Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:2** (2010), 131–142 DOI: 10.2298/FIL1002131X

CLASSIFICATION OF THE CROSSED PRODUCT $\mathbf{C(M)\times_\theta \mathbb{Z}_p}$ FOR CERTAIN PAIRS $(\mathbf{M},\theta)$

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Abstract

Let M be a separable compact Hausdorff space with $\dim M \le 2$ and $\theta\colon M\to M$ be a homeomorphism with prime period p $(p\ge 2)$. Set $M_\theta = \{x\in M\,|\,\theta(x)=x\}\neq\varnothing$ and $M_0=M\backslash M_\theta$. Suppose that M_0 is dense in M and $\mathrm{H}^2(M_0/\theta,\mathbb{Z})\cong 0$, $\mathrm{H}^2(\chi(M_0/\theta),\mathbb{Z})\cong 0$. Let M' be another separable compact Hausdorff space with $\dim M'\le 2$ and θ' be the self–homeomorphism of M' with prime period p. Suppose that $M'_0=M'\backslash M'_{\theta'}$ is dense in M'. Then $C(M)\times_\theta\mathbb{Z}_p\cong C(M')\times_{\theta'}\mathbb{Z}_p$ iff there is a homeomorphism F from M/θ onto M'/θ' such that $F(M_\theta)=M'_{\theta'}$. Thus, if (M,θ) and (M',θ') are orbit equivalent, then $C(M)\times_\theta\mathbb{Z}_p\cong C(M')\times_{\theta'}\mathbb{Z}_p$.

1 Introduction

The classification of dynamical systems and corresponding C^*–crossed products have became one of most important research subjects for more than a decade. One of the most important results about the classification of the minimal Cantor dynamical system was obtained by T. Giordano, I.F. Putnam and F. Skau in [5]. They proved that two minimal Cantor crossed products $C(X_1)\times_{\alpha_1}\mathbb{Z}$ and $C(X_2)\times_{\alpha_2}\mathbb{Z}$ are $*$–isomorphic iff (X_1,α_1) and (X_2,α_2) are strong orbit equivalent (cf. [5, Theorem 2.1]). Recently, H. Lin and H. Matui used their notation so–called approximate conjugacy of dynamical systems to obtain many equivalent conditions that make $C(X_1)\times_{\alpha_1}\mathbb{Z}\cong C(X_2)\times_{\alpha_2}\mathbb{Z}$ in [13].

For the type of dynamical systems such as $(C(X),\alpha,\mathbb{Z}_n)$ where X is a compact Hausdorff space and $\alpha\colon X\to X$ is homeomorphism, there is little known when $C(X_1)\times_{\alpha_1}\mathbb{Z}_n\cong C(X_2)\times_{\alpha_2}\mathbb{Z}_n$. We only know if α_1 acts on X_1 freely and $\mathrm{H}^2(X_1/\alpha_1,\mathbb{Z})$ has no element annihilated by n, then $C(X_1)\times_{\alpha_1}\mathbb{Z}\cong C(X_2)\times_{\alpha_2}\mathbb{Z}$ iff α_2 acts on X_2 freely and $X_1/\alpha_1\cong X_2/\alpha_2$ (cf. [11, Proposition 5]).

2010 *Mathematics Subject Classifications.* 46L05
Key words and Phrases. matrix bundle, Crossed product, Busby invariant
Received: February 8, 2010
Communicated by Vladimir Rakocevic
Project supported by Natural Science Foundation of China (no.10771069) and Shanghai Leading Academic Discipline Project(no.B407)

But there is a special example showed by Elliott in [3] which could enlighten one on the classification of $C(X)\times_\alpha \mathbb{Z}_n$ when α has fixed points. The example is: $C(\mathbf{S}^1)\times_\alpha \mathbb{Z}_2\cong\{f\colon[0,1]\to \mathrm{M}_2(\mathbb{C})$ continuous$\,|\, f(0), f(1)$ are diagonal$\}$, where $\alpha(z)=\overline{z}$, $\forall\, z\in\mathbf{S}^1$ (cf. [1, 6.10.4, 10.11.5]).

Inspired by this example, we try to find when the crossed product $C(X)\times_\alpha\mathbb{Z}_p$ has the similar form and try to classify it. So, in this paper, we will consider the classification of the crossed product $C(M)\times_\theta\mathbb{Z}_p$, here M is a separable compact Hausdorff space and θ is a self–homeomorphism of M with prime period p ($p\geq 2$) such that $M_\theta=\{x\in M|\,\theta(x)=x\}\neq\varnothing$. We let (M,θ) denote the pair which satisfy conditions mentioned above throughout the paper.

Based on the Extension theory of C^*–algebras and author's previous work on $C(M)\times_\theta\mathbb{Z}_p$, we obtain that if $\dim M\leq 2$, $M_0=M\backslash M_\theta$ is dense in M and $\mathrm{H}^2(M_0/\theta,\mathbb{Z})\cong 0$, $\mathrm{H}^2(\chi(M_0/\theta),\mathbb{Z})\cong 0$, then

$$C(M)\times_\theta\mathbb{Z}_p\cong\{(a_{ij})_{p\times p}\in\mathrm{M}_p(C(M/\theta))|\,a_{ij}(x)=0,\ \forall\, x\in M_\theta,\ i\neq j\}.$$

where M/θ (or M_0/θ) is the orbit space of θ and $\chi(M_0/\theta)$ is the corona set of M_0/θ. Moreover, let (M',θ') be another pair with $\dim M'\leq 2$ and $\overline{M\backslash M'_{\theta'}}=M'$. Then

$$C(M)\times_\theta\mathbb{Z}_p\cong C(M')\times_{\theta'}\mathbb{Z}_p$$

iff there is a homeomorphism F from M/θ onto M'/θ' such that $F(M_\theta)=M'_{\theta'}$.

