$

Corollary 1. *Let the decay function $\lambda(t)$ satisfy the conditions in Theorem 1 and let (H_2) hold for the functions $\theta_1(t)$ and $\theta_2(t)$. Let also there exist constants $\mu_1, \mu_2 > 0$ and $l_0, l_i, \bar{l}_i, \rho_i, \bar{\rho}_i \geq 0$, $i = 1, \ldots, n$, such that*

$$x^T f(x,t) \leq -l_0|x|^2, \tag{26}$$

$$|h(y_1, y_2, \ldots, y_n, t)| \leq \sum_{i=1}^{n} l_i \lambda^{-\rho_i}(t)|y_i| + \theta_1(t)\, \lambda^{-\mu_1}(t), \tag{27}$$

$$||G(x, y_1, \ldots, y_n, t)||^2 \leq \sum_{i=1}^{n} \bar{l}_i \lambda^{-\bar{\rho}_i}(t)|y_i|^2 + \theta_2(t)\, \lambda^{-\mu_2}(t) \tag{28}$$

for all $x, y_1, \ldots, y_n \in R^d$ and $t \geq 0$, and let

$$l_0 > \frac{p-1}{2} + \sum_{i=1}^{n} \left[l_i + \frac{p-1}{2}\, \bar{l}_i \right]. \tag{29}$$

Then, Eq. (23) is pth moment stable with decay $\lambda(t)$ of order γ, where

$$\gamma = \mu_1 p \wedge \frac{\mu_2 p}{2} \wedge \rho_1 \wedge \bar{\rho}_1 \wedge \ldots \wedge \rho_n \wedge \bar{\rho}_n \wedge \left\{ p \left(l_0 - \sum_{i=1}^{n} \left[l_i + \frac{p-1}{2}\, \bar{l}_i \right] - \frac{p-1}{2} \right) \right\}. \tag{30}$$

Proof. For $V(x,t) \equiv |x|^p$, we find from (20) and (26), (27), (28) that

$$\begin{aligned}
& L\tilde{V}(x, y_1, y_2, \ldots, y_n, t) \\
&\quad = p|x|^{p-2} \left[x^T f(x,t) + x^T h(y_1, \ldots, y_n, t) \right] \\
&\qquad + \frac{p(p-1)}{2}\, |x|^{p-2} ||G(x, y_1, \ldots, y_n, t)||^2 \\
&\quad \leq -p\, l_0 |x|^p + p|x|^{p-1} \left[\sum_{i=1}^{n} l_i \lambda^{-\rho_i}(t)|y_i| + \theta_1(t)\, \lambda^{-\mu_1}(t) \right] \\
&\qquad + \frac{p(p-1)}{2}\, |x|^{p-2} \left[\sum_{i=1}^{n} \bar{l}_i \lambda^{-\bar{\rho}_i}(t)|y_i|^2 + \theta_2(t)\, \lambda^{-\mu_2}(t) \right].
\end{aligned}$$

By applying the inequalities (25), we compute finally that

$$
\begin{aligned}
&L\tilde{V}(x, y_1, y_2, \dots, y_n, t)\\
&\quad \le -\left(p\, l_0 - (p-1)\sum_{i=1}^{n}\left[l_i\lambda^{-\rho_i}(t) + \frac{p-2}{2}\,\bar{l}_i\lambda^{-\bar{\rho}_i}(t)\right] - \frac{p(p-1)}{2}\right)|x|^p\\
&\quad +\sum_{i=1}^{n}\left[l_i\lambda^{-\rho_i}(t) + (p-1)\,\bar{l}_i\lambda^{-\bar{\rho}_i}(t)\right]|y_i|^p\\
&\quad +\theta_1^p(t)\,\lambda^{-\mu_1 p}(t) + (p-1)\theta_2^{\frac{p}{2}}(t)\,\lambda^{-\frac{\mu_2 p}{2}}(t).
\end{aligned}
$$

