

A METHOD OF CONSTRUCTING BRAIDED HOPF ALGEBRAS

Tianshui Ma[†], Shuanhong Wang^{*†} and Shaoxian Xu

Abstract

Let A and B be two Hopf algebras and $R \in \text{Hom}(B \otimes A, A \otimes B)$, the twisted tensor product Hopf algebra $A\#_R B$ was introduced by S. Caenepeel et al in [3] and further studied in our recent work [6]. In this paper we give the necessary and sufficient conditions for $A\#_R B$ to be a Hopf algebra with a projection. Furthermore, a braided Hopf algebra \bar{A} is constructed by twisting the multiplication of A through a (γ, R) -pair (A, B) . Finally we give a method to construct Radford's biproduct directly by defining the module action and comodule action from the twisted tensor biproduct.

1 Introduction and Preliminaries

In Hopf algebra theory, one of the celebrated result is the Radford's biproducts construction [10] which is equivalent to a Hopf algebra in the Yetter-Drinfel'd category over some Hopf algebra and is related to the classification of finite-dimensional pointed Hopf algebras [1]. A Hopf algebra in the Yetter-Drinfel'd category is called *braided Hopf algebra*. Drinfel'd [5] observed that for a finite dimensional quasitriangular Hopf algebra H , its double $D(H)$ is a Hopf algebra with a projection. Majid [8] proved that the converse of Drinfel'd's result also holds, and computed on H^* the braided Hopf algebra structure associated to this projection [10]. Other references about braided Hopf algebras are [12, 14].

Let A and B be two Hopf algebras, $\sigma : A \otimes B \rightarrow k$ an invertible skew pair, then we have Doi-Takeuchi double $A \bowtie_\sigma B$ [4]. In [2] Beattie and Bulacu gave the necessary and sufficient conditions for $A \bowtie_\sigma B$ to be a bialgebra with a projection generalizing the Majid's result in [8]. While the twisted tensor product $A\#_R B$ is a generalization of Doi-Takeuchi double, it is a natural question that when the twisted

*Corresponding author.

[†]Supported by the NNSF of CHINA (10871042) and NSF of Jiangsu Province (BK2009258).
2010 *Mathematics Subject Classifications*. 16W30

Key words and Phrases. Braided Hopf algebra; Yetter-Drinfel'd category; Twisted tensor product.

Received: January 20, 2010.

Communicated by Dragan S. Djordjević.

tensor product $A\#_R B$ is a bialgebra with a projection and whether we can find a Hopf algebra \bar{A} in ${}^B_B\mathcal{YD}$ (i.e. a braided Hopf algebra \bar{A}) such that $\bar{A} \#_B B \cong A\#_R B$ as Hopf algebra by twisting the multiplication on A . In this note we mainly give the answers to the above questions. Some relevant references on the twisting (or “deformation”) are [2, 8, 7, 12, 13, 14].

The present paper is organized as follows. In Section 1, we review some known notions of twisted tensor products, Yetter-Drinfel’d categories, braided Hopf algebra and Radford’s biproduct. Let A and B be two Hopf algebras and $R \in \text{Hom}(B \otimes A, A \otimes B)$. In section 2 we obtain the necessary and sufficient conditions for twisted tensor product $A\#_R B$ to be a bialgebra with a projection (see Theorem 2.3) which generalizes the Beattie-Bulacu’s results [2]. Furthermore we give the concept of (γ, R) -pair (A, B) by using twisted tensor product Hopf algebra $A\#_R B$ (see Definition 2.5). Section 3 is devoted to constructing a braided Hopf algebra \bar{A} by twisting the multiplication of A through a (γ, R) -pair (A, B) such that $\bar{A} \#_B B \cong A\#_R B$ as Hopf algebras (see Theorem 3.2, Theorem 3.3). In last section another braided Hopf algebra is derived directly by defining the module action and comodule action from the twisted tensor biproduct (see Theorem 4.3).

Throughout the paper, we follow the definitions and terminologies in Montgomery’s book [9] and all algebraic systems are supposed to be over the field k . Let C be a coalgebra. Then we use the simple Sweedler’s notation for the comultiplication [11]: $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$. We denote by ${}^C\mathcal{M}$ the category of left C -comodules and for any $V \in {}^C\mathcal{M}$, we will use a simple Sweedler’s notations: $\rho(v) = v_{-1} \otimes v_0$ for all $v \in V$. Given a k -space M , we write i_M for the identity map on M .