2 Preliminaries

Let $\mathcal{A}$ be a C^*–algebra with unit 1. We denote by $\mathcal{U}(\mathcal{A})$ the group of unitary elements of $\mathcal{A}$ and $\mathcal{U}_0(\mathcal{A})$ the connected component of the unit 1 in $\mathcal{U}(\mathcal{A})$. Let $\mathrm{M}_m(\mathcal{A})$ be the matrix algebra of order m over $\mathcal{A}$. Set (cf. [16, 17] or [20])

$$\mathrm{S}_n(\mathcal{A})=\{\,(a_1,\cdots,a_n)|\,\sum_{k=1}^n a_k^*a_k=1\}$$
$$\mathrm{csr}(\mathcal{A})=\min\{\,n|\,\mathcal{U}_0(\mathrm{M}_n(\mathcal{A}))\text{ acts transitively on }\mathrm{S}_m(\mathcal{A})\ \forall\, m\geq n\,\}$$

Lemma 2.1. *Let X be a compact Hausdorff space with covering dimension $\dim X\leq 2$. Given $U\in\mathcal{U}(\mathrm{M}_p(C(X)))$, there are $U_0\in\mathcal{U}_0(\mathrm{M}_p(C(X)))$ and $v\in\mathcal{U}((C(X)))$ such that $U=U_0\,\mathrm{diag}\,(1_{p-1},v)$.*

Proof. We have from [14], $\mathrm{csr}(C(X))\leq\left[\dfrac{\dim X+1}{2}\right]+1\leq 2$, where $[x]$ stands for the greatest integer which is less than or equal to x.

Let u_1 be the first column of U. Then $u_1\in\mathrm{S}_p(C(X))$ and consequently, there is $U_1\in\mathcal{U}_0(\mathrm{M}_p(C(X))$ such that u_1 becomes the first column of U_1 for $\mathrm{csr}(C(X))\leq 2\leq p$. Put $W=U_1^*U$. Then the unitary element W has the form $W=\mathrm{diag}\,(1,V_1)$ for some $V_1\in\mathcal{U}(\mathrm{M}_{p-1}(C(X)))$.

Repeating above procedure to V_1, there are $U_2\in\mathcal{U}_0(\mathrm{M}_{p-1}(C(X)))$ and $V_2\in\mathcal{U}(\mathrm{M}_{p-2}(C(X)))$ such that $V_1=U_2\,\mathrm{diag}\,(1,V_2)$. So $U=U_1U_2\,\mathrm{diag}\,(1_2,V_2)$. By induction, we can finally find $U_0\in\mathcal{U}_0(\mathrm{M}_p(C(X)))$ and $v\in\mathcal{U}(C(X))$ such that $U=U_0\,\mathrm{diag}\,(1_{p-1},v)$. □

According to [21], $\{x\in M|\,\theta^k(x)=x\}=M_\theta,\ k=2,\cdots,p-1$. Now set $M_0=M\backslash M_\theta$. Let M/θ (resp. M_0/θ)) denote the orbit space of θ and let P be the canonical projective map of M (or M_0) onto M/θ (or M_0/θ). Let $O_\theta(x)=\{x,\theta(x),\cdots,\theta^{p-1}(x)\}$ denote the orbit of x in M or M_0. For the pair (M,θ), the dynamical system $(C(M),\theta,\mathbb{Z}_p)$ (resp. $(C_0(M_0),\theta,\mathbb{Z}_p)$) yields a crossed product C^*–algebra $C(M)\times_\theta\mathbb{Z}_p$ (resp. $C_0(M_0)\times_\theta\mathbb{Z}_p$). Set

$$\mathrm{D}(M,\theta)=\left\{\begin{pmatrix} f_0 & f_1 & \cdots & f_{p-1}\\ \theta(f_{p-1}) & \theta(f_0) & \cdots & \theta(f_{p-2})\\ \cdots & \cdots & \cdots & \cdots\\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \cdots & \theta^{p-1}(f_0)\end{pmatrix};\ f_0,\cdots,f_{p-1}\in C(M)\right\}$$

$$\mathrm{D}(M_0,\theta)=\left\{\begin{pmatrix} f_0 & f_1 & \cdots & f_{p-1}\\ \theta(f_{p-1}) & \theta(f_0) & \cdots & \theta(f_{p-2})\\ \cdots & \cdots & \cdots & \cdots\\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \cdots & \theta^{p-1}(f_0)\end{pmatrix};\ f_0,\cdots,f_{p-1}\in C_0(M_0)\right\},$$

where $\theta(f)(x)=f(\theta(x)),\forall x\in M$ (resp. M_0), $f\in C(M)$ (resp. $C_0(M_0)$). By 7.6.1 and 7.6.5 of [15], we have $C(M)\times_\theta\mathbb{Z}_p\cong\mathrm{D}(M,\theta)\subset\mathrm{M}_n(C(M))$ and $C_0(M_0)\times_\theta\mathbb{Z}_p\cong\mathrm{D}(M_0,\theta)\subset\mathrm{M}_n(C_0(M_0))$.

Let $\omega=\mathrm{e}^{2\pi i/p}$. Put $e_j=\mathrm{diag}\,(\underbrace{0,\cdots,0}_{j-1},1,\underbrace{0,\cdots,0}_{p-j})$ and

$$\Omega_p=\begin{pmatrix} \frac{1}{\sqrt{p}} & \frac{\omega}{\sqrt{p}} & \cdots & \frac{\omega^{p-1}}{\sqrt{p}}\\ \frac{1}{\sqrt{p}} & \frac{\omega^2}{\sqrt{p}} & \cdots & \frac{(\omega^2)^{p-1}}{\sqrt{p}}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{1}{\sqrt{p}} & \frac{\omega^{p-1}}{\sqrt{p}} & \cdots & \frac{(\omega^{p-1})^{p-1}}{\sqrt{p}}\\ \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} & \cdots & \frac{1}{\sqrt{p}}\end{pmatrix},\quad P_j=\Omega_p^*e_j\Omega_p,\ j=1,\cdots,p.$$

