Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:1** (2010), 47–61 DOI: 10.2298/FIL1001047O

GLOBAL APPROXIMATION PROPERTIES OF MODIFIED SMK OPERATORS

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Abstract

In this paper, introducing a general modification of the classical Szász-Mirakjan-Kantorovich (SMK) operators, we study their global approximation behavior. Some special cases are also presented.

1 Introduction

As usual, for $n \in \mathbb{N}$ and $x \in [0, \infty)$, the classical Szász-Mirakjan (SM) operators and the Szász-Mirakjan-Kantorovich (SMK) operators are defined respectively by

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

and

$$K_n(f;x) := ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int\limits_{I_{n,k}} f(t)dt \tag{1.1}$$

where $I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and f belongs to an appropriate subspace of $C[0, \infty)$ for which the above series is convergent. Assume now that (u_n) is a sequence of functions such that, for a fixed $a \geq 0$,

$$0 \leq u_n(x) \leq x \text{ for every } x \in [a, \infty) \text{ and } n \in \mathbb{N}. \tag{1.2}$$

In recent years, in order to get more powerful approximations some modifications of the above operators have been introduced as follows (see [7, 9]):

$$D_n(f;x) := e^{-nu_n(x)} \sum_{k=0}^{\infty} \frac{(nu_n(x))^k}{k!} f\left(\frac{k}{n}\right)$$

2010 *Mathematics Subject Classifications.* 41A25, 41A36.
Key words and Phrases. Szász-Mirakjan operators, Szász-Mirakjan-Kantorovich operators, weighted space, Lipschitz classes, global approximation.
Received: November 1, 2009
Communicated by Gradimir Milovanović

and

$$L_n(f;x) := ne^{-nu_n(x)} \sum_{k=0}^{\infty} \frac{(nu_n(x))^k}{k!} \int_{I_{n,k}} f(t)dt. \tag{1.3}$$

In particular, considering a non-trivial sequence (u_n) defined by

$$u_n(x) := u_n^{[1]}(x) = \frac{-1+\sqrt{4n^2x^2+1}}{2n}, \ x \in [0,\infty), \ n \in \mathbb{N},$$

the authors have shown in [7] that the operators D_n have a better error estimation on the interval $[0,\infty)$ than the operators S_n while in [9] it was shown by considering

$$u_n(x) := u_n^{[2]}(x) = x - \frac{1}{2n}, \ x \in [\frac{1}{2},\infty), \ n \in \mathbb{N}$$

that the operators L_n enable better error estimation on $[1/2,\infty)$ than the operators K_n. Some applications of this idea to other well-known positive linear operators may be found in the papers [1, 2, 6, 8, 11, 12, 13, 14, 15].

Let $p \in \mathbb{N}_0 := \{0,1,...\}$ and define the weight function μ_p as follows:

$$\mu_0(x) := 1 \text{ and } \mu_p(x) := \frac{1}{1+x^p} \text{ for } x \geq 0 \text{ and } p \in \mathbb{N}. \tag{1.4}$$

Then, we consider the following (weighted) subspace $C_p[0,\infty)$ of $C[0,\infty)$ generated by μ_p :

$$C_p[0,\infty) := \{f \in C[0,\infty) : \mu_p f \text{ is uniformly continuous and bounded on } [0,\infty)\}$$

endowed with the norm

$$\|f\|_p := \sup_{x\in[0,\infty)} \mu_p(x)\,|f(x)| \quad \text{for} \quad f \in C_p[0,\infty).$$

If A is a subinterval of $[0,\infty)$, then by $\|f\|_{p|A}$ we denote the restricted norm to A, i.e.,

$$\|f\|_{p|A} := \sup_{x\in A} \mu_p(x)\,|f(x)|\,.$$

We also consider the following Lipschitz classes:

$$\begin{aligned}
\Delta_h^2 f(x) &: = f(x+2h) - 2f(x+h) + f(x), \\
\omega_p^2(f,\delta) &: = \sup_{h\in(0,\delta]} \left\|\Delta_h^2 f\right\|_p, \\
\omega_p^1(f,\delta) &: = \sup\{\mu_p(x)\,|f(t)-f(x)| : |t-x| \leq \delta \text{ and } t,x \geq 0\} \\
Lip_p^2\alpha &: = \left\{f \in C_p[0,\infty) : \omega_p^2(f;\delta) = O(\delta^\alpha) \text{ as } \delta \to 0^+\right\},
\end{aligned}$$

where $h > 0$ and $0 < \alpha \leq 2$.

In the present paper we study the global approximation behavior of the operators L_n given by (1.3). More precisely, we prove the following result.