1.1. The twisted tensor product Hopf algebras.

Let A and B be two algebras and suppose that one has a linear map $R : B \otimes A \longrightarrow A \otimes B$. Then $A\#_R B$ is defined to be a vector space $A \otimes B$ with the product defined by

$$m_{A\#_R B} = (m_A \otimes m_B)(i_A \otimes R \otimes i_B)$$

or

$$(a\#_R b)(a'\#_R b') = aa'_R\#_R b_R b'. \quad (1. 1)$$

for $a, a' \in A$ and $b, b' \in B$, where we write $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$. If $A\#_R B$ is an associative algebra with unit $1_A\#_R 1_B$, we call $A\#_R B$ a *twisted tensor product algebra*. If A and B are Hopf algebras, then twisted tensor product algebra $A\#_R B$ is called a *twisted tensor product Hopf algebra* if $A\#_R B$ is a Hopf lagbera with the usual tensor coproduct.

Let A and B be two algebras and suppose that we have given a linear map $R : B \otimes A \longrightarrow A \otimes B$. Then by [3] or [6] we have that $A\#_R B$ is a twisted tensor product algebra is equivalent to the following conditions hold:

$$R(m_B \otimes i_A) = (i_A \otimes m_B)(R \otimes i_B)(i_B \otimes R), \quad (1. 2)$$

$$R(i_B \otimes m_A) = (m_A \otimes i_B)(i_A \otimes R)(R \otimes i_A), \quad (1. 3)$$

$$R(b \otimes 1_A) = 1_A \otimes b; \quad R(1_B \otimes a) = a \otimes 1_B. \quad (1. 4)$$

Let A and B be two Hopf algebras, $\sigma : A \otimes B \longrightarrow k$ an invertible skew pair, then we have Doi-Takeuchi double $A \bowtie_{\sigma} B$ [4]. If we define the linear map $R : B \otimes A \longrightarrow A \otimes B$ by $R(b \otimes a) = \sigma(a_1, b_1)a_2 \otimes b_2\sigma^{-1}(a_3, b_3)$, then we get $A \bowtie_{\sigma} B$ with the product given by

$$(a\#b)(a'\#b') = \sigma(a'_1, b_1)aa'_2 \otimes b_2b'\sigma^{-1}(a'_3, b_3)$$

for $a, a' \in A$ and $b, b' \in B$.

Let A and B be Hopf algebras, then the twisted tensor product algebra with the usual tensor product coalgebra structure is a Hopf algebra if and only if R is a coalgebra map, i.e. for $\forall a \in A, b \in B$ and $R = r$

$$a_{R1} \otimes b_{R1} \otimes a_{R2} \otimes b_{R2} = a_{1R} \otimes b_{1R} \otimes a_{2r} \otimes b_{2r} \quad (1.5)$$

$$\varepsilon_A(a_R)\varepsilon_B(b_R) = \varepsilon_A(a)\varepsilon_B(b). \quad (1.6)$$

If the extra condition

$$\varepsilon_A(a_R)b_R = \varepsilon_A(a)b \quad (1.7)$$

holds, then we call $A\#_R B$ a *conormal twisted tensor product Hopf algebra*.

If $A\#_R B$ is a conormal twisted tensor product Hopf algebra, then by Eq.(1.5) \sim Eq.(1.7), for all $a \in A$ and $b \in B$, we have

$$b_R \otimes a_R = b_1\varepsilon_B(b_{2R}) \otimes a_R = \varepsilon_B(b_{1R})b_2 \otimes a_R; \quad (1.8)$$

$$a_{R1} \otimes a_{R2} \otimes b_R = a_{1R}\varepsilon_B(b_{1R}) \otimes a_{2r} \otimes b_{2r}; \quad (1.9)$$

$$b_{R1} \otimes a_R \otimes b_{R2} = b_1 \otimes a_R \otimes b_{2R}. \quad (1.10)$$

1.2. Braided Hopf algebras and Radford's biproduct.

Let B be a Hopf algebra with a bijective antipode S . Then a left Yetter-Drinfel'd module (V, \cdot, ρ) is a left B -module and a left B -comodule such that the following condition

$$(b_1 \cdot v)_{-1}b_2 \otimes (b_1 \cdot v)_0 = b_1v_{-1} \otimes b_2 \cdot v_0 \quad (1.11)$$

is satisfied for all $b \in B$ and $v \in V$.

We denote the category of left Yetter-Drinfel'd modules and the morphisms that are both B -linear and B -colinear by ${}^B_B\mathcal{YD}$. Then ${}^B_B\mathcal{YD}$ is a braided monoidal category with the braiding τ given by

$$\tau : M \otimes N \longrightarrow N \otimes M; \quad m \otimes n \mapsto m_{(-1)} \cdot n \otimes m_0$$

for any $m \in M \in {}^B_B\mathcal{YD}$ and $n \in N \in {}^B_B\mathcal{YD}$.