It is easy to check that Ω_p is unitary in $\mathrm{M}_p(\mathbb{C})$ and $P_1,\cdots,P_p$ are projections in $\mathrm{D}(M,\theta)$. Define $*$–homomorphism $\pi\colon\mathrm{D}(M,\theta)\to\bigoplus_{j=0}^{p-1}C(M_\theta)$ as

$$\pi\left(\begin{pmatrix} f_0 & f_1 & \cdots & f_{p-1}\\ \theta(f_{p-1}) & \theta(f_0) & \cdots & \theta(f_{p-2})\\ \cdots & \cdots & \cdots & \cdots\\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \cdots & \theta^{p-1}(f_0)\end{pmatrix}\right)=\left(\sum_{j=0}^{p-1}\omega^{j(p-1)}f_j|_{M_\theta},\cdots,\sum_{j=0}^{p-1}f_j|_{M_\theta}\right).$$

π induces following exact sequence of C^*–algebras:

$$0\longrightarrow\mathrm{D}(M_0,\theta)\xrightarrow{\ l\ }\mathrm{D}(M,\theta)\xrightarrow{\ \pi\ }\bigoplus_{j=0}^{p-1}C(M_\theta)\longrightarrow 0.\tag{2.1}$$

Lemma 2.2. *For the pair (M,θ), we have*

(1) Every irreducible representation of $\mathrm{D}(M_0,\theta)$ *is equivalent to the representation* π_x, *where* $\pi_x(a)=a(x)$ *for some* $x\in M_0$ *and* $\forall a\in \mathrm{D}(M_0,\theta)$ *and* $P(x)\to[\pi_x]$ *gives a homeomorphism of* M_0/θ *onto* $\widehat{\mathrm{D}(M_0,\theta)}$*–the spectrum of* $\mathrm{D}(M_0,\theta)$, *where we identify* $\mathrm{D}(M_0,\theta)$ *with* $\{\,a\in \mathrm{D}(M,\theta)|\,a|_{M_\theta}=0\}$;

(2) $\mathrm{D}(M_0,\theta)$ *is a p–homogeneous algebra which is* $*$*–isomorphic to* $C_0(M_0/\theta,E)$, *where* E *is a matrix bundle over* M_0/θ *with fiber* $\mathrm{M}_p(\mathbb{C})$;

(3) Let σ *be a pure state on* $\mathrm{D}(M,\theta)$. *Then* σ *is multiplicable iff there is* $x_0\in M_\theta$ *such that*

$$\sigma\left(\begin{pmatrix} f_0 & f_1 & \dots & f_{p-1}\\ \theta(f_{p-1}) & \theta(f_0) & \dots & \theta(f_{p-2})\\ \dots & \dots & \dots & \dots\\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \dots & \theta^{p-1}(f_0)\end{pmatrix}\right)=\sum_{j=0}^{p-1}\omega^{jk}f_j(x_0)$$

for some $k\in\{0,\cdots,p-1\}$. *We let* $\sigma_{x_0,k}$ *denote the* σ.

Proof. Let (Π,K) be an irreducible representation of $\mathrm{D}(M_0,\theta)$. Then by [2, Proposition 2.10.2], there is an irreducible representation (ρ,H) of $\mathrm{M}_p(C_0(M_0))$ such that $K\subset H$ and $\Pi(\cdot)=\rho(\cdot)|_K$. It is well-known that (ρ,H) is equivalent to $(\rho_{x_0},\mathbb{C}^p)$ for some $x_0\in M_0$, where $\rho_x((f_{ij})_{p\times p})=(f_{ij}(x))_{p\times p}$, $x\in M_0$ and $f_{ij}\in C_0(M_0)$, $i,j=1,\cdots,p$.

Put $\pi_x=\rho_x|_{\mathrm{D}(M_0,\theta)}$, $x\in M_0$. Given $A=(a_{ij})_{p\times p}\in \mathrm{M}_p(\mathbb{C})$. Since $x_0,\theta(x_0),\cdots,\theta^{p-1}(x_0)$ are mutually different in M_0, we can find $f_0,\cdots,f_{p-1}\in C(M)$ such that $f_j|_{M_\theta}=0,\ j=0,\cdots,p-1$ and

$$\pi_{x_0}\left(\begin{pmatrix} f_0 & f_1 & \dots & f_{p-1}\\ \theta(f_{p-1}) & \theta(f_0) & \dots & \theta(f_{p-2})\\ \dots & \dots & \dots & \dots\\ \theta^{p-1}(f_1) & \theta^{p-1}(f_2) & \dots & \theta^{p-1}(f_0)\end{pmatrix}\right)=A.$$

This means that $(\pi_{x_0},\mathbb{C}^p)$ is an irreducible representation of $\mathrm{D}(M_0,\theta)$ and hence (Π,K) is equivalent to $(\pi_{x_0},\mathbb{C}^p)$. Thus $\mathrm{D}(M_0,\theta)$ is a p–homogeneous C^*–algebra.

Let $x_0\in M_0$ and $x_1=\theta^j(x_0)$ for some $j\in\{1,\cdots,p-1\}$. It is easy to check that there is a unitary matrix U in $\mathrm{M}_p(\mathbb{C})$ such that $\pi_{x_0}(a)=U\pi_{x_1}(a)\,U^*$, $\forall\,a\in \mathrm{D}(M_0,\theta)$, i.e., $(\pi_{x_0},\mathbb{C}^p)$ and $(\pi_{x_1},\mathbb{C}^p)$ are equivalent; On the other hand, if $(\pi_{x_0},\mathbb{C}^p)$ and $(\pi_{x_1},\mathbb{C}^p)$ are equivalent and $O_\theta(x_0)\cap O_\theta(x_1)=\varnothing$, then we can choose $g_0,\cdots,g_{p-1}\in C(M)$ such that $g_j(x)=0,\ \forall\,x\in M_\theta\cup O_\theta(x_0)$ and $g_j|_{O_\theta(x_1)}\neq 0$, $j=0,\cdots,p-1$. Set

$$G=\begin{pmatrix} g_0 & g_1 & \dots & g_{p-1}\\ \theta(g_{p-1}) & \theta(g_0) & \dots & \theta(g_{p-2})\\ \dots & \dots & \dots & \dots\\ \theta^{p-1}(g_1) & \theta^{p-1}(g_2) & \dots & \theta^{p-1}(g_0)\end{pmatrix}\in \mathrm{D}(M_0,\theta).$$

Then $\pi_{x_0}(G) = 0$, while $\pi_{x_1}(G) \neq 0$. Therefore, we have $O_\theta(x_0) \cap O_\theta(x_1) \neq \varnothing$, that is, $x_0 = \theta^j(x_1)$ for some $j \in \{0, 1, \cdots, p-1\}$. So the map $P(x) \mapsto [\pi_x]$ gives a homeomorphism of M_0/θ onto $\widehat{\mathrm{D}(M_0, \theta)}$ by using [2, Lemma 3.3.3].