Theorem 1.1. *Let (u_n) be a sequence of functions satisfying (1.2) for a fixed $a \geq 0$. Assume that $u_n'(x)$ exists and $u_n' \neq 0$ on $[a,\infty)$. Then, for every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in [a,\infty)$, there exists an absolute constant $M_p > 0$ such that*

$$\mu_p(x)\left|L_n(f;x) - f(x)\right| \leq M_p\omega_p^2\left(f, \sqrt{(u_n(x)-x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}}\right) + \omega_p^1(f; x - u_n(x) + \frac{1}{2n}),$$

where μ_p is the same as in (1.4). Particularly, if $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$, then

$$\mu_p(x)\left|L_n(f;x) - f(x)\right| \leq M_p\left((u_n(x)-x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}\right)^{\frac{\alpha}{2}} + \omega_p^1(f; x - u_n(x) + \frac{1}{2n})$$

holds.

Remark. If the sequence (u_n) in Theorem 1.1 also satisfies

$$\lim_{n\to\infty} u_n(x) = x \quad \text{for every } x \in [a,\infty), \tag{1.5}$$

then we can get that

$$\lim_{n\to\infty} \mu_p(x)\left|L_n(f;x) - f(x)\right| = 0 \quad \text{for every } x \in [0,\infty)$$

holds true provided that $f \in C_p[0,\infty)$ or $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$. Furthermore, we will see that our operators L_n map $C_p[0,\infty)$ into $C_p[a,\infty)$ (see Lemma 2.5). Hence, if the convergence in (1.5) is uniform on $[a,\infty)$, then we have

$$\lim_{n\to\infty} \|L_n f - f\|_{p|[a,\infty)} = 0.$$

2 Auxiliary Results

In this section, we will get some lemmas which are quite effective in proving our Theorem 1.1. Throughout the paper we use the following test functions

$$e_i(y) = y^i, \ i = 0,1,2,3,4,$$

and the moment function

$$\psi_x(y) = y - x.$$

Now, by the definition (1.3), we first get the following two results.

Lemma 2.1. *For the operators L_n, we have*

(*i*) $L_n(e_0;x) = 1$,

(ii) $L_n(e_1;x) = u_n(x) + \frac{1}{2n}$,

(iii) $L_n(e_2;x) = u_n^2(x) + \frac{2u_n(x)}{n} + \frac{1}{3n^2}$,

(iv) $L_n(e_3;x) = u_n^3(x) + \frac{9u_n^2(x)}{2n} + \frac{7u_n(x)}{2n^2} + \frac{1}{4n^3}$.

We should note that the proof of Lemma 2.1 can be obtained from the papers [9] and [10].

Lemma 2.2. *For the operators L_n, we have*

(i) $L_n(\psi_x;x) = u_n(x) - x + \frac{1}{2n}$,

(ii) $L_n(\psi_x^2;x) = (u_n(x) - x)^2 + \frac{2u_n(x) - x}{n} + \frac{1}{3n^2}$,

(iii) $$L_n(\psi_x^3;x) = (u_n(x) - x)^3 + \frac{3\,(3u_n(x) - x)\,(u_n(x) - x)}{2n} + \frac{7u_n(x) - 2x}{2n^2} + \frac{1}{4n^3}.$$

Now we get the next result.

Lemma 2.3. *Let (u_n) be a sequence of functions satisfying* (1.2) *for a fixed $a \geq 0$. For the operators L_n, we have*

$$\begin{aligned} L_n(e_m;x) &= \sum_{j=0}^{m} c_{m,j} u_n^j(x) n^{j-m} \\ : &= u_n^m(x) + \frac{m^2}{2n} u_n^{m-1}(x) + ... + \frac{2\,(2^m - 1)}{(m+1)n^{m-1}} u_n(x) + \frac{n^{-m}}{m+1}, \end{aligned}$$

where $c_{m,j}$'s are positive coefficients.

Proof. Since

$$\begin{aligned} \int\limits_{I_{n,k}} t^m dt &= \frac{1}{(m+1)\,n^{m+1}} \left\{ (k+1)^{m+1} - k^{m+1} \right\} \\ &= \frac{1}{(m+1)\,n^{m+1}} \sum_{j=0}^{m} \binom{m+1}{j} k^j, \end{aligned}$$

we get

$$\begin{aligned} L_n(e_m;x) &= \sum_{k=0}^{\infty} p_{k,n}(x) \int\limits_{I_{n,k}} t^m dt \\ &= \frac{1}{(m+1)} \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{n^{m-j}} \left\{ e^{-nu_n(x)} \sum_{k=0}^{\infty} \left(\frac{k}{n}\right)^j \frac{(nu_n(x))^k}{k!} \right\} \\ &= \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{n^{m-j}} D_n(e_j;x) \\ &= \frac{n^{-m}}{m+1} + \frac{n^{-m}}{m+1} \sum_{j=1}^{m} \binom{m+1}{j} n^j D_n(e_j;x). \end{aligned}$$