A Hopf algebra in ${}^B_B\mathcal{YD}$ is called a *braided Hopf algebra* [1].

The biproduct construction associated a smash product and a smash coproduct of Hopf algebras is due to D. E. Radford [10].

Let B be a Hopf algebra, A both an algebra and a coalgebra such that there exists a linear map $S_A : A \longrightarrow A$ satisfying $m_A(S_A \otimes i_A)\Delta_A = i_A$ and $m_A(i_A \otimes S_A)\Delta_A = i_A$. Then the following statements are equivalent:

1. $(A \times B, m_{A \# B}, 1_A \otimes 1_B, \Delta_{A \times B}, \varepsilon_A \otimes \varepsilon_B, S_{A \times B})$ is a Hopf algebra.
2. The following conditions (1) ~ (7) hold:
 - (1) $A \# B$ is a smash product;
 - (2) $A \times B$ is a smash coproduct;
 - (3) A is a left B -module coalgebra;
 - (4) A is a left B -comodule algebra;
 - (5) $\Delta_A(aa') = a_1(a_{2(-1)} \cdot a'_1) \otimes a_0 a'_2$, $\Delta_A(1_A) = 1_A \otimes 1_A$; (1. 12)
 - (6) $(b_1 \cdot a)_{(-1)} b_2 \otimes (b_1 \cdot a)_0 = b_1 a_{(-1)} \otimes b_2 \cdot a_0$;
 - (7) ε_A is an algebra map.
3. A is a Hopf algebra in Yetter-Drinfel'd ${}^B_B \mathcal{YD}$, i.e., A is a braided Hopf algebra.

2 (γ, R) -compatible pair (A, B)

In this section, we obtain the necessary and sufficient conditions for the twisted tensor product to be a bialgebra with a projection which generalizes Beattie-Bulacu's result in [2]. Furthermore the notion of (γ, R) -compatible pair is given.

Lemma 2.1. Let A, B be Hopf algebras and $A \#_R B$ a twisted tensor product Hopf algebra. Let D be any Hopf algebra and $\alpha : A \rightarrow D$ and $\beta : B \rightarrow D$ Hopf algebra maps. Then the map $F : A \#_R B \rightarrow D$ defined by $F(a \otimes b) = \alpha(a)\beta(b)$ is a Hopf algebra map if

$$\alpha(a_R)\beta(b_R) = \beta(b)\alpha(a) \quad (2. 1)$$

for all $a \in A$ and $b \in B$.

Proof. Since $A \#_R B = A \otimes B$ as coalgebras and the multiplication m_D is a coalgebra map, F is a coalgebra map.

In what follows, we compute F is an algebra map as follows:

$$\begin{aligned} F((a \otimes b)(a' \otimes b')) &= F(aa'_R \otimes b_R b') = \alpha(a)\alpha(a'_R)\beta(b_R)\beta(b') \\ &\stackrel{(2.1)}{=} \alpha(a)\beta(b)\alpha(a')\beta(b') = F(a \otimes b)F(a' \otimes b') \end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$. ■

Example 2.2. Let $R(b \otimes a) = \tau(a_1, b_1)a_2 \otimes b_2\tau^{-1}(a_3, b_3)$. Then Lemma 2.1 is exactly the Proposition 2.4 in [4].

Theorem 2.3. Let A, B be Hopf algebras and $A \#_R B$ a twisted tensor product Hopf algebra. Then we have the following bialgebra morphisms:

$$i : A \rightarrow A \#_R B, \quad i(a) = a \otimes 1; \quad j : B \rightarrow A \#_R B, \quad j(b) = 1 \otimes b$$

for all $a \in A$ and $b \in B$.

(1) There exists a bialgebra projection $\pi : A\#_R B \longrightarrow A$ such that $\pi \circ i = i_A$ if and only if there is a bialgebra map $\gamma : B \longrightarrow A$ such that

$$\gamma(b)a = a_R\gamma(b_R) \quad (2. 2)$$

for all $a \in A$ and $b \in B$.

(2) There exists a bialgebra projection $\pi' : A\#_R B \longrightarrow B$ such that $\pi' \circ j = j_B$ if and only if there is a bialgebra map $\gamma' : A \longrightarrow B$ such that

$$b\gamma'(a) = \gamma'(a_R)b_R$$

for all $a \in A$ and $b \in B$.

Proof. We only verify that Part 1 is satisfied. Similarly for Part 2.