Set $[x] = P(x) \in M_0/\theta$ and $D([x]) = \mathrm{D}(M_0, \theta)/\mathrm{Ker}\,\pi_x \cong \mathrm{M}_p(\mathbb{C})$, $\forall\, x \in M_0$. Let $a([x])$ be the canonical image of $a \in \mathrm{D}(M_0, \theta)$ in $D([x])$. Put $E = \underset{[x] \in M_0/\theta}{\cup} D([x])$. Then E is a fiber bundle (matrix bundle) over M_0/θ with fiber $\mathrm{M}_p(\mathbb{C})$ (cf. [4, §3.2, P.249]). The $*$–isomorphism from $\mathrm{D}(M_0, \theta)$ onto $C_0(M_0/\theta, E)$ is defined by $\phi(a)([x]) = a([x])$, $\forall\, a \in \mathrm{D}(M_0, \theta)$ and $[x] \in M_0/\theta$.

The proof of (3) comes from Corollary 1, Case 1 and case 2 on P77 and case 2 on P79 of [11]. □

Assume that the matrix bundle E above is trivial, i.e., there is a homeomorphism $\Gamma\colon E \to M_0/\theta \times \mathrm{M}_p(\mathbb{C})$ such that $\Gamma(D([x])) = [x] \times \mathrm{M}_p(\mathbb{C})$, $\forall\, [x] \in M_0/\theta$. Then Γ induces an isomorphism $\Gamma^*\colon C_0(M_0/\theta, E) \to \mathrm{M}_p(C_0(M_0/\theta))$ given by $\Gamma^*(f)([x]) = \Gamma(f([x]))$, $\forall\, f \in C_0(M_0/\theta, E)$ and $[x] \in M_0/\theta$. Set $\Phi = l \circ \phi^{-1} \circ \Gamma^{-1}$. Applying Lemma 2.2 to (2.1), we have following:

Corollary 2.3. *Let (M, θ) be the pair such that all matrix bundles over M_0/θ is trivial. Then we have following exact sequence of C^*–algebras*

$$0 \longrightarrow \mathrm{M}_p(C_0(M_0/\theta)) \xrightarrow{\Phi} \mathrm{D}(M, \theta) \xrightarrow{\pi} \bigoplus_{j=0}^{p-1} C(M_\theta) \longrightarrow 0. \tag{2.2}$$

3 Main results

Consider the exact sequence of C^*–algebras

$$0 \longrightarrow \mathcal{B} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \longrightarrow 0, \tag{3.1}$$

where $\mathcal{B}$ is a separable C^*–algebra and $i(\mathcal{B})$ is the essential ideal of $\mathcal{E}$. Let $\{u_n\}$ be an approximate identity of $\mathcal{B}$ and $M(\mathcal{B})$ be the multiplier algebra of $\mathcal{B}$. Define $*$–monomorphism $\rho\colon \mathcal{E} \to M(\mathcal{B})$ by $\mu(e) = \lim\limits_n i^{-1}(i(u_n)e)$, $\forall\, e \in \mathcal{E}$, where the limit is taken as the strict limit in $M(\mathcal{B})$. Then Busby invariant $\tau_q\colon \mathcal{A} \to M(\mathcal{B})/\mathcal{B}$, associated with (3.1) is given by $\tau_q(a) = \Omega(\mu(e))$, where $e \in \mathcal{E}$ such that $q(e) = a$, $\Omega\colon M(\mathcal{B}) \to M(\mathcal{B})/\mathcal{B}$ is the quotient map. τ_q is well–defined and is monomorphic since $i(\mathcal{B})$ is essential. Please see [10, Chapter 3] or [12, Chapter 5], or [19, Chapter 3]) for details. In order to obtain our main result, we need following lemma, which comes from [10, Lemma 3.2.2].

Lemma 3.1. *Let τ_q (resp. $\tau_{q'}$) be the Busby invariant associated following exact sequence of C^*–algebras:*

$$0 \longrightarrow \mathcal{B} \xrightarrow{i} \mathcal{E} \xrightarrow{q} \mathcal{A} \longrightarrow 0 \quad (\textit{resp. } 0 \longrightarrow \mathcal{B} \xrightarrow{i'} \mathcal{E}' \xrightarrow{q'} \mathcal{A} \longrightarrow 0),$$

where $i(\mathcal{B})$ (resp. $i'(\mathcal{B})$) is the essential ideal of $\mathcal{E}$ (resp. $\mathcal{E}'$). Assume that there is a unitary element $u \in M(\mathcal{B})$ such that $\tau_q(a) = \Omega(u)\tau_{q'}(a)\Omega(u^*)$, $\forall\, a \in \mathcal{A}$. Then $\mathcal{E}$ is $*$–isomorphic to $\mathcal{E}'$.

Let X be a locally compact Hausdorff space. Let X^+ (resp. $\beta(X)$) denote the one–point (resp. Stone–Cěch) compactification of X. Set $\chi(X) = \beta(X)\backslash X$ (the corona set of X).