If we take $\nu_i = \rho_i \wedge \bar{\rho}_i$, $\mu = \mu_1 p \wedge \frac{\mu_2 p}{2}$ and $\theta(t) = p\big(\theta_1^p(t) \vee \theta_2^{\frac{p}{2}}(t)\big)$, we derive that (21) holds, or equivalently, (6) holds with

$$
\begin{aligned}
\lambda_0 &= p\, l_0 - (p-1)\sum_{i=1}^{n}\left[l_i + \frac{p-2}{2}\,\bar{l}_i\right] - \frac{p(p-1)}{2}, \qquad (31)\\
\lambda_i &= l_i + (p-1)\bar{l}_i, \;\; i = 1, \dots, n.
\end{aligned}
$$

Thus, all the conditions of Theorem 1 are valid and, therefore, what remains is to apply it to obtain the desired results. ♢

Corollary 2. *Let the decay function $\lambda(t)$ satisfy the conditions in Theorem 2 and let (H_2) hold for the functions $\theta_1(t)$ and $\theta_2(t)$. Let also there exist constants $\mu_1, \mu_2 > 0$ and $l_0, l_i, \bar{l}_i, \rho_i, \bar{\rho}_i \ge 0$, $i = 1, \dots, n$, such that conditions (26), (27), (28) hold and let*

$$
p\, l_0 > \sum_{i=1}^{n}\left[l_i + (p-1)\bar{l}_i\right]\lambda^{-\nu_i}(\tau) + (p-1)\sum_{i=1}^{n}\left[l_i + \frac{p-2}{2}\,\bar{l}_i\right] + \frac{p(p-1)}{2}, \qquad (32)
$$

where $\nu_i = \rho_i \wedge \bar{\rho}_i$. Then, Eq. (23) is pth moment stable with decay $\lambda(t)$ of order γ, where $\gamma = \mu_1 p \wedge \frac{\mu_2 p}{2} \wedge \alpha^*$ and α^* is the unique root of the equation $\alpha + \lambda^{\alpha}(\tau)\sum_{i=1}^{n}\lambda_i\lambda^{-\nu_i}(\tau) - \lambda_0 = 0$, and λ_0, λ_i are given with (31).

Proof. The proof follows straightforwardly by applying Theorem 2, where condition (32) is obtained from the condition $\sum_{i=1}^{n}\lambda_i\lambda^{-\nu_i}(\tau) < \lambda_0$. ♢

Note that consequences analogous to Theorem 3 and Theorem 4 could be also stated for Eq. (23).

Let us now give some examples to illustrate the above features explicitly.

Example 1 Let us investigate the pth moment stability, $p \ge 2$, of the following nonlinear one-dimensional SDDE with time-varying delays and dependent on parameters $a > 0$ and $q > 0$,

$$
\begin{aligned}
dx(t) &= \left[-a|x(t)| + \frac{t\cdot \sin x(\rho_i(t))}{(t+1)^3(\ln(t+1)+1)^{\frac{1}{2}} + |x(t)|}\right]dt \qquad (33)\\
&\quad + \left[\frac{1 - e^{-|x(\rho_2(t))|}}{(t+1)^2 + x^2(t)} + \frac{\cos t\cdot\sqrt{|x(\rho_1(t))|}}{(\ln(t+1)+1)^q + 1}\right]dw(t),\;\; t \ge 0,
\end{aligned}
$$

with an initial condition $x_0 = \xi \in L^p(\Omega, \mathcal{F}_0, C([-\tau, 0], R))$. It is supposed here that $\rho_1(t)$ and $\rho_2(t)$ satisfy conditions (1) and that $\tau = \max\{-\rho_1(0), -\rho_2(0)\}$.

