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COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS SATISFYING GENERALIZED WEAK CONTRACTIVE CONDITION

Mujahid Abbas and Dragan Ðorić

Abstract

Contractive conditions introduced in [Q. Zhang and Y. Song, Fixed point theory for generalized φ-weak contraction, Appl. Math. Lett. 22(2009), 75-78] and [D. Ðorić, Common fixed point for generalized (ψ, φ)-weak contractions, Applied Mathematics Letters, 22(2009), 1896-1900] are employed to obtain a new common fixed point theorem for four maps. Our result substantially generalizes comparable results in the literature.

1 Introduction and preliminaries

Alber and Guerre-Delabriere [1] defined weakly contractive maps on a Hilbert space and established a fixed point theorem for such map. Afterwards, Rhoades [8] using the notion of weakly contractive maps, obtained a fixed point theorem in a complete metric space. Dutta and Choudhury [5] generalized the weak contractive condition and proved a fixed point theorem for a selfmap, which in turn generalizes Theorem 1 in [8] and the corresponding result in [1]. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. The area of common fixed point theory, involving four single valued maps, began with the assumption that all of the maps are commuted. Introducing weakly commuting maps, Sessa [9] generalized the concept of commuting maps. Then Jungck generalized this idea, first to compatible mappings [11] and then to weakly compatible mappings [12]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. On the other hand, Beg and Abbas [3] obtained a common fixed point theorem extending weak contractive condition for two maps. In this direction, Zhang and Song [13] introduced the concept of a generalized φ-weak contraction

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condition and obtained a common fixed point for two maps. Recently, Đorić [4] proved a common fixed point theorem for generalized (ψ, φ)-weak contractions.

The purpose of this paper is to obtain a common fixed point theorem for four maps that satisfy contractive condition which is more general than that given in [13]. Our result extend, unify and generalize the comparable results in [3], [4], [5] and [13].

Throughout this paper X is nonempty set, (X, d) is a metric space and $f : X \to X$ is selfmapping on X. A point $x \in X$ is called *fixed point of* f if $f(x) = x$. To state our result we need definition of weakly compatible pair of maps.

Definition 1.1. *Let f and g be selfmappings on X.*

(1) A point $u \in X$ is coincidence point of f and g if $fu = gu$,

(2) A pair of f and g is called weakly compatible pair if they commute at coincidence points.

We also use two classes of functions,

$\Phi = \{\varphi \mid \varphi : [0, \infty) \to [0, \infty)$ is lower semi continuous, $\varphi(t) > 0$ for all $t > 0$, $\varphi(0) = 0\}$,

$\Psi = \{\psi \mid \psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0\}$.

2 Main theorem

Following is the main result of this paper.

Theorem 2.1. *Suppose that f, g, S and T are selfmaps of a complete metric space (X, d), $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and that the pairs $\{f, S\}$ and $\{g, T\}$ are weakly compatible. If*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{1}$$

for each $x, y \in X$, where $\varphi \in \Phi$, $\psi \in \Psi$ and where

$$M(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\} \tag{2}$$

then f, g, S and T have a unique common fixed point in X provided one of the ranges $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is closed.

Proof. Let x_0 be an arbitrary point in X. Choose a point $x_1 \in X$ such that $y_0 = fx_0 = Tx_1$. This can be done, since the range of T contains the range of f. Similarly, a point $x_2 \in X$ can be chosen such that $y_1 = gx_1 = Sx_2$ as $g(X) \subseteq S(X)$. Continuing this process, we obtain a sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$.

First, we show that $\{y_n\}$ is a Cauchy sequence in X. Consider two cases.

1. If for some n, $y_n = y_{n+1}$, then $y_{n+1} = y_{n+2}$. If not, then for $n = 2m$, where $m \in Z^+$, we have

$$\begin{aligned}
M(x_{2m+2}, x_{2m+1}) &= \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(fx_{2m+2}, Sx_{2m+2}), \\
&\quad d(gx_{2m+1}, Tx_{2m+1}), \\
&\quad \frac{1}{2}(d(Sx_{2m+2}, gx_{2m+1}) + d(fx_{2m+2}, Tx_{2m+1}))\} \\
&= \max\{d(y_{2m+1}, y_{2m}), d(y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m}), \\
&\quad \frac{1}{2}(d(y_{2m+1}, y_{2m+1}) + d(y_{2m+2}, y_{2m}))\} \\
&= \max\{d(y_{2m+2}, y_{2m+1}), \frac{1}{2}d(y_{2m+2}, y_{2m})\} \\
&= d(y_{2m+2}, y_{2m+1}).
\end{aligned}$$