Lemma 3.2. *Let X be a locally compact Hausdorff space with $\dim X \leq 2$. If $\mathrm{H}^2(X, \mathbb{Z}) \cong 0$, then every matrix bundle over X is trivial.*

Proof. Let $M(X^+)$ denote the collection of all (isomorphism classes of) matrix bundles of arbitrary degree over X^+ and $SM(X^+)$ denote the the distinguished subsemigroup of $M(X)$, consisting of matrix bundles of the form $V \otimes V^*$, where V is an arbitrary complex vector bundle over X^+, and $*$ denotes adjoints. When $\dim X^+ = \dim X \leq 2$ and $\mathrm{H}^2(X, \mathbb{Z}) = \mathrm{H}^2(X^+, \mathbb{Z}) \cong 0$, $M(X^+) = SM(X^+)$ by [7, Corollary 1] and all complex vector bundles over X^+ are trivial. Therefore, all matrix bundles of arbitrary degree over X^+ is trivial. Because every matrix bundle over X with fiber $\mathrm{M}_n(\mathbb{C})$ can be extended to an matrix bundle over X^+ with fiber $\mathrm{M}_n(\mathbb{C})$ (using the same methods described in [9, P110–P112]), we get the assertion. □

By Lemma 3.2, we may assume that the pair (M, θ) satisfies condition (A):

$$\dim M \leq 2 \quad \text{and} \quad \mathrm{H}^2(M_0/\theta, \mathbb{Z}) \cong 0.$$

Lemma 3.3. *Let the pair (M, θ) satisfy Condition (A). If $M_0 = M\backslash M_\theta$ is dense in M, then $\Phi(\mathrm{M}_p(C_0(M_0/\theta))$ is an essential ideal of $\mathrm{D}(M, \theta)$.*

Proof. Let $a \in \mathrm{D}(M, \theta)$ such that $ac = 0$, $\forall\, c \in \mathrm{D}(M_0, \theta) = \Phi(\mathrm{M}_p(C_0(M_0/\theta))$. If there is $x_0 \in M_0$ such that $a(x_0) \neq 0$, then we can find a $b \in \mathrm{D}(M_0, \theta)$ such that $b(x_0) = (a(x_0))^*$. It follows that $a(x_0)(a(x_0))^* = 0$, i.e., $a(x_0) = 0$, a contradiction. So $a|_{M_0} = 0$. Since M_0 is dense in M and all elements in a are continuous, we have $a = 0$. □

It is known that

$$\begin{aligned} M(\mathrm{M}_p(C_0(M_0/\theta))) = \mathrm{M}_p(C_b(M_0/\theta)) &\cong \mathrm{M}_p(C(\beta(M_0/\theta))), \\ \mathrm{M}_p(C_b(M_0/\theta))/\mathrm{M}_p(C_0(M_0/\theta)) &\cong \mathrm{M}_p(C(\chi(M_0/\theta))). \end{aligned}$$

Assume that (M, θ) satisfies Condition (A) and the condition that $M\backslash M_\theta$ is dense in M. Then the $*$–monomorphism $\mu\colon \mathrm{D}(M, \theta) \to \mathrm{M}_p(C_b(M_0/\theta))$ is given by $\mu(a) = \lim\limits_{n\to\infty} \Phi^{-1}(\Phi(u_n)a)$, $\forall\, a \in \mathrm{D}(M, \theta)$, where $\{u_n\}$ is an approximate identity for $\mathrm{M}_p(C_0(M_0/\theta))$. We now construct the Busby invariant τ_π associated with (2.2) as follows.

Let $f_0, \cdots, f_{p-1} \in C(M_\theta)$. Extends them to $g_0, \cdots, g_{p-1} \in C(M)$, respectively, such that $g_j|_{M_\theta} = f_j$, $j = 0, \cdots, p-1$. Put $\hat{f}_j = \dfrac{1}{p}\sum\limits_{k=0}^{p-1} \theta^k(g_j)$, $j = 0, \cdots, p-1$. Then $\theta(\hat{f}_j) = \hat{f}_j$ and $\hat{f}_j|_{M_\theta} = f_j$, $j = 0, \cdots, p-1$. Thus, functions $h_0, \cdots, h_{p-1}$ given by

$h_j([x])=\hat{f}_j(x)$, $x\in M_0$, $j=0,\cdots,p-1$ are all in $C_b(M_0/\theta)$. Set $a(f_0,\cdots,f_{p-1})=\sum_{j=0}^{p-1} P_{j+1}\hat{f}_j$. Then $\pi(a(f_0,\cdots,f_{p-1}))=(f_0,\cdots,f_{p-1})\in\bigoplus_{j=0}^{p-1}C(M_\theta)$.

Let $\Omega\colon \mathrm{M}_p(C_b(M_0/\theta))\to \mathrm{M}_p(C_b(M_0/\theta))/\mathrm{M}_p(C_0(M_0/\theta))$ be the canonical homomorphism. Note that for any $b\in\mathrm{M}_p(C_0(M_0/\theta))$ and $[x]\in M_0/\theta$,

$$\begin{aligned}(\mu(P_{j+1}\hat{f}_j)b)([x]) =&\Phi^{-1}(P_{j+1}\hat{f}_j\Phi(b))([x])=\Gamma^*\circ\phi(P_{j+1}\hat{f}_j\Phi(b))([x])\\ =&\Gamma((P_{j+1}\hat{f}_j\Phi(b))([x])).\end{aligned}$$

since $P_{j+1}\hat{f}_j\Phi(b)-P_{j+1}\Phi(b)\hat{f}_j(x)\in\operatorname{Ker}\pi_x$, it follows that

$$\Gamma((P_{j+1}\hat{f}_j\Phi(b))([x]))=\Gamma((P_{j+1}\Phi(b))([x]))\hat{f}_j(x)=(\mu(P_{j+1})b)([x])h_j([x]),$$

that is, $\mu(P_{j+1}\hat{f}_j)=\mu(P_{j+1})h_j$, $j=0,\cdots,p-1$. Set $Q_j=\Omega\circ\mu(P_j)$, $j=1,\cdots,p$. Then τ_π is given by

$$\tau_\pi(f_0,\cdots,f_{p-1})=\Omega\circ\mu(a(f_0,\cdots,f_{p-1}))=\sum_{j=0}^{p-1}Q_{j+1}\Omega(h_j1_p).$$

Let $\mathcal{A}(M_\theta)=\{(a_{ij})_{p\times p}\in\mathrm{M}_p(C(M/\theta))|\,a_{ij}(x)=0,\ x\in M_\theta,\ i\neq j\}$. Define the $*$–homomorphism $\Lambda\colon\mathcal{A}(M_\theta)\to\bigoplus_{j=0}^{p-1}C(M_\theta)$ by $\Lambda((a_{ij})_{p\times p})=(a_{11}|_{M_\theta},\cdots,a_{pp}|_{M_\theta})$. Then we have following exact sequence:

$$0\longrightarrow \mathrm{M}_p(C_0(M_0/\theta))\xrightarrow{\ i\ }\mathcal{A}(M_\theta)\xrightarrow{\ \Lambda\ }\bigoplus_{j=0}^{p-1}C(M_\theta)\longrightarrow 0. \tag{3.2}$$

The Busby invariant τ_Λ associated with (3.2) is given by

$$\tau_\Lambda(f_0,\cdots,f_{p-1})=\sum_{j=0}^{p-1}\Omega(e_{j+1})\Omega(h_j1_p),$$

where $h_0,\cdots,h_{p-1}$ are given as above.