Also, by Lemma 2.4 of [6], we know that

$$D_n(e_j;x) = \sum_{i=1}^{j} b_{j,i} u_n^i(x) n^{i-j} := u_n^j(x) + \frac{j(j-1)}{2n} u_n^{j-1}(x) + ... + n^{1-j} u_n(x),$$

where $b_{j,i}$'s are positive coefficients. Then, we may write that

$$\begin{aligned} L_n(e_m;x) &= \frac{n^{-m}}{m+1} + \frac{n^{-m}}{m+1} \sum_{j=1}^{m} \binom{m+1}{j} n^j \sum_{i=1}^{j} b_{j,i} u_n^i(x) n^{i-j} \\ &= \frac{n^{-m}}{m+1} + \frac{n^{-m}}{m+1} \sum_{i=1}^{m} \left(\sum_{j=i}^{m} \binom{m+1}{j} b_{j,i} \right) n^i u_n^i(x) \\ &= \frac{n^{-m}}{m+1} + u_n^m(x) + \frac{m^2}{2n} u_n^{m-1}(x) + ... + \frac{n^{1-m}}{m+1} \left(\sum_{j=1}^{m} \binom{m+1}{j} \right) u_n(x) \\ &= u_n^m(x) + \frac{m^2}{2n} u_n^{m-1}(x) + ... + \frac{2\left(2^m-1\right)}{(m+1)n^{m-1}} u_n(x) + \frac{1}{(m+1)\, n^m}. \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} L_n(e_m;x) &= \sum_{j=0}^{m} c_{m,j} u_n^j(x) \\ &: = u_n^m(x) + \frac{m^2}{2n} u_n^{m-1}(x) + ... + \frac{2\left(2^m-1\right)}{(m+1)n^{m-1}} u_n(x) + \frac{1}{(m+1)\, n^m}, \end{aligned}$$

which completes the proof. □

We now give some useful estimations for the operators L_n.

Lemma 2.4. *Let (u_n) be a sequence of functions satisfying (1.2) for a fixed $a \geq 0$. Then, for the operators L_n, we have* (*i*) $L_n(\psi_x^2;x) \le (u_n(x)-x)^2 + \dfrac{u_n(x)}{n} + \dfrac{1}{3n^2}$,

(*ii*) $L_n(\psi_x^3;x) \le \dfrac{5}{2}\left\{(u_n(x)-x)^2 + \dfrac{u_n(x)}{n} + \dfrac{1}{3n^2}\right\}$.

Proof. (*i*) By (1.2) and Lemma (*ii*), we easily get

$$\begin{aligned} L_n(\psi_x^2;x) &= (u_n(x)-x)^2 + \frac{u_n(x)}{n} + \frac{u_n(x)-x}{n} + \frac{1}{3n^2} \\ &\le (u_n(x)-x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}. \end{aligned}$$

(*ii*) Since $0 \le u_n(x) \le x$ for every $x \in [a,\infty)$ and $n \in \mathbb{N}$, it follows from Lemma 2.2 (*iii*) that

$$\begin{aligned} L_n(\psi_x^3;x) &= (u_n(x)-x)^3 + \frac{6u_n(x)(u_n(x)-x)}{2n} + \frac{3(u_n(x)-x)^2}{2n} \\ &\quad + \frac{5u_n(x)}{2n^2} + \frac{u_n(x)-x}{n^2} + \frac{1}{4n^3} \\ &\le \frac{3(u_n(x)-x)^2}{2n} + \frac{5u_n(x)}{2n^2} + \frac{1}{4n^3} \\ &\le \frac{5(u_n(x)-x)^2}{2} + \frac{5u_n(x)}{2n} + \frac{5}{6n^2}, \end{aligned}$$

whence the result. □

Lemma 2.5. *Let* (u_n) *be a sequence of functions satisfying* (1.2) *for a fixed* $a \ge 0$. *Assume that* $u_n'(x)$ *exists and* $u_n'(x) \ne 0$ *on* $[a,\infty)$. *Then, for the operators* L_n, *there exists a constant* $M_p \ge 0$ *such that*

$$\mu_p(x) L_n\left(\frac{1}{\mu_p};x\right) \le M_p. \tag{2.1}$$

Furthermore, for all $f \in C_p[0,\infty)$, *we have*

$$\|L_n(f)\|_{p|[a,\infty)} \le M_p \|f\|_p, \tag{2.2}$$

which guarantees that L_n *maps* $C_p[0,\infty)$ *into* $C_p[a,\infty)$.