From (1)

$$\begin{aligned}
\psi(d(y_{n+2}, y_{n+1})) &= \psi(d(y_{2m+2}, y_{2m+1})) \\
&= \psi(d(fx_{2m+2}, gx_{2m+1})) \\
&\leq \psi((M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1})) \\
&= \psi((d(y_{2m+2}, y_{2m+1})) - \varphi(d(y_{2m+2}, y_{2m+1})) \\
&< \psi(d(y_{2m+2}, y_{2m+1})),
\end{aligned}$$

which is a contradiction. Hence we must have $y_{n+1} = y_{n+2}$, when n is even. By similar arguments we can show that this equality holds also when n is odd. Therefore, in any case for all those n for which $y_n = y_{n+1}$ holds, we always obtain $y_{n+1} = y_{n+2}$. Repeating above process inductively, one obtains $y_n = y_{n+k}$, for all $k \geq 1$. Therefore, in this case $\{y_n\}$ turns out to be eventually a constant sequence and hence a Cauchy one.

2. If $y_n \neq y_{n+1}$, for every positive integer n, then for $n = 2m+1$, for some $m \in Z^+$,

$$\begin{aligned}
M(x_{2m+2}, x_{2m+1}) &= \max\{d(Sx_{2m+2}, Tx_{2m+1}), d(fx_{2m+2}, Sx_{2m+2}), \\
&\quad d(gx_{2m+1}, Tx_{2m+1}), \\
&\quad \frac{1}{2}(d(Sx_{2m+2}, gx_{2m+1}) + d(fx_{2m+2}, Tx_{2m+1}))\} \\
&= \max\{d(y_{2m+1}, y_{2m}), d(y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m}), \\
&\quad \frac{1}{2}(d(y_{2m+1}, y_{2m+1}) + d(y_{2m+2}, y_{2m}))\} \\
&= \max\{d(y_{2m+1}, y_{2m}), d(y_{2m+2}, y_{2m+1}), \\
&\quad \frac{1}{2}(d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2m+2}))\} \\
&= \max\{d(y_{2m+1}, y_{2m}), d(y_{2m+2}, y_{2m+1})\}.
\end{aligned}$$

Now if $M(x_{2m+2}, x_{2m+1}) = d(y_{2m+2}, y_{2m+1})$, then

$$\begin{aligned}
\psi(d(y_{n+1}, y_n)) &= \psi(d(fx_{n+1}, gx_{n+2})) \\
&\leq \psi(M(x_{n+1}, x_n)) - \varphi(M(x_{n+1}, x_n)) \\
&= \psi(M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1})) \\
&= \psi(d(y_{2m+2}, y_{2m+1})) - \varphi(d(y_{2m+2}, y_{2m+1})) \\
&= \psi(d(y_{n+1}, y_n)) - \varphi(d(y_{n+1}, y_n)) \\
&< \psi(d(y_{n+1}, y_n)),
\end{aligned}$$

gives a contradiction. Therefore

$$M(x_{2m+2}, x_{2m+1}) = d(y_{2m+1}, y_{2m}). \tag{3}$$

Hence from (1), we obtain

$$\begin{aligned}
\psi(d(y_{n+1}, y_n)) &= \psi(d(fx_{n+1}, gx_{n+2})) \\
&\leq \psi(M(x_{n+1}, x_n)) - \varphi(M(x_{n+1}, x_n)) \\
&= \psi(M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1})) \\
&= \psi(d(y_{2m+1}, y_{2m})) - \varphi(d(y_{2m+1}, y_{2m})) \\
&= \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})) \\
&< \psi(d(y_n, y_{n-1})).
\end{aligned}$$