Now we present the main results of the paper as follows

Theorem 3.4. *Suppose that the pair (M,θ) satisfies condition (A) and Condition (B):*

$$M_0=M\backslash M_\theta \text{ is dense in } M,\quad \mathrm{H}^2(\chi(M_0/\theta),\mathbb{Z})\cong 0.$$

Then $C(M)\times_\theta\mathbb{Z}_p\cong\mathcal{A}(M_\theta)$.

Proof. Since $\dim(M_0/\theta)=\dim M_0\le\dim M\le 2$ by [21, Lemma 1.3] and $\mathrm{H}^2(M_0/\theta,\mathbb{Z})\cong 0$, it follows from Lemma 3.2 that every matrix bundle over M_0/θ is trivial. Since

$$\dim(\chi(M_0/\theta))\le\dim(\beta(M_0/\theta))\le\dim(M_0/\theta))\le 2,\quad \mathrm{H}^2(\chi(M_0/\theta),\mathbb{Z})\cong 0,$$

all complex vector bundles over $\chi(M_0/\theta)$ are trivial.

It is easy to check that for any $b \in \mathrm{M}_p(C_0(M_0/\theta))$, there is $f_j \in C_0(M_0)$ with $\theta(f_j) = f_j$ such that $P_j\Phi(b)P_j = f_jP_j$, $j = 1, \cdots, p$. So, $\mu(P_j)\mathrm{M}_p(C_0(M_0/\theta))\mu(P_j)$ is a commutative C^*–algebra, $j = 1, \cdots, p$. Since $\mathrm{M}_p(C_0(M_0/\theta))$ is dense in $\mathrm{M}_p(C_b(M_0/\theta))$ in the sense of strict topology, $\mu(P_j)\mathrm{M}_p(C_b(M_0/\theta))\mu(P_j)$ is also a commutative C^*–algebra, $j = 1, \cdots, p$. Assume that $Q_1, \cdots, Q_p$ in $\mathrm{M}_p(C(\chi(M_0/\theta))$. Then $Q_j\mathrm{M}_p(C(\chi(M_0/\theta)))\,Q_j$ is commutative and hence $\operatorname{rank} Q_j(x) \leq 1$, $\forall\, x \in \chi(M_0/\theta)$ by [15, Lemma 6.1.3], $j = 1, \cdots, p$.

Note that $Q_1, \cdots, Q_p$ are mutually orthogonal and $\sum\limits_{j=1}^{p} Q_j = 1_p$. Therefore, we have $\operatorname{rank} Q_j(x) = 1$, $\forall\, x \in \chi(M_0/\theta)$, $j = 1, \cdots, p$. So there are partial isometries $V_1, \cdots, V_p$ in $\mathrm{M}_p(C(\chi(M_0/\theta))$ such that $S_j = V_j^*V_j$ and $e_j = V_jV_j^*$, $j = 1, \cdots, p$. Put $V = \sum\limits_{j=1}^{p} V_j$. Then V is a unitary element in $\mathrm{M}_p(C(\chi(M_0/\theta))$ and $S_j = V^*e_jV$, $j = 1, \cdots, p$. Applying Lemma 2.1 to V^*, we can find $V_0 \in \mathcal{U}_0(\mathrm{M}_p(C(\chi(M_0/\theta)))$ and $v \in \mathcal{U}(C(\chi(M_0/\theta)))$ such that $V^* = V_0 \operatorname{diag}(1_{p-1}, v)$. Consequently, $S_j = V_0e_jV_0^*$, $j = 1, \cdots, p$.

Similarly, there is $V_0' \in \mathcal{U}_0(\mathrm{M}_p(C(\chi(M_0/\theta))))$ such that $\Omega(e_j) = V_0'e_jV_0'^*$. Choose $U \in \mathcal{U}_0(\mathrm{M}_p(C(\beta(M_0/\theta))))$ such that $\Omega(U) = V_0V_0'^*$. Since $\Omega(h_j1_p)$ commutes with every element in $\mathrm{M}_p(C(\chi(M_0/\theta)))$, $j = 0, \cdots, p-1$, it follows that

$$\tau_\pi(f_0, \cdots, f_{p-1}) = \Omega(U)\tau_\Lambda(f_0, \cdots, f_{p-1})\Omega(U^*),$$

$\forall\, f_0, \cdots, f_{p-1} \in C(M_\theta)$. Therefore, $C(M) \times_\theta \mathbb{Z}_p \cong \mathcal{A}(M_\theta)$ by Lemma 3.1. □

For the pair (M, θ) with $\dim M \leq 1$, we have $\dim(M_0/\theta) \leq 1$ and $\dim \chi(M_0/\theta) \leq 1$. In this case, $\mathrm{H}^2(M_0/\theta, \mathbb{Z}) \cong \mathrm{H}^2(\chi(M_0/\theta), \mathbb{Z}) \cong 0$. Therefore, we have following corollary according to Theorem 3.4:

Corollary 3.5. *Let (M, θ) be the pair with $\dim M \leq 1$ and $\overline{M_0} = M$, where $\overline{M_0}$ is the closure of M_0 in M. Then $C(M) \times_\theta \mathbb{Z}_p \cong \mathcal{A}(M_\theta)$.*