Assume that there is a bialgebra map $\gamma : B \longrightarrow A$ such that Eq.(2.2) holds. Define the map $\pi : A\#_R B \longrightarrow A$ as $\pi(a \otimes b) = a\gamma(b)$ for all $a \in A$ and $b \in B$. Then

$$\pi \circ i(a) = \pi(a \otimes 1) = a\gamma(1) = a$$

for all $a \in A$. Moreover, set $D = A$, $\alpha = i_A$ and $\beta = \gamma : B \longrightarrow A$ in Lemma 2.1, then by Eq.(2.2) we conclude that π is an algebra map.

On the other hand, given a bialgebra projection $\pi : A\#_R B \longrightarrow A$. Define the map $\gamma : B \longrightarrow A$ by $\gamma(b) = \pi \circ j(b) = \pi(1 \otimes b)$ for all $b \in B$. Then γ is a bialgebra map since π and j are. It is easy to verify that Eq.(2.2) holds. In fact, for all $a \in A$ and $b \in B$, we have

$$\begin{aligned} \gamma(b)a &= (\pi \circ j(b))a = \pi(1 \otimes b)\pi(a \otimes 1) = \pi((1 \otimes b)(a \otimes 1)) \\ &= \pi(a_R \otimes b_R) = a_R\gamma(b_R). \end{aligned} \quad \blacksquare$$

Example 2.4. Let $R(b \otimes a) = \sigma(a_1, b_1)a_2 \otimes b_2\sigma^{-1}(a_3, b_3)$. Then Proposition 3.1 in [2] is obtained.

Definition 2.5. Suppose that A and B are Hopf algebras and $A\#_R B$ is a conormal twisted tensor product Hopf algebra. If $\gamma : A \longrightarrow B$ is a Hopf algebra morphism such that

$$b\gamma(a) = \gamma(a_R)b_R \quad (2. 3)$$

holds, then we call (A, B) a (γ, R) -compatible pair.

Remark 2.6. 1. If (A, B) is a (γ, R) -compatible pair, then we have the following useful equations:

$$b_1\gamma(a_R)(S_B(b_2))_R = \varepsilon_B(b)\gamma(a); \quad (2. 4)$$

$$bS_B(\gamma(a)) = \gamma((S_A(a))_R)b_R. \quad (2. 5)$$

2. If (A, B) is a (γ, R) -compatible pair, then there exists a Hopf algebra \bar{A} in ${}^B_B\mathcal{YD}$ such that $\bar{A} \#_R B \cong A\#_R B$ as Hopf algebra by Theorem 2.3.

Proposition 2.7. Let (A, B) be a (γ, R) -compatible pair and (V, ρ) a left B -comodule. Set $\bar{\rho} : V \longrightarrow B \otimes V$ such that $\bar{\rho}(v) = \gamma(v_{-1}) \otimes v_0$ for all $v \in V$, then $(V, \bar{\rho})$ is a left B -comodule.

Proof. It is obvious. \blacksquare

3 Construct braided Hopf algebra through (γ, R) -compatible pair

In this section we construct a braided Hopf algebra \bar{A} through (γ, R) -compatible pair by twisting the multiplication of A such that $\bar{A} \#_{\times} B \cong A \#_R B$ as Hopf algebras.

Lemma 3.1. Let (A, B) be a (γ, R) -compatible pair. For all $a \in A$ and $b \in B$, define

$$\rightharpoonup: B \otimes A \longrightarrow A, \quad b \rightharpoonup a = \varepsilon_B(b_R)a_R;$$

and

$$\rho: A \longrightarrow B \otimes A, \quad \rho(a) = a_{-1} \otimes a_0 = \gamma(a_1)S_B(\gamma(a_3)) \otimes a_2,$$

then

- (1) (A, \rightharpoonup) is a left B -module coalegebra;
- (2) (A, ρ) is a left B -comodule coalegebra;
- (3) $(A, \rightharpoonup, \rho)$ is a Yetter-Drinfel'd module if the following condition holds:

$$(S_A(a))_R \otimes b_R = S_A(a_R) \otimes b_R. \quad (3.1)$$

Proof. (1) Since

$$b \rightharpoonup (b' \rightharpoonup a) = \varepsilon_B(b_r)\varepsilon_B(b'_R)a_{Rr} \stackrel{(1.2)}{=} \varepsilon_B((bb')_R)a_R = (bb') \rightharpoonup a$$

and $1_B \rightharpoonup a = \varepsilon_B(1_{BR})a_R \stackrel{(1.4)}{=} a$, (A, \rightharpoonup) is a left B -module.