Proof. For $p=0$, (2.1) follows immediately. Assume now that $p \ge 1$. By (1.2), (1.3) and (1.4), we get

$$\begin{aligned} &\mu_p(x) L_n\left(\frac{1}{\mu_p};x\right) \\ =\; &\mu_p(x)\left\{L_n(e_0;x) + L_n(e_p;x)\right\} \\ =\; &\mu_p(x)\left\{1 + u_n^p(x) + \frac{p^2}{2n}u_n^{p-1}(x) + \dots + \frac{2\,(2^p-1)}{(p+1)n^{p-1}}u_n(x) + \frac{n^{-p}}{m+1}\right\} \\ \le\; &\mu_p(x)\left\{x^p + \frac{p^2}{2n}x^{p-1} + \dots + \frac{2\,(2^p-1)}{(p+1)n^{p-1}}x + \left(1 + \frac{n^{-p}}{m+1}\right)\right\}. \end{aligned}$$

Now, since $p \geq 1$, we can find a constant C_p depending on p such that the inequalities

$$\begin{aligned} \frac{x^p}{1+x^p} &\leq C_p, \ \frac{p^2 x^{p-1}}{2n(1+x^p)} \leq C_p, ..., \\ \frac{2\left(2^p-1\right)}{(p+1)n^{p-1}} \frac{x}{1+x^p} &\leq C_p, \ \left(1+\frac{n^{-p}}{m+1}\right) \frac{1}{1+x^p} \leq C_p \end{aligned}$$

hold for every $x \in [a, \infty)$ and $n \in \mathbb{N}$. So, letting $M_p := (p+1)C_p$, we may write that

$$\mu_p(x) L_n\left(\frac{1}{\mu_p}; x\right) \leq M_p,$$

which gives (2.1). On the other hand, for all $f \in C_p[0, \infty)$ and every $x \in [a, \infty)$, it follows that

$$\begin{aligned} \mu_p(x)\left|L_n(f;x)\right| &\leq \mu_p(x) \sum_{k=0}^{\infty} p_{k,n}(x) \int\limits_{I_{n,k}} |f(t)|\, dt \\ &= \mu_p(x) \sum_{k=0}^{\infty} p_{k,n}(x) \int\limits_{I_{n,k}} \mu_p(t)\, |f(t)| \frac{1}{\mu_p(t)} dt \\ &\leq \|f\|_p \, \mu_p(x) L_n\left(\frac{1}{\mu_p}; x\right) \\ &\leq M_p \|f\|_p . \end{aligned}$$

Now taking supremum over $x \in [a, \infty)$, the last inequality implies (2.2). □

Lemma 2.6. *Let (u_n) be a sequence of functions satisfying* (1.2) *for a fixed $a \geq 0$. Assume that $u_n'(x)$ exists and $u_n'(x) \neq 0$ on $[a, \infty)$. Then, for the operators L_n, there exists a constant $M_p \geq 0$ such that*

$$\mu_p(x) L_n\left(\frac{\psi_x^2}{\mu_p}; x\right) \leq M_p \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2} \right\}.$$

Proof. For $p = 0$ the result follows from Lemma 2.4 (i). Now let $p = 1$. Then, using Lemma 2.4 (i)-(ii) we can write that

$$\begin{aligned} \mu_1(x) L_n\left(\frac{\psi_x^2}{\mu_1}; x\right) &= \mu_1(x) \left\{ (1+x) L_n\left(\psi_x^2; x\right) + L_n(\psi_x^3; x) \right\} \\ &\leq (u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2} \\ &\quad + \frac{5}{2\,(1+x)} \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2} \right\} \\ &\leq \frac{7}{2} \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2} \right\}. \end{aligned}$$