Following the similar arguments to those given above, we conclude the same inequality when n is taken as even integer. Consequently, we have

$$\psi(d(y_{n+1}, y_n)) < \psi(d(y_n, y_{n-1})), \text{ for all } n \geq 0$$

which further implies that $d(y_{n+1}, y_n) \leq d(y_n, y_{n-1})$. Therefore $\{d(y_{n+1}, y_n)\}$ is monotone decreasing sequence which is bounded below by 0. By theorem of monotone and bounded sequence there exists $r \geq 0$ such that $d(y_{n+1}, y_n) \to r$ as $n \to \infty$. From equation (2), for $x = x_{n+1}$ and $y = x_n$ we obtain,

$$\lim_{n\to\infty} \psi(M(x_{n+1}, x_n)) = \psi(r). \tag{4}$$

Now (1), (4) and lower semicontinuity of φ give

$$\limsup_{n\to\infty} \psi(d(y_{n+1}, y_n)) \leq \limsup_{n\to\infty} \psi(M(x_{n+1}, x_n)) - \liminf_{n\to\infty} \varphi(M(x_{n+1}, x_n))$$

which implies that $\psi(r) \leq \psi(r) - \varphi(r)$. Therefore $r = 0$, and

$$\lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{5}$$

Because of (5), to show $\{y_n\}_{n\geq 1}$ to be a Cauchy sequence in X, it is sufficient to show that $\{y_{2n}\}_{n\geq 1}$ is Cauchy in X. If not, there is an $\varepsilon > 0$, and there exists even integers $2m_k$ and $2n_k$ with $2m_k > 2n_k > k$ such that $d(y_{2m_k}, y_{2n_k}) \geq \varepsilon$ and $d(y_{2m_k-2}, y_{2n_k}) < \varepsilon$. Now (5) and inequality

$$\begin{aligned} \varepsilon &\leq d(y_{2m_k}, y_{2n_k}) \\ &\leq d(y_{2n_k}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k-2}) + d(y_{2m_k-1}, y_{2m_k}) \end{aligned}$$

implies that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varepsilon. \tag{6}$$

Also, (5) and inequality

$$d(y_{2m_k}, y_{2n_k}) \leq d(y_{2m_k}, y_{2m_k+1}) + d(y_{2m_k+1}, y_{2n_k})$$

gives that $\varepsilon \leq \lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k})$, while (5) and inequality

$$d(y_{2m_k+1}, y_{2n_k}) \leq d(y_{2m_k+1}, y_{2m_k}) + d(y_{2m_k}, y_{2n_k})$$

yields $\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) \leq \varepsilon$. Hence

$$\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) = \varepsilon. \tag{7}$$

By the similar way we obtain

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k-1}) = \lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k+1}) = \varepsilon. \tag{8}$$

Now from definition of M (equation (2)) and from (5), (6), (7), (8) we have

$$\begin{aligned} M(x_{2n_k}, x_{2m_k+1}) &= \max\{d(Sx_{2n_k}, Tx_{2m_k+1}), d(fx_{2n_k}, Sx_{2n_k}), \\ &\quad d(gx_{2m_k+1}, Tx_{2m_k+1}), \\ &\quad \frac{1}{2}(d(Sx_{2n_k}, gx_{2m_k+1}) + d(fx_{2n_k}, Tx_{2m_k+1}))\} \\ &= \max\{d(y_{2n_k-1}, y_{2m_k}), d(y_{2n_k}, y_{2n_k-1}), \\ &\quad d(y_{2m_k+1}, y_{2m_k}), \frac{1}{2}(d(y_{2n_k-1}, y_{2m_k+1}) + \\ &\quad d(y_{2n_k}, y_{2m_k}))\}. \end{aligned}$$

Thus

$$\lim_{k\to\infty} M(x_{2n_k}, x_{2m_k+1}) = \max\left\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon+\varepsilon)\right\} = \varepsilon.$$

Putting $x = x_{2n_k}$ and $y = x_{2m_k+1}$ in (1) we obtain

$$\begin{aligned} \psi(d(y_{2n_k}, y_{2m_k+1})) &= \psi(d(fx_{2n_k}, gx_{2m_k+1})) \\ &\leq \psi(M(x_{2n_k}, x_{2m_k+1})) - \varphi(M(x_{2n_k}, x_{2m_k+1})) \end{aligned}$$

which, on taking limit as $k \to \infty$, implies that

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon). \tag{9}$$

As (9) is a contradiction with $\varepsilon > 0$, it follows that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence in X. Since X is complete, there exists, a point $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

The second step of proof is to show that z is the fixed point for maps f and S. It is clear that