It is easy to deduce that the Lipschitz and growth conditions hold for the coefficients of this equation,

$$F(x, y_1, y_2, t) = -a|x| + \frac{t \sin y_1}{(t+1)^3(\ln(t+1)+1)^{\frac{1}{2}} + |x|},$$
$$G(x, y_1, y_2, t) = \frac{1 - e^{-|y_2|}}{(t+1)^2 + x^2} + \frac{\cos t \sqrt{|y_1|}}{(\ln(t+1)+1)^q + 1},$$

so that there exists a unique solution $x(t;\xi), t \in [-\tau, \infty)$ of Eq. (33) satisfying $E \sup_{t\in[-\tau,\infty)} |x(t;\xi)|^p < \infty$. The trivial solution also exists since $F(0,0,0,t) \equiv 0$, $G(0,0,0,t) \equiv 0$.

Let us take $V(x,t) = |x|^p$. Then, (15) holds with $c_1 = c_2 = 1$. In view of (20), we have

$$L\tilde{V}(x, y_1, y_2, t) \le -ap\,|x|^p + \frac{p|x|^{p-1}|y_1|}{(t+1)^2(\ln(t+1)+1)^{\frac{1}{2}}} + p(p-1)\left[\frac{|x|^{p-2}|y_2|^2}{(t+1)^4} + \frac{|x|^{p-2}\cos^2 t\,|y_1|}{(\ln(t+1)+1)^{2q}}\right],$$

where we used $1 - e^{-x} \le x$ and $(a+b)^2 \le 2a^2 + 2b^2$. Having in mind conditions (6) and (15), it is logical to take $\lambda(t) = \ln(t+1)+1$. Then, $\lambda'(t) = \frac{1}{t+1}$, $0 < \lambda'(t) \le \lambda(t)$, $\lambda''(t) < 0$ and, moreover, $\lambda(t+s) \le \lambda(t) \cdot \lambda(s)$ for all $t, s \ge 0$. Thus, the decay rate $\lambda(t)$ satisfies the conditions from all the assertions in Section 2.

In order to apply Theorems 1–4 to prove the pth moment stability of Eq. (33) with respect to the decay rate $\lambda(t) = \ln(t+1)+1$, we first need to verify that conditions (21) and (22) hold. By applying the inequalities in (25), we see that

$$\begin{aligned} &L\tilde{V}(x, y_1, y_2, t) \\ &\quad\le -[ap - (p-1)(2p-3)]\,|x|^p + \lambda^{-\frac{1}{2}}(t)\,\lambda'^2(t)\,|y_1|^p \\ &\qquad +2(p-1)\lambda'^4(t)\,|y_2|^p(p-1)\lambda^{-2q}(t)\,|y_1|^p + (p-1)\cos^{2p} t\,\lambda^{-2q}(t), \end{aligned} \tag{34}$$

and since $\lambda'(t) \le \lambda(t)$, then

$$\begin{aligned} &L\tilde{V}(x, y_1, y_2, t) \\ &\quad\le -[ap - (p-1)(2p-3)]\,|x|^p + p\,\lambda^{-\left(\frac{5}{2}\wedge 2q\right)}(t)\,|y_1|^p \\ &\qquad +2(p-1)\lambda^{-4}(t)\,|y_2|^p + (p-1)\cos^{2p} t\,\lambda^{-2q}(t). \end{aligned}$$

Therefore, condition (21) holds with $\theta(t) = (p-1)\cos^{2p} t$ and $\mu = 2q$, $\lambda_0 = ap - (p-1)(2p-3)$, $\lambda_1 = p$, $\lambda_2 = 2(p-1)$, $\nu_1 = \frac{5}{2} \wedge 2q$, $\nu_2 = 4$.