While

$$\begin{aligned} (b_1 \rightharpoonup a_1) \otimes (b_2 \rightharpoonup a_2) &= \varepsilon_B(b_{1R})a_{1R} \otimes \varepsilon_B(b_{2R})a_{2R} \\ &\stackrel{(1.9)}{=} \varepsilon_B(b_R)a_{R1} \otimes a_{R2} = (b \rightharpoonup a)_1 \otimes (b \rightharpoonup a)_2 \end{aligned}$$

and $\varepsilon_A(b \rightharpoonup a) = \varepsilon_A(a_R)\varepsilon_B(b_R) \stackrel{(1.6)}{=} \varepsilon_A(a)\varepsilon_B(b)$, then (A, \rightharpoonup) is a left B -module coalgebra.

(2) It is straightforward.

(3) First by Eq.(2.3) and Eq.(3.1), we have

$$b\gamma(S_A(a)) = \gamma((S_A(a))_R)b_R = \gamma(S_A(a_R))b_R. \quad (3.2)$$

Next we only need to check Eq.(1.11) is satisfied. In fact, for all $a \in A, b \in B$ and

$R = r = \overline{R}$, one can obtain

$$\begin{aligned}
b_1 a_{-1} \otimes (b_2 \rightharpoonup a_0) &= b_1 \gamma(a_1) S_B(\gamma(a_3)) \otimes \varepsilon_B(b_{2R}) a_{2R} \\
&\stackrel{(2.3)}{=} \gamma(a_{1\overline{R}}) b_{1\overline{R}} S_B(\gamma(a_3)) \otimes \varepsilon_B(b_{2R}) a_{2R} \\
&\stackrel{(1.8)}{=} \gamma(a_{1\overline{R}}) b_2 \gamma(S_A(a_3) \varepsilon_B(b_{1\overline{R}}) \otimes \varepsilon_B(b_{3R}) a_{2R}) \\
&\stackrel{(1.8)}{=} \gamma(a_{1\overline{R}}) b_3 \gamma(S_A(a_3) \varepsilon_B(b_{1\overline{R}}) \otimes \varepsilon_B(b_{2R}) a_{2R}) \\
&\stackrel{(3.2)}{=} \gamma(a_{1\overline{R}}) \gamma(S_A(a_{3r}) b_{3r} \varepsilon_B(b_{1\overline{R}}) \otimes \varepsilon_B(b_{2R}) a_{2R}) \\
&\stackrel{(1.8)}{=} \gamma(a_{1\overline{R}}) S_B(\gamma(a_{3r}) b_4 \varepsilon_B(b_{1\overline{R}}) \varepsilon_B(b_{3r}) \otimes \varepsilon_B(b_{2R}) a_{2R}) \\
&\stackrel{(1.9)}{=} \gamma(a_{1\overline{R}}) S_B(\gamma(a_{2r2}) b_3 \varepsilon_B(b_{1\overline{R}}) \varepsilon_B(b_{2r}) \otimes \varepsilon_B(b_{2R}) a_{2r1}) \\
&\stackrel{(1.9)}{=} \gamma(a_{R1}) S_B(\gamma(a_{r3}) b_2 \varepsilon_B(b_{1r}) \otimes a_{r2}) \\
&= (b_1 \rightharpoonup a)_{-1} b_2 \otimes (b_1 \rightharpoonup a)_0.
\end{aligned}$$

Thus $(A, \rightharpoonup, \rho)$ is a Yetter-Drinfel'd module. \blacksquare

Theorem 3.2. Let (A, B) be a (γ, R) -compatible pair. Assume that Eq.(3.1) holds. Then there exists a bialgebra \overline{A} in ${}^B_B\mathcal{YD}$, where $\overline{A} = A$ as a vector space over k , the left B -comodule structure and left B -module structure of \overline{A} are the same as in Lemma 3.1, and the comultiplication, counit and unit of \overline{A} are the same as of A , with new multiplication structure

$$\overline{m}(a \otimes a') = a \cdot_R a' = a_1 a'_{R1} \varepsilon_B((S_B(\gamma(a_2)))_R)$$

for all $a, a' \in A$.

Proof. By Lemma 3.1 we know that \overline{A} is in ${}^B_B\mathcal{YD}$, $(\overline{A}, \rightharpoonup)$ is a left B -module coalgebra and (\overline{A}, ρ) is a left B -comodule coalgebra.

Next we take four steps to finish the proof of the theorem.