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z,$$

and

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Assuming $S(X)$ is closed, there exists $u \in X$ such that $z = Su$. We claim that $fu = z$. If not, then

$$\begin{aligned} M(u, x_{2n+1}) &= \max\{d(Su, Tx_{2n+1}), d(fu, Su), d(gx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{1}{2}(d(Su, gx_{2n+1}) + d(fu, Tx_{2n+1}))\} \\ &= \max\{d(z, Tx_{2n+1}), d(fu, z), d(gx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{1}{2}(d(z, gx_{2n+1}) + d(fu, Tx_{2n+1}))\} \\ &\to d(fu, z) \end{aligned}$$

as $n \to \infty$. From (1),

$$\psi(d(fu, gx_{2n+1})) \leq \psi(M(u, x_{2n+1})) - \varphi(M(u, x_{2n+1})),$$

which, on taking limit as $n \to \infty$ implies that

$$\psi(d(fu, z)) \leq \psi(d(fu, z)) - \varphi(d(fu, z)),$$

a contradiction with $d(fu, z) > 0$. Hence $fu = z$. Therefore $fu = Su = z$. Since the maps f and S are weakly compatible, we have $fz = fSu = Sfu = Sz$. Next we claim that $fz = z$. If not, then

$$\begin{aligned} M(z, x_{2n+1}) &= \max\{d(Sz, Tx_{2n+1}), d(fz, Sz), d(gx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{1}{2}(d(Sz, gx_{2n+1}) + d(fz, Tx_{2n+1}))\} \\ &= \max\{d(fz, Tx_{2n+1}), d(fz, fz), d(gx_{2n+1}, Tx_{2n+1}), \\ &\qquad \frac{1}{2}(d(fz, gx_{2n+1}) + d(fz, Tx_{2n+1}))\} \\ &\to d(fz, z) \ (n \to \infty) \end{aligned}$$

and again by (1)

$$\psi(d(fz, gx_{2n+1})) \leq \psi(M(z, x_{2n+1})) - \varphi(M(z, x_{2n+1})),$$

which, on taking limit as $n \to \infty$ gives the contradiction

$$\psi(d(fz, z)) \leq \psi(d(fz, z)) - \varphi(d(fz, z)).$$

Therefore $fz = z$.

The next step is to show that z is also fixed point for maps g and T. Since $f(X) \subseteq T(X)$, there is some v in X such that $fz = Tv$. Then $fz = Tv = Sz = z$. We claim that $gv = z$. If $gv \neq z$, then from (1) we have

$$d(z, gv) = d(fz, gv) \leq \psi(M(z, v)) - \varphi(M(z, v)),$$

where

$$\begin{aligned} M(z, v) &= \max\{d(Sz, Tv), d(fz, Sz), d(gv, Tv), \frac{1}{2}(d(Sz, gv) + d(fz, Tv))\} \\ &= d(gv, z). \end{aligned}$$

Thus

$$\psi(d(z, gv)) \leq \psi(d(z, gv)) - \varphi(d(z, gv))$$

gives a contradiction. Therefore $z = gv$. Hence $gv = Tv = z$. By weak compatibility of mappings g and T we obtain $gz = gTv = TTv = Tz$. Finally, we claim that $gz = z$. If $gz \neq z$, then by (1)

$$\psi(d(z, gz)) = \psi(d(fz, gz)) \leq \psi(M(z, z)) - \varphi(M(z, z)),$$

where

$$\begin{aligned} M(z, z) &= \max\{d(Sz, Tz), d(fz, Sz), d(gz, Tz), \frac{1}{2}(d(Sz, gz) + d(fz, Tz))\} \\ &= d(z, gz). \end{aligned}$$

Therefore

$$\psi(d(z, gz)) \leq \psi(d(z, gz)) - \varphi(d(z, gz))$$

gives a contradiction. Hence, $fz = gz = Sz = Tz = z$. Similar analysis is valid for the case in which $T(X)$ is closed, as well as for the cases in which $f(X)$ or $g(X)$ is closed, since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$.

As uniqueness of common fixed point z easily follows from the inequality (1), the proof is completed. □

Now we give an example to support our result.