Theorem 3.6. *Suppose that the pair (M, θ) satisfy Condition (A), Condition (B). Let (M', θ') be another pair with $\overline{M_0'} = M'$. Then $C(M) \times_\theta \mathbb{Z}_p \cong C(M') \times_{\theta'} \mathbb{Z}_p$ iff there exists a homeomorphism $F\colon M/\theta \to M'/\theta'$ such that $F(M_\theta) = M'_{\theta'}$.*

Proof. ($\Leftarrow$) Put $\alpha = F|_{M_0/\theta}$. Then α has a unique homeomorphic extension $\bar{\alpha}\colon \beta(M_0/\theta) \to \beta(M_0'/\theta')$ (cf. [8, §44 Corollary 10]). Thus $\bar{\alpha}\colon \chi(M_0/\theta) \to \chi(M_0'/\theta')$ is a homeomorphism. Thus (M', θ') Satisfies Condition (A) and Condition (B). Consequently, $C(M) \times_\theta \mathbb{Z}_p \cong \mathcal{A}(M_\theta)$ and $C(M') \times_{\theta'} \mathbb{Z}_p \cong \mathcal{A}(M'_{\theta'})$ by Theorem 3.4. Clearly, $\Psi((a_{ij})_{p\times p}) = (a_{ij} \circ F)_{p\times p}$ gives a $*$–isomorphism from $\mathcal{A}(M'_{\theta'})$ onto $\mathcal{A}(M_\theta)$. The assertion follows.

($\Rightarrow$) Let Δ be the $*$–isomorphism from $\mathrm{D}(M', \theta')$ onto $\mathrm{D}(M, \theta)$. Let $a' \in \mathrm{D}(M_0', \theta')$ and put $a = \Delta(a')$. If $a \notin \mathrm{D}(M_0, \theta)$, we can pick $y_0 \in M_\theta$ such that $a(y_0) \neq 0$. Then $\sigma_{y_0,1} \circ \Delta$ is multiplicable on $\mathrm{D}(M', \theta')$. Thus, there exist $x_0 \in M'_{\theta'}$ and $k \in \{1, \cdots, p\}$ such that $\sigma_{y_0,1} \circ \Delta = \sigma_{x_0,k}$ and hence $\sigma_{y_0,1}(a) = \sigma_{x_0,k}(a') = 0$, a contradiction. So, Δ induces a $*$–isomorphism $\Delta_0\colon \mathrm{D}(M_0', \theta') \to \mathrm{D}(M_0, \theta)$ and so that $\Delta_0' = \Phi^{-1} \circ \Delta_0 \circ \Phi'^{-1}$ gives a $*$–isomorphism of $\mathrm{M}_p(C_0(M_0'/\theta'))$ onto $\mathrm{M}_p(C_0(M_0/\theta))$. Thus, we can find a homeomorphism $\alpha\colon M_0/\theta \to M_0'/\theta'$. This shows that (M', θ') satisfies Condition (A) and Condition (B) too.

Now we have $\mathcal{A}(M'_{\theta'}) \cong \mathcal{A}(M_\theta)$ via the $*$–isomorphism Θ by Theorem 3.4. Since $\Theta(f1_p)$ commutes with every element in $\mathcal{A}(M_\theta)$, $\forall\, f \in C(M'/\theta')$, it follows that there is $h \in C(M/\theta)$ such that $\Theta(f1_p) = h1_p$. Thus, $f \mapsto h$ yields a $*$–isomorphism from $C(M'/\theta')$ onto $C(M/\theta)$ so that there is a homeomorphism $F\colon M/\theta \to M'/\theta'$ such that $\Theta(f1_p) = (f \circ F)1_p = h1_p$.

Let ϕ be a character on $\mathcal{A}(M_\theta)$ with $\phi(1_p) = 1$. Since $e_1, \cdots, e_p \in \mathcal{A}(M_\theta)$ and $\sum\limits_{j=1}^{p} e_j = 1_p$, there is e_{i_0} such that $\phi(e_{i_0}) = 1$ and $\phi(e_j) = 0$, $j \neq i_0$. Without losing the generality, we may assume $i_0 = 1$. Then $f \mapsto \phi(fe_1)$ is a character on $C(M/\theta)$. Thus, there is $x_0 \in M/\theta$ such that $\phi(fe_1) = f(x_0)$, $\forall\, f \in C(M/\theta)$. For any $g \in C_0(M_0/\theta)$, let $B = (b_{ij})_{p\times p} \in \mathcal{A}(M_\theta)$ be given by $b_{p1} = g$ and $b_{ij} = 0$, $i \neq p$, $j \neq 1$. Then $B^*B = g^*g\, e_1$, $e_pB = B$ and hence $|g(x_0)|^2 = 0$, $\forall\, g \in C_0(M_0/\theta)$. Consequently, $x_0 \in M_\theta$.

For any $x \in M_\theta$, define the character ϕ_x on $\mathcal{A}(M_\theta)$ by $\phi_x((a_{ij})_{p\times p}) = a_{11}(x)$. Note that $\phi_x \circ \Theta^{-1}$ is a character on $\mathcal{A}(M'_{\theta'})$. So by above arguments, there is $y \in M'_{\theta'}$ such that

$$g(y) = \phi_x \circ \Theta(g1_p) = \phi_x((g \circ F)1_p) = g(F(x)), \quad \forall\, g \in C(M'/\theta').$$

This means that $F(M_\theta) \subset M'_{\theta'}$. Similarly, we have $F^{-1}(M'_{\theta'}) \subset M_\theta$. Thus, $F(M_\theta) = M'_{\theta'}$. □

Corollary 3.7. *Suppose that the pair (M, θ) satisfy Condition (A) and Condition (B). Let (M', θ') be another pair. If (M, θ) and (M', θ') are orbit equivalent, that is, there is a homeomorphism $F\colon M \to M'$ such that $F(O_\theta(x)) = O_{\theta'}(F(x)), \forall\, x \in M$, then $C(M) \times_\theta \mathbb{Z}_p \cong C(M') \times_{\theta'} \mathbb{Z}_p$.*

Proof. F induces a homeomorphism $\tilde{F}$ of M/θ onto M'/θ' given by $\tilde{F}(P(x)) = P'(F(x))$ by the assumption, where $P'\colon M' \to M'/\theta'$ is the canonical projective map. Obviously, $\tilde{F}(M_\theta) = M'_{\theta'}$ and $\overline{M_0'} = F(\overline{M_0}) = M'$. So $C(M) \times_\theta \mathbb{Z}_p \cong C(M') \times_{\theta'} \mathbb{Z}_p$ by Theorem 3.6. □

4 Some examples

Example 4.1. Consider (M, θ), where $M = \mathbf{S}^1$ and $\theta(z) = \bar{z}$, $\forall\, z \in \mathbf{S}^1$. Then

$$M_\theta = \{-1, 1\},\ \overline{M_0} = M,\ M_0/\theta \cong (-1, 1),\ M/\theta \cong [-1, 1].$$

(M, θ) satisfies Condition (A) and (B) for $\dim M = 1$. Thus $C(M) \times_\theta \mathbb{Z}_p \cong \mathcal{A}(M_\theta)$ by Theorem 3.4.