Finally, assume that $p \geq 2$. Then, we get from Lemma 2.3 that

$$\begin{aligned}
& L_n\left(\frac{\psi_x^2}{\mu_p};x\right) \\
= \; & L_n(e_{p+2};x) - 2xL_n(e_{p+1};x) + x^2 L_n(e_p;x) \\
= \; & u_n^{p+2}(x) + \frac{(p+2)^2}{2n}u_n^{p+1}(x) + ... + \frac{2\left(2^{p+2}-1\right)}{(p+3)n^{p+1}}u_n(x) + \frac{1}{(p+3)\,n^{p+2}} \\
& -2x\left\{u_n^{p+1}(x) + \frac{(p+1)^2}{2n}u_n^{p}(x) + ... + \frac{2\left(2^{p+1}-1\right)}{(p+2)n^{p}}u_n(x) + \frac{1}{(p+2)\,n^{p+1}}\right\} \\
& +x^2\left\{u_n^{p}(x) + \frac{p^2}{2n}u_n^{p-1}(x) + ... + \frac{2\left(2^{p}-1\right)}{(p+1)n^{p-1}}u_n(x) + \frac{1}{(p+1)\,n^{p}}\right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
L_n\left(\frac{\psi_x^2}{\mu_p};x\right) = \; & (u_n(x)-x)^2 u_n^p(x) \\
& +\frac{u_n(x)}{n}\left\{\frac{(p+2)^2}{2}u_n^{p}(x) + ... + \frac{2\left(2^{p+2}-1\right)}{(p+3)n^{p}}\right\} \\
& -\frac{2xu_n(x)}{n}\left\{\frac{(p+1)^2}{2}u_n^{p-1}(x) + ... + \frac{2\left(2^{p+1}-1\right)}{(p+2)n^{p-1}}\right\} \\
& +\frac{x^2u_n(x)}{n}\left\{\frac{p^2}{2}u_n^{p-2}(x) + ... + \frac{2\left(2^{p}-1\right)}{(p+1)n^{p-2}}\right\} \\
& +\frac{1}{n^p}\left\{\frac{1}{(p+3)\,n^2} - \frac{2x}{(p+2)\,n} + \frac{x^2}{(p+1)}\right\}.
\end{aligned}$$

Therefore, since $0 \leq u_n(x) \leq x$ for every $x \in [a,\infty)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
& \mu_p(x)L_n\left(\frac{\psi_x^2}{\mu_p};x\right) \\
\leq \; & (u_n(x)-x)^2\left(\frac{x^p}{1+x^p}\right) \\
& +\frac{u_n(x)}{n}\left\{\frac{(p+2)^2}{2}\left(\frac{x^p}{1+x^p}\right) + ... + \frac{2\left(2^{p+2}-1\right)}{(p+3)n^{p}}\left(\frac{1}{1+x^p}\right)\right\} \\
& +\frac{u_n(x)}{n}\left\{\frac{p^2}{2}\left(\frac{x^p}{1+x^p}\right) + ... + \frac{2\left(2^{p}-1\right)}{(p+1)n^{p-2}}\left(\frac{x^2}{1+x^p}\right)\right\} \\
& +\frac{1}{3n^2}\left\{\frac{3}{(p+3)\,n^2}\left(\frac{1}{1+x^p}\right) + \frac{3}{(p+1)}\left(\frac{x^2}{1+x^p}\right)\right\}.
\end{aligned}$$

Thus, since $p \geq 2$, it is possible to find a constant M_p depending on p such that

$$\mu_p(x)L_n\left(\frac{\psi_x^2}{\mu_p};x\right) \leq M_p\left\{(u_n(x)-x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}\right\},$$

whence the result. □

Now, for $p \in \mathbb{N}$, consider the space

$$C_p^2[0,\infty) := \left\{ f \in C_p[0,\infty) : f'' \in C_p[0,\infty) \right\}.$$

Then we have the following result.

Lemma 2.7. *Let* (u_n) *be a sequence of functions satisfying* (1.2) *for a fixed* $a \geq 0$ *and let* $g \in C_p^2[0,\infty)$. *Assume that* $u_n'(x)$ *exists and* $u_n' \neq 0$ *on* $[a,\infty)$. *For the operators* L_n, *if* $\Omega_n(f;x) := L_n(f;x) - f\left(u_n(x) + \frac{1}{2n}\right) + f(x)$, *then there exists a positive constant* M_p *such that, for all* $x \in [a,\infty)$ *and* $n \in \mathbb{N}$, *we have*

$$\mu_p(x) \left| \Omega_n(g;x) - g(x) \right| \leq M_p \left\| g'' \right\|_p \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2} \right\}.$$

Proof. By, the Taylor formula, using $\psi_x(y) = y - x$, we may write that

$$g(y) - g(x) = \psi_x(y) g'(x) + \int_x^y \psi_t(y) g''(t) dt, \quad y \in [0,\infty).$$

Then, since $\Omega_n(\psi_x(y);x) = 0$, we get

$$\begin{aligned} \left| \Omega_n(g;x) - g(x) \right| &= \left| \Omega_n(g(y) - g(x);x) \right| \\ &= \left| \Omega_n\left(\int_x^y \psi_t(y) g''(t) dt; x \right) \right| \\ &= \left| L_n\left(\int_x^y \psi_t(y) g''(t) dt; x \right) \right. \\ &\quad \left. - \int_x^{u_n(x)+\frac{1}{2n}} \psi_t\left(u_n(x) + \frac{1}{2n}\right) g''(t) dt \right|. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \left| \Omega_n(g;x) - g(x) \right| &\leq L_n\left(\left| \int_x^y \psi_t(y) g''(t) dt \right|; x \right) \\ &\quad + \left| \int_x^{u_n(x)+\frac{1}{2n}} \psi_t\left(u_n(x) + \frac{1}{2n}\right) g''(t) dt \right|. \end{aligned}$$