The application of Theorem 1, precisely, condition $\frac{c_2}{c_1}(\lambda_1 + \lambda_2) < \lambda_0$, implies that it must be

$$a > \frac{(p+1)(2p-1)}{p}. \tag{35}$$

Hence, Eq. (33) is pth moment stable with decay $\lambda(t)$ in the sense that

$$\limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{\ln[\ln(t+1)+1]} = \limsup_{t\to\infty} \frac{\ln E|x(t;\xi)|^p}{\ln \ln t} \le -\gamma,$$

where, in view of (7),

$$\gamma = 2q \wedge 5/2 \wedge \big[ap - (p+1)(2p-1)\big]/2. \tag{36}$$

To apply Theorem 2, we must specify $\rho_1(t)$ and $\rho_2(t)$. Let $\rho_1(t) = t - e^{-t}$ and $\rho_2(t) = t - \frac{e-1}{t+1}$. Then, $\tau = e-1$ and $\lambda(\tau) = 2$. Hence, $\frac{c_2}{c_1}\big[\lambda_1 \lambda^{-\nu_1}(\tau)\lambda'^{\eta_1}(0) + \lambda_2 \lambda^{-\nu_2}(\tau)\lambda'^{\eta_2}(0)\big] < \lambda_0$ yields

$$a > 2^{-\left(\frac{5}{2}\wedge 2q\right)} + \frac{(p-1)(2p-\frac{23}{8})}{p}.$$

In view of (10), Eq. (33) is pth moment stable with $\gamma = 2q \wedge \alpha^*$, where α^* is the unique root of the equation (11), that is, α^* is such that

$$\alpha^* + 2^{\alpha^*}\left[p\, 2^{-\left(\frac{5}{2}\wedge 2q\right)} + \frac{p-1}{8}\right] - ap + (p-1)(2p-3)] = 0.$$

Theorem 3 and Theorem 4 could be applied analogously since (22) follows from (34),

$$\begin{aligned} &L\tilde{V}(x, y_1, y_2, t) \\ &\quad \le -[ap - (p-1)(2p-3)]\,|x|^p + p\,\lambda^{-\left(\frac{5}{2}\wedge 2q\right)}(t)\,|y_1|^p \\ &\qquad +2(p-1)\lambda^{-1}(t)\,\lambda'^3(t)\,|y_2|^p + (p-1)\cos^{2p} t\,\lambda^{-2q}(t), \end{aligned}$$

where $\nu_2 = 1$, $\eta_1 = 0$, $\eta_2 = 3$ and $\theta(t)$, $\mu, l_0, \lambda_1, \lambda_2, \nu_1$ are as above.

Since $\lambda'(0) = 1$, the application of Theorem 3 yields to the same conditions (35) and (36) as in Theorem 1. It is easy to deduce, however, that the application of Theorem 4 requires stronger conditions than Theorem 2. $\diamondsuit$

Example 2 Let us consider the pth moment stability, $p \ge 2$, of the following two-dimensional SDDE with time-varying delays and dependent on parameters $a_1, a_2, \alpha, \beta > 0$,

$$\begin{aligned} dx_1(t) &= \left[-a_1 x_1(t) + \frac{e^{-\alpha t}\sqrt{1+x_1^2(\rho_1(t))}}{(t+1)^3} \cdot \frac{x_2(\rho_2(t))}{1+x_2^2(\rho_2(t))}\right] dt \\ &\quad + \frac{\ln(1+|x_2(\rho_1(t))|)}{t+1}\, dw_1(t)) \\ dx_2(t) &= \left[-a_2 x_2(t) - \frac{x_1(\rho_2(t))}{(t+1)^2}\right] dt \\ &\quad + \frac{e^{-\beta t} x_1^{\frac{1}{3}}(\rho_2(t))}{(t+1)^2 + |x_1(t)|}\, dw_1(t) - \frac{x_2(\rho_2(t))}{(t+1)^3}\, dw_2(t) \end{aligned} \tag{37}$$

with an initial condition $x_0 = \xi \in L^p(\Omega, \mathcal{F}_0, C([-\tau, 0], R))$. It is easy to deduce the existence of a unique solution $x(t;\xi)$, as well as of the trivial solution.