Define $\gamma(< z >) = \dfrac{x+1}{2}$ for $z = x + i\,y \in \mathbf{S}^1$, where $< z >= P(z) \in M/\theta$. Clearly, γ is a homeomorphism from M/θ onto $[0, 1]$ and $\gamma(M_\theta) = \{0, 1\}$. So

$$\begin{aligned}C(M) \times_\theta \mathbb{Z}_2 \cong & \mathcal{A}(\{0,1\}) \\ &= \{f \colon [0,1] \to \mathrm{M}_2(\mathbb{C}) \text{ continuous} \mid f(0),\, f(1) \text{ are diagonal}\}.\end{aligned}$$

Example 4.2. Let $M = \mathbf{S}^1 \times \mathbf{S}^1$ and $\theta(z_1, z_2) = (z_2, z_1)$, $\forall\, z_1, z_2 \in \mathbf{S}^1$. Then $M_\theta = \{(z, z) \mid z \in \mathbf{S}^1\} \cong \mathbf{S}^1$ and $\overline{M_0} = M$. Set

$$S = \{(z_1 + z_2, z_1 z_2) \mid z_1,\, z_2 \in \mathbf{S}^1\} \subset \mathbb{C} \times \mathbb{C}.$$

It is easy to check that S is a closed and bounded subset in $\mathbb{C} \times \mathbb{C}$, that is, S is compact. Define the continuous map $\xi \colon M/\theta \to S$ and $\beta \colon [0, 1] \times \mathbf{S}^1 \to S$, respectively, by $\xi(< z_1, z_2 >) = (z_1 + z_2, z_1 z_2)$ and $\beta(t, z) = (2zt, z^2)$, where $< z_1, z_2 >= P(z_1, z_2) \in M/\theta$. Then ξ and β are all homeomorphic (cf. [21, Exmple 4.3]). Therefore the homeomorphism $\delta = \beta^{-1} \circ \xi \colon M/\theta \to [0, 1] \times \mathbf{S}^1$ sends M_θ to $\{1\} \times \mathbf{S}^1$.

Claim 1. (M, θ) satisfies Condition (A) and Condition (B).

Since $\delta(M_\theta) = \{1\} \times \mathbf{S}^1$, $\delta(M_0/\theta) = [0, 1) \times \mathbf{S}^1$. Let $i \colon \{1\} \times \mathbf{S}^1 \to [0, 1] \times \mathbf{S}^1$ be the inclusion. Then i is homotopic equivalence map. Since $\mathrm{H}^2([0, 1] \times \mathbf{S}^1) \cong 0$, it follows from the exact sequence of the reduced coholomological groups (cf. [18]) that

$$\tilde{\mathrm{H}}^1([0,1] \times \mathbf{S}^1, \mathbb{Z}) \xrightarrow{i^*} \mathrm{H}^1(\{1\} \times \mathbf{S}^1, \mathbb{Z}) \longrightarrow \tilde{\mathrm{H}}^2(([0,1) \times \mathbf{S}^1)^+, \mathbb{Z}) \longrightarrow 0.$$

So $\tilde{\mathrm{H}}^2(([0, 1) \times \mathbf{S}^1)^+, \mathbb{Z}) \cong 0$ and consequently, (M, θ) satisfies Condition (A).

Noting that $[0, 1) \times \mathbf{S}^1 \cong [0, +\infty) \times \mathbf{S}^1$, we have $\mathrm{H}^2(\chi([0, 1) \times \mathbf{S}^1), \mathbb{Z}) \cong 0$ by [6, Corollary 4.7]. So (M, θ) satisfies Condition (B).

Let $M' = [0, 2] \times \mathbf{S}^1$ and $\theta'(t, z) = (2 - t, z)$. Then $M'_{\theta'} = \{1\} \times \mathbf{S}^1$. Define the homeomorphism $\delta' \colon M'/\theta' \to [0, 1] \times \mathbf{S}^1$ by $\delta'(< t, z >) = (1 - |1 - t|, z)$.

Claim 2. $C(M) \times_\theta \mathbb{Z}_2 \cong C(\mathbf{S}^1) \otimes \mathcal{A}(1)$, where

$$\mathcal{A}(1) = \left\{ \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathrm{M}_2(C([0, 1])) \,\middle|\, f_{12}(1) = f_{21}(1) = 0 \right\}.$$

Since $M/\theta \cong [0, 2] \times \mathbf{S}^1/\theta'$ via $\delta'^{-1} \circ \delta$ and $(\delta'^{-1} \circ \delta)(M_\theta) = M'_{\theta'}$, it follows from Theorem 3.6 that

$$C(M) \times_\theta \mathbb{Z}_2 \cong C(M') \times_{\theta'} \mathbb{Z}_2 \cong C(\mathbf{S}^1) \otimes (C([0, 2]) \times_{\theta_1} \mathbb{Z}_2),$$

where $\theta_1 \colon [0, 2] \to [0, 2]$ given by $\theta_1(t) = 2 - t$.

Note that $[0, 2]/\theta_1 \cong [0, 1]$ via $< t > \mapsto |1 - t|$, $\forall\, t \in [0, 1]$. We have $C([0, 2]) \times_{\theta_1} \mathbb{Z}_2 \cong \mathcal{A}(1)$ by Theorem 3.4.

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