Since

$$\left| \int_x^y \psi_t(y) g''(t) dt \right| \leq \frac{\|g''\|_p \, \psi_x^2(y)}{2} \left(\frac{1}{\mu_p(x)} + \frac{1}{\mu_p(y)} \right)$$

and

$$\left| \int_x^{u_n(x)+\frac{1}{2n}} \psi_t \left(u_n(x) + \frac{1}{2n} \right) g''(t) dt \right| \leq \frac{\|g''\|_p}{2\mu_p(x)} \left(u_n(x) - x + \frac{1}{2n} \right)^2,$$

it follows from Lemmas 2.4-2.6 that

$$\begin{aligned} \mu_p(x) \left| \Omega_n(g;x) - g(x) \right| &\leq \frac{\|g''\|_p}{2} \left\{ L_n(\psi_x^2; x) + \mu_p(x) L_n \left(\frac{\psi_x^2}{\mu_p}; x \right) \right\} \\ &\quad + \frac{\|g''\|_p}{2} \left(u_n(x) - x + \frac{1}{2n} \right)^2 \\ &\leq M_p \left\| g'' \right\|_p \left\{ (u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2} \right\}. \end{aligned}$$

Lemma is proved. □

3 Proof of Theorem 1.1

In this section we prove our main result Theorem 1.1.

We first consider the modified Steklov means (see [4, 5]) of a function $f \in C_p[0, \infty)$ as follows:

$$f_h(y) := \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \left\{ 2f(y+s+t) - f(y + 2(s+t)) \right\} dsdt,$$

where $h > 0$ and $y \geq 0$. In this case, it is clear that

$$f(y) - f_h(y) = \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} \Delta_{s+t}^2 f(y) dsdt,$$

which guarantees that

$$\|f - f_h\|_p \leq \omega_p^2(f; h). \tag{3.1}$$

Furthermore, we have

$$f_h''(y) = \frac{1}{h^2} \left(8\Delta_{h/2}^2 f(y) - \Delta_h^2 f(y) \right),$$

which implies

$$\|f_h''\|_p \leq \frac{9}{h^2} \omega_p^2(f; h). \tag{3.2}$$

Then, combining (3.1) with (3.2) we conclude that the Steklov means f_h corresponding to $f \in C_p[0,\infty)$ belongs to $C_p^2[0,\infty)$.

Now let $p \in \mathbb{N}_0$, $f \in C_p[0,\infty)$ and $x \in [a,\infty)$ be fixed. Assume that, for $h > 0$, f_h denotes the Steklov means of f. For any $n \in \mathbb{N}$, the following inequality holds:

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \Omega_n\left(|f(y) - f_h(y)|\,;x\right) + |f(x) - f_h(x)| \\ &\quad + |\Omega_n\left(f_h;x\right) - f_h(x)| + \left| f\left(u_n(x) + \frac{1}{2n}\right) - f(x)\right|. \end{aligned}$$

In the following, M_p denotes a positive constant depending on p which may assume different values in different formulas. Since $f_h \in C_p^2[0,\infty)$, it follows from Lemma 2.7 that

$$\begin{aligned} \mu_p(x)\,|L_n(f;x) - f(x)| &\leq \|f - f_h\|_p \left\{\mu_p(x)\Omega_n\left(\frac{1}{\mu_p};x\right) + 1\right\} \\ &\quad + M_p\,\|f_h''\|_p \left\{(u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}\right\} \\ &\quad + \mu_p(x)\left| f(u_n(x) + \frac{1}{2n}) - f(x)\right| \\ &\leq \|f - f_h\|_p \left\{\mu_p(x)L_n\left(\frac{1}{\mu_p};x\right) + 3\right\} \\ &\quad + M_p\,\|f_h''\|_p \left\{(u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}\right\} \\ &\quad + \mu_p(x)\left| f(u_n(x) + \frac{1}{2n}) - f(x)\right|. \end{aligned}$$

By (2.1), (3.1) and (3.2), the last inequality yields that

$$\begin{aligned} \mu_p(x)\,|L_n(f;x) - f(x)| &\leq M_p\omega_p^2(f;h)\left\{1 + \frac{1}{h^2}\left((u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}\right)\right\} \\ &\quad + \omega_p^1(f;x - u_n(x) + \frac{1}{2n}). \end{aligned}$$

Thus, choosing $h = \sqrt{(u_n(x) - x)^2 + \frac{u_n(x)}{n} + \frac{1}{3n^2}}$, the first part of the proof is completed. The remain part of the proof can be easily obtained from the definition of the space $Lip_p^2\alpha$.