Example 2.1. *Let $X = [0, 1]$ with the usual metric. Define f, g, S and T on X by*

$$fx = \begin{cases} \frac{1}{2}, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}, \quad gx = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases},$$

$$Sx = \begin{cases} 0, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ \frac{2}{3}, & \frac{1}{2} < x \leq \frac{2}{3} \\ 1, & \frac{2}{3} < x \leq 1 \end{cases}, \quad Tx = \begin{cases} 1, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ \frac{1}{10}, & \frac{1}{2} < x \leq \frac{2}{3} \\ 0, & \frac{2}{3} < x \leq 1 \end{cases}$$

and $\varphi, \psi : [0,\infty) \to [0,\infty)$ *by* $\varphi(t) = \frac{1}{10}t^2$ *and* $\psi(t) = \sqrt{t}$*. Then* $\varphi \in \Phi$ *and* $\psi \in \Psi$*. Obviously,* $f(X) \subseteq T(X)$ *and* $g(X) \subseteq S(X)$*. Furthermore the pairs* $\{f, S\}$*, and* $\{g, T\}$ *are weakly compatible and satisfy (1). Thus* f*,* g*,* S*, and* T *satisfy the conditions given in Theorem 2.1 and* $\frac{1}{2}$ *is the unique common fixed point of* f*,* g*,* S*, and* T*.*

Special cases of the theorem 2.1 give results obtained earlier in different papers.

Corolary 2.1. ([4]) *Let* (X, d) *be a complete metric space and let* $f, g : X \to X$ *be two mappings such that*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \text{for all } x, y \in X,$$

where $\varphi \in \Phi$*,* $\psi \in \Psi$ *and*

$$M(x, y) = \max\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, gy) + d(fx, y))\}.$$

Then there exists a unique point $u \in X$ *such that* $u = fu = gu$*.*

Proof. If we take S and T as identity maps on X, then from Theorem 2.1 follows that f and g have a unique common fixed point.

Corolary 2.2. ([13]) *Let* (X, d) *be a complete metric space and let* $f, g : X \to X$ *be two mappings such that for all* $x, y \in X$

$$d(fx, gy) \leq M(x, y) - \varphi(M(x, y)),$$

where $\varphi \in \Phi$ *and*

$$M(x, y) = \max\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, gy) + d(fx, y))\}.$$

Then f *and* g *have a unique common fixed point.*

Proof. If we take S and T as identity maps on X and $\psi(t) = t$ for $t \in [0, \infty)$, then from Theorem 2.1 follows that f and g have a unique common fixed point.

Corolary 2.3. ([4]) *Let* (X, d) *be a complete metric space and let* $f : X \to X$ *be a mapping such that for all* $x, y \in X$

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

where $\varphi \in \Phi$ *and*

$$M(x, y) = \max\{d(x, y), d(fx, x), d(fy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\}.$$

Then f *has a unique fixed point in* X*.*

Example 2.2. *Let* $X = [0,1] \cup [\frac{3}{2}, 2]$ *with the usual metric and let self map* f *on* X *be defined as follows:*

$$fx = \begin{cases} \frac{3}{4}, & 0 \leq x \leq 1 \\ \frac{1}{4}, & \frac{3}{2} \leq x \leq 2 \end{cases}.$$

If we take $\psi(t) = t$ *and* $\varphi(t) = \frac{1}{4}t$ *for* $t \in [0, \infty)$*, then all conditions of Corollary 2.3 are satisfied and* f *has a unique fixed point* $\frac{3}{4}$*. Note that if we take* $x = 1$ *and* $y = \frac{3}{2}$*, then for any choice of functions* ψ *and* ϕ*, mapping* f *does not satisfy the contractive condition given in [5].*

Corolary 2.4. ([13]) *Let* (X, d) *be a complete metric space and let* $f : X \to X$ *be a mapping such that for all* $x, y \in X$

$$d(fx, fy) \leq M(x, y) - \varphi(M(x, y)),$$

where $\varphi \in \Phi$ *and*

$$M(x, y) = \max\{d(x, y), d(fx, x), d(fy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\}.$$

Then f *has a unique fixed point in* X*.*

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Mujahid Abbas:
Centre for Advanced Studies in Mathematics and Department of Mathematics, Lahore University of Management Sciences, 54792-Lahore, Pakistan
E-mail: `mujahid@lums.edu.pk`

Dragan Đorić:
Faculty of Organizational Sciences, University of Belgrade, 11000 Beograd, Jove Ilića 154, Serbia
E-mail: `djoricd@fon.rs`