Step 1. $(\overline{A}, \overline{m})$ is a k -algebra. In fact, for all $a, a', a'' \in \overline{A}$, we have

$$\begin{aligned}
(a \cdot_R a') \cdot_R a'' &= (a_1 a'_{R1})_1 a''_{r1} \varepsilon_B((S_B(\gamma((a_1 a'_{R2})))_r) \varepsilon_B((S_B(\gamma(a_2)))_R)) \\
&= a_1 a'_{R1} a''_{r1} \varepsilon_B((S_B(\gamma(a_2 a'_{R2})))_r) \varepsilon_B((S_B(\gamma(a_3)))_R) \\
&= a_1 a'_{R1} a''_{r1} \varepsilon_B((S_B(\gamma(a'_{R2})) S_B(\gamma(a_2)))_r) \varepsilon_B((S_B(\gamma(a_3)))_R) \\
&\stackrel{(1.8)}{=} a_1 a'_{R1} a''_{r1} \varepsilon_B((S_B(\gamma(a'_{R2})) (S_B(\gamma(a_2)))_R)_r) \\
&\stackrel{(1.9)}{=} a_1 a'_{1\overline{R}} a''_{r\overline{r}} \varepsilon_B((S_B(\gamma(a'_{2R})) (S_B(\gamma(a_2)))_R)_r) \varepsilon_B((S_B(\gamma(a_3)))_{\overline{R}}) \\
&\stackrel{(2.4)}{=} a_1 a'_{1\overline{R}} a''_{r\overline{r}} \varepsilon_B((S_B(\gamma(a_2)) S_B(\gamma(a'_{2'})))_r) \varepsilon_B((S_B(\gamma(a_3)))_{\overline{R}}) \\
&\stackrel{(1.2)}{=} a_1 a'_{1\overline{R}} a''_{r\overline{r}} \varepsilon_B((S_B(\gamma(a_2)))_{\overline{r}}) \varepsilon_B((S_B(\gamma(a'_{2'})))_r) \varepsilon_B((S_B(\gamma(a_3)))_{\overline{R}}) \\
&\stackrel{(1.8)}{=} a_1 a'_{1\overline{R}} a''_{r\overline{r}} \varepsilon_B((S_B(\gamma(a_2)))_{\overline{Rr}}) \varepsilon_B((S_B(\gamma(a'_{2'})))_r) \\
&\stackrel{(1.2)}{=} a_1 (a'_{1r})_R \varepsilon_B((S_B(\gamma(a_2)))_R) \varepsilon_B((S_B(\gamma(a'_{2'})))_r) \\
&= a \cdot_R (a' \cdot_R a'')
\end{aligned}$$

One can easily check that the unit of \bar{A} is the same as the unit of A .

Step 2. (\bar{A}, \dashv) is a left B -module algebra. Indeed, for all $a, a' \in \bar{A}$ and $b \in B$, one can get

$$\begin{aligned}
(b_1 \dashv a) \cdot_R (b_2 \dashv a') &= \varepsilon_B(b_{1R})a_{R1}a'_{r\bar{R}}\varepsilon_B((S_B(\gamma(a_{R2})))_{\bar{R}})\varepsilon_B(b_{2r}) \\
&\stackrel{(1.8)}{=} a_{R1}a'_{r\bar{R}}\varepsilon_B((S_B(\gamma(a_{R2})))_{\bar{R}})\varepsilon_B(b_{Rr}) \\
&\stackrel{(1.2)}{=} a_{R1}a'_r\varepsilon_B((S_B(\gamma(a_{R2}))b_R)_r) \\
&\stackrel{(1.9)}{=} a_{1R}\varepsilon_B(b_{1R})a'_r\varepsilon_B((S_B(\gamma(a_{2\bar{R}}))b_{2\bar{R}})_r) \\
&\stackrel{(3.2)}{=} a_{1R}\varepsilon_B(b_{1R})a'_r\varepsilon_B((b_2S_B(\gamma(a_2)))_r) \\
&\stackrel{(1.2)}{=} a_{1R}\varepsilon_B(b_{1R})a'_{r\bar{R}}\varepsilon_B(b_{2\bar{R}})\varepsilon_B((S_B(\gamma(a_2)))_r) \\
&\stackrel{(1.3)}{=} (a_1a'_r)_{R\varepsilon_B(b_R)}\varepsilon_B((S_B(\gamma(a_2)))_r) \\
&= b \dashv (a \cdot_R a')
\end{aligned}$$

and

$$b \dashv 1_{\bar{A}} = \varepsilon_B(b_R)1_{\bar{A}R} \stackrel{(1.4)}{=} \varepsilon_B(b)1_{\bar{A}}.$$