4 Concluding Remarks

In this section, we give some special cases of Theorem 1.1 by choosing some appropriate function sequences (u_n).

For example, if we take $a = 0$ and

$$u_n(x) = x, \quad x \in [0,\infty),\ n \in \mathbb{N},$$

then our operators in (1.3) turn out to be the classical SMK operators K_n defined by (1.1). In this case, we get the following global approximation result for these operators.

Corollary 4.1. *For every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in [0,\infty)$, there exists an absolute constant $M_p > 0$ such that*

$$\mu_p(x) \left| K_n(f;x) - f(x) \right| \le M_p \omega_p^2 \left(f, \sqrt{\frac{x}{n} + \frac{1}{3n^2}} \right) + \omega_p^1 \left(f; \frac{1}{2n} \right).$$

Also, if $f \in Lip_p^2 \alpha$ for some $\alpha \in (0,2]$, then

$$\mu_p(x) \left| K_n(f;x) - f(x) \right| \le M_p \left(\frac{x}{n} + \frac{1}{3n^2} \right)^{\frac{\alpha}{2}} + \omega_p^1 \left(f; \frac{1}{2n} \right).$$

Now, if take $a = 0$ and

$$u_n(x) := u_n^{[1]}(x) = \frac{-1 + \sqrt{4n^2x^2 + 1}}{2n}, \ x \in [0,\infty), \ n \in \mathbb{N},$$

then our operators L_n in (1.3) turn out to be

$$L_n^{[1]}(f;x) := n e^{-\left(-1+\sqrt{4n^2x^2+1}\right)/2} \sum_{k=0}^{\infty} \frac{\left(-1 + \sqrt{4n^2x^2+1}\right)^k}{2^k k!} \int_{I_{n,k}} f(t)dt.$$

Then we have

Corollary 4.2. *For every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in [0,\infty)$, there exists an absolute constant $M_p > 0$ such that*

$$\begin{aligned} \mu_p(x) \left| L_n^{[1]}(f;x) - f(x) \right| &\le M_p \omega_p^2 \left(f, \sqrt{\left(u_n^{[1]}(x) - x \right)^2 + \frac{u_n^{[1]}(x)}{n} + \frac{1}{3n^2}} \right) \\ &\quad + \omega_p^1 \left(f; x - u_n^{[1]}(x) + \frac{1}{2n} \right). \end{aligned}$$

Also, if $f \in Lip_p^2 \alpha$ for some $\alpha \in (0,2]$, then

$$\begin{aligned} \mu_p(x) \left| L_n^{[1]}(f;x) - f(x) \right| &\le M_p \left(\left(u_n^{[1]}(x) - x \right)^2 + \frac{u_n^{[1]}(x)}{n} + \frac{1}{3n^2} \right)^{\frac{\alpha}{2}} \\ &\quad + \omega_p^1 \left(f; x - u_n^{[1]}(x) + \frac{1}{2n} \right). \end{aligned}$$

Furthermore, if we choose $a = \frac{1}{2}$ and

$$u_n(x) := u_n^{[2]}(x) = x - \frac{1}{2n}, \ x \in [\frac{1}{2}, \infty), \ n \in \mathbb{N},$$

then the operators in (1.3) reduce to the following operators (see [9]):

$$L_n^{[2]}(f;x) := ne^{\frac{1-2nx}{2}} \sum_{k=0}^{\infty} \frac{(2nx-1)^k}{2^k k!} \int\limits_{I_{n,k}} f(t)dt.$$

Then, we know from [9] that the operators $L_n^{[2]}$ preserve the test functions e_0 and e_1. Hence, we get the next result.

Corollary 4.3. *For every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in [\frac{1}{2},\infty)$, there exists an absolute constant $M_p > 0$ such that*

$$\mu_p(x)\left|L_n^{[2]}(f;x) - f(x)\right| \leq M_p\omega_p^2\left(f, \sqrt{\frac{x}{n} + \frac{1}{12n^2}}\right) + \omega_p^1\left(f; \frac{1}{n}\right).$$

Also, if $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$, then

$$\mu_p(x)\left|L_n^{[2]}(f;x) - f(x)\right| \leq M_p\left(\frac{x}{n} + \frac{1}{12n^2}\right)^{\frac{\alpha}{2}} + \omega_p^1\left(f; \frac{1}{n}\right).$$