If we take $x(t) = (x_1(t), x_2(t))^T$, $x(\rho_i(t)) = (x_1(\rho_i(t)), x_2(\rho_i(t)))^T$, $i = 1, 2$, and denote that $x = (x_1, x_2)^T$, $y_i = (y_{1i}, y_{2i})^T$, respectively, we see that

$$F(x, y_1, y_2, t) = \begin{bmatrix} -a_1 x_1 \\ -a_2 x_2 \end{bmatrix} + \begin{bmatrix} \frac{e^{-\alpha t}\sqrt{1+y_{11}^2}}{(t+1)^3} \cdot \frac{y_{22}}{1+y_{22}^2} \\ -\frac{y_{12}}{(t+1)^2} \end{bmatrix} \equiv f(x,t) + h(y_1, y_2, t)$$

$$G(x, y_1, y_2, t) = \begin{bmatrix} \frac{\ln(1+|y_{21}|)}{t+1} & 0 \\ \frac{e^{-\beta t} y_{12}^{\frac{1}{3}}}{(t+1)^2+|x_1|} & -\frac{y_{22}}{(t+1)^3} \end{bmatrix}.$$

Obviously, Eq. (37) could be understood as a stochastic perturbation of the deterministic asymptotically stable system

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -a_1 & 0 \\ 0 & -a_2 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

To find conditions under which Eq. (37) remains stable in the sense of pth moment, we will apply Corollary 1 and choose $\lambda(t) = t + 1$. Then,

$$\begin{aligned} x^T f(x,t) &\le -(a_1 \wedge a_2)|x|^2, \\ |h(y_1, y_2, t)|^2 &\le \frac{1}{4}\, e^{-2\alpha t}(1 + y_{11}^2)\, \lambda^{-6}(t) + y_{12}^2\, \lambda^{-4}(t) \\ &\le \frac{1}{4}\, |y_1|^2 \lambda^{-6}(t) + |y_2|^2\, \lambda^{-4}(t) + \frac{1}{4}\, e^{-2\alpha t} \lambda^{-6}(t). \end{aligned}$$

By using the elementary inequality $(a + b + c)^r \le (3^r \wedge 1)(|a|^r + |b|^r + |c|^r)$, $r \ge 0, a, b, c \in R$, we find that

$$|h(y_1, y_2, t)| \le \frac{1}{2}\, |y_1|\, \lambda^{-3}(t) + |y_2|\, \lambda^{-2}(t) + \frac{1}{2}\, e^{-\alpha t} \lambda^{-3}(t).$$

Similarly,

$$\begin{aligned} ||G(x, y_1, y_2, t)||^2 &\le y_{21}^2 \lambda^{-2}(t) + e^{-2\beta t} y_{12}^{\frac{2}{3}} \lambda^{-4}(t) + y_{22}^2 \lambda^{-6}(t) \\ &\le |y_1|^2 \lambda^{-2}(t) + |y_2|^2 \lambda^{-4}(t) + e^{-2\beta t} \lambda^{-4}(t). \end{aligned}$$

It is now easy to apply condition (29) to conclude that Eq. (37) is pth moment stable with polynomial decay if

$$a_1 \wedge a_2 > \frac{3}{2}\, p.$$

Then,

$$\limsup_{t \to \infty} \frac{\ln E|x(t;\xi)|^p}{\ln(t+1)} \le -\gamma,$$

where, from (30), $\gamma = 2 \wedge p\alpha \wedge p\beta \wedge p\big(a_1 \wedge a_2 - \frac{3}{2}p\big)$. $\diamondsuit$

Let us close the paper with the following comments:

Note that all the previous assertions remain to be valid for SDDE with constant delays, that is, if $\rho_i(t) \equiv t - \tau_i$, $\tau_i = \text{const} > 0$, $\tau = \max\{-\tau_1, \dots, -\tau_n\}$.

Moreover, all the assertions could be appropriately stated for stochastic differential equations (without delay) by taking $\rho_i(t) \equiv t$ and, to the authors' best knowledge, these assertions would represent new criteria to verify the pth moment stability with decay for these equations.

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