Step 3. We show that (\bar{A}, ρ) is a left B -comodule algebra. In fact, for all $a, a' \in \bar{A}$

$$\begin{aligned}
a_{-1}a'_{-1} \otimes a_0 \cdot_R a'_0 &= \gamma(a_1)S_B(\gamma(a_4))\gamma(a'_1)S_B(\gamma(a'_3)) \otimes a_2a'_{2R}\varepsilon_B((S_B(\gamma(a_3)))_R) \\
&\stackrel{(2.3)}{=} \gamma(a_1)\gamma(a'_{1r})(S_B(\gamma(a_4)))_rS_B(\gamma(a'_3)) \otimes a_2a'_{2R}\varepsilon_B((S_B(\gamma(a_3)))_R) \\
&\stackrel{(3.2)}{=} \gamma(a_1)\gamma(a'_{1r})\gamma(S_A(a'_{3\bar{R}}))(S_B(\gamma(a_4)))_{r\bar{R}} \otimes a_2a'_{2R}\varepsilon_B((S_B(\gamma(a_3)))_R) \\
&\stackrel{(1.8)}{=} \gamma(a_1)\gamma(a'_{1r})\gamma(S_A(a'_{3\bar{R}}))(S_B(\gamma(a_3)))_{4\varepsilon_B((S_B(\gamma(a_3)))_{2\bar{R}})} \\
&\quad \varepsilon_B((S_B(\gamma(a_3)))_{3r}) \otimes a_2a'_{2R}\varepsilon_B((S_B(\gamma(a_3)))_{1R}) \\
&\stackrel{(1.8)}{=} \gamma(a_1)\gamma(a'_{1r})\gamma(S_A(a'_{3\bar{R}}))(S_B(\gamma(a_3)))_{4\varepsilon_B((S_B(\gamma(a_3)))_{3\bar{R}})} \\
&\quad \varepsilon_B((S_B(\gamma(a_3)))_{1r}) \otimes a_2a'_{2R}\varepsilon_B((S_B(\gamma(a_3)))_{2R}) \\
&\stackrel{(1.9)}{=} \gamma(a_1)\gamma(a'_{1r1})\gamma(S_A(a'_{2\bar{R}}))(S_B(\gamma(a_3)))_{3\varepsilon_B((S_B(\gamma(a_3)))_{2\bar{R}})} \\
&\quad \otimes a_2a'_{1r2}\varepsilon_B((S_B(\gamma(a_3)))_{1r}) \\
&\stackrel{(1.9)}{=} \gamma(a_1)\gamma(a'_{r1})\gamma(S_A(a'_{r3}))(S_B(\gamma(a_3)))_2 \otimes a_2a'_{r2}\varepsilon_B((S_B(\gamma(a_3)))_{1r}) \\
&= \gamma(a_1a'_{r1})S_B(\gamma(a_3a'_{r3})) \otimes a_2a'_{r2}\varepsilon_B((S_B(\gamma(a_4)))_r) \\
&= (a \cdot_R a')_{-1} \otimes (a \cdot_R a')_0
\end{aligned}$$

and

$$1_{\bar{A}-1} \otimes 1_{\bar{A}0} = \gamma(1_{\bar{A}})S_B(\gamma(1_{\bar{A}})) \otimes 1_{\bar{A}} = 1_B \otimes 1_{\bar{A}}.$$

Step 4. Since for all $a, a' \in \bar{A}$,

$$\begin{aligned}
a_1 \cdot_R (a_{2-1} \rightarrow a'_1) \otimes a_{20} \cdot_R a'_2 &= a_1 a'_{1Rr} \varepsilon_B((S_B(\gamma(a_2)))_r) \varepsilon_B((\gamma(a_3) S_B(\gamma(a_6))))_R) \\
&\quad \otimes a_4 a'_{2\bar{R}} \varepsilon_B((S_B(\gamma(a_5)))_{\bar{R}}) \\
&\stackrel{(1.2)}{=} a_1 a'_{1R} \varepsilon_B((S_B(\gamma(a_2))) \gamma(a_3) S_B(\gamma(a_6)))_R) \\
&\quad \otimes a_4 a'_{2\bar{R}} \varepsilon_B((S_B(\gamma(a_5)))_{\bar{R}}) \\
&= a_1 a'_{1R} \varepsilon_B((S_B(\gamma(a_4)))_R) \otimes a_2 a'_{2\bar{R}} \varepsilon_B((S_B(\gamma(a_3)))_{\bar{R}}) \\
&\stackrel{(1.9)}{=} a_1 a'_{R1} \otimes a_2 a'_{R2} \varepsilon_B((S_B(\gamma(a_3)))_R) \\
&= \Delta(a \cdot_R a'),
\end{aligned}$$

Then Eq.(1.12) is satisfied. Therefore (\bar{A}, \bar{m}) is a bialgebra in ${}^B_B\mathcal{YD}$. \blacksquare

Theorem 3.3. Under the assumptions of Theorem 3.2, \bar{A} is a Hopf algebra in ${}^B_B\mathcal{YD}$ with an antipode \bar{S} , where

$$\bar{S}(a) = \varepsilon((\gamma(a_1))_R) (S_A(a_2))_R$$

for all $a \in A$.