Finally, taking $a = \frac{1}{\sqrt{3}}$ and

$$u_n(x) := u_n^{[3]}(x) := \frac{\sqrt{3n^2x^2+2} - \sqrt{3}}{n\sqrt{3}}, \quad x \in [\frac{1}{\sqrt{3}},\infty),\ n \in \mathbb{N}, \tag{4.1}$$

we get the following positive linear operators:

$$L_n^{[3]}(f;x) := ne^{\left(\sqrt{3}-\sqrt{3n^2x^2+2}\right)/\sqrt{3}} \sum_{k=0}^{\infty} \frac{\left(\sqrt{3n^2x^2+2} - \sqrt{3}\right)^k}{3^{k/2}k!} \int\limits_{I_{n,k}} f(t)dt. \tag{4.2}$$

In this case, observe that the operators $L_n^{[3]}$ preserve the test functions e_0 and e_2. Hence, we get the following result.

Corollary 4.4. *For every $p \in \mathbb{N}_0$, $n \in \mathbb{N}$, $f \in C_p[0,\infty)$ and $x \in [\frac{1}{\sqrt{3}},\infty)$, there exists an absolute constant $M_p > 0$ such that*

$$\begin{aligned}\mu_p(x)\left|L_n^{[3]}(f;x) - f(x)\right| &\leq M_p\omega_p^2\left(f, \sqrt{\left(u_n^{[3]}(x) - x\right)^2 + \frac{u_n^{[3]}(x)}{n} + \frac{1}{3n^2}}\right) \\ &\quad + \omega_p^1\left(f; x - u_n^{[3]}(x) + \frac{1}{2n}\right).\end{aligned}$$

Also, if $f \in Lip_p^2\alpha$ for some $\alpha \in (0,2]$, then

$$\begin{aligned}\mu_p(x)\left|L_n^{[3]}(f;x) - f(x)\right| &\leq M_p\left(\left(u_n^{[3]}(x) - x\right)^2 + \frac{u_n^{[3]}(x)}{n} + \frac{1}{3n^2}\right)^{\frac{\alpha}{2}} \\ &\quad + \omega_p^1\left(f; x - u_n^{[3]}(x) + \frac{1}{2n}\right).\end{aligned}$$

References

[1] O. Agratini, *Linear operators that preserve some test functions*, Int. J. Math. Math. Sci. Art. ID 94136 (2006), 11 pp.

[2] O. Agratini, *On the iterates of a class of summation-type linear positive operators*, Comput. Math. Appl. 55 (2008), 1178–1180.

[3] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its application*, Walter de Gruyter Studies in Math. 17, de Gruyter & Co., Berlin, 1994.

[4] M. Becker, *Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J. 27 (1978), 127–142.

[5] P.L. Butzer and K. Scherer, *Jackson and Bernstein-type inequalities for families of commutative operators in Banach spaces*, J. Approx. Theory 5 (1972), 308–342.

[6] O. Duman and M.A. Özarslan, *Global approximation results for modified Szász-Mirakjan operators*, Taiwanese J. Math.; (*accepted for publication*).

[7] O. Duman and M.A. Özarslan, *Szász-Mirakjan type operators providing a better error estimation*, Appl. Math. Lett. 20 (2007), 1184–1188.

[8] O. Duman, M.A. Özarslan and H. Aktuğlu, *Better error estimation for Szász-Mirakjan-Beta operators*, J. Comput. Anal. Appl. 10 (2008), 53–59.

[9] O. Duman, M.A. Özarslan and B. Della Vecchia, *Modified Szász-Mirakjan-Kantorovich operators preserving linear functions*, Turkish J. Math. 33 (2009), 151–158.

[10] V. Gupta, V. Vasishtha and M.K. Gupta, *Rate of convergence of the Szász-Kantorovitch-Bezier operators for bounded variation functions*, Publ. Inst. Math. (Beograd) (N.S.) 72 (2002), 137–143.

[11] J.P. King, *Positive linear operators which preserve* x^2, Acta. Math. Hungar. 99 (2003), 203–208.

[12] N.I. Mahmudov, *Korovkin-type theorems and applications*, Cent. Eur. J. Math. 7 (2009), 348–356.

[13] M.A. Özarslan and O. Duman, *Local approximation results for Szasz-Mirakjan type operators*, Arch. Math. (Basel) 90 (2008), 144–149.

[14] M.A. Özarslan and O. Duman, *MKZ type operators providing a better estimation on* $[1/2, 1)$, Canadian Math. Bull. 50 (2007), 434–439.

[15] L. Rempulska and K. Tomczak, *Approximation by certain linear operators preserving* x^2, Turkish J. Math. 33 (2009), 273–281.

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