Proof. Firstly, we will prove \bar{S} is a morphism in ${}^B_B\mathcal{YD}$.

$$\begin{aligned}
b \rightarrow \bar{S}(a) &= (S_A(a_2))_{Rr} \varepsilon_B((\gamma(a_1))_R) \varepsilon_B(b_r) \\
&\stackrel{(1.2)}{=} (S_A(a_2))_R \varepsilon_B((b\gamma(a_1))_R) \\
&\stackrel{(2.3)}{=} (S_A(a_2))_R \varepsilon_B((\gamma(a_{1r}) b_r)_R) \\
&\stackrel{(1.2)}{=} (S_A(a_2))_{\bar{R}\bar{R}} \varepsilon_B((\gamma(a_{1r}))_{\bar{R}}) \varepsilon_B(b_{rR}) \\
&\stackrel{(3.1)}{=} (S_A(a_{2R}))_{\bar{R}} \varepsilon_B((\gamma(a_{1r}))_{\bar{R}}) \varepsilon_B(b_{rR}) \\
&\stackrel{(1.8)}{=} (S_A(a_{2R}))_{\bar{R}} \varepsilon_B((\gamma(a_{1r}))_{\bar{R}}) \varepsilon_B(b_{1r}) \varepsilon_B(b_{2R}) \\
&\stackrel{(1.9)}{=} (S_A(a_{R2}))_{\bar{R}} \varepsilon_B((\gamma(a_{R1}))_{\bar{R}}) \varepsilon_B(b_R) \\
&= \bar{S}(b \rightarrow a),
\end{aligned}$$

hence \bar{S} is a left B -module morphism.

For all $a \in \bar{A}$, since

$$\begin{aligned}
\rho(\bar{S}(a)) &= \gamma((S_A(a_2))_{R1}) \gamma(S_A((S_A(a_2))_{R3})) \otimes (S_A(a_2))_{R2} \varepsilon_B((\gamma(a_1))_R) \\
&\stackrel{(1.9)}{=} \gamma((S_A(a_4))_{r1}) S_B(\gamma((S_A(a_3))_R)) \otimes (S_A(a_4))_{r2} \varepsilon_B((\gamma(a_1))_r) \varepsilon_B((\gamma(a_2))_R) \\
&\stackrel{(3.1)}{=} \gamma((S_A(a_4))_{r1}) \gamma((S_A(S_A(a_3))_R)) \varepsilon_B((\gamma(a_2))_R) \otimes (S_A(a_4))_{r2} \varepsilon_B((\gamma(a_1))_r) \\
&\stackrel{(2.4)}{=} \gamma((S_A(a_4))_{r1}) (\gamma(a_2))_{R1} \gamma((S_A(S_A(a_3))_{\bar{R}\bar{R}})) (S_B((\gamma(a_2))_{R2}))_{\bar{R}} \\
&\quad \otimes (S_A(a_4))_{r2} \varepsilon_B((\gamma(a_1))_r)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(1.10)}{=} \gamma((S_A(a_5))_{r1})\gamma(a_2)\gamma((S_A(S_A(a_4))_{R\bar{R}}))(S_B((\gamma(a_3))_R))_{\bar{R}} \\
&\quad \otimes (S_A(a_5))_{r2}\varepsilon_B((\gamma(a_1))_r) \\
&\stackrel{(2.3)}{=} \gamma((S_A(a_5))_{r1})\gamma(a_2)S_B((\gamma(a_3))_R)\gamma((S_A(S_A(a_4)))_R) \\
&\quad \otimes (S_A(a_5))_{r2}\varepsilon_B((\gamma(a_1))_r) \\
&\stackrel{(3.1)}{=} \gamma((S_A(a_5))_{r1})\gamma(a_2)S_B((\gamma(a_3))_R)S_B(\gamma((S_A(a_4))_R)) \\
&\quad \otimes (S_A(a_5))_{r2}\varepsilon_B((\gamma(a_1))_r) \\
&= \gamma((S_A(a_5))_{r1})\gamma(a_2)S_B(\gamma((S_A(a_4))_R)(\gamma(a_3))_R) \\
&\quad \otimes (S_A(a_5))_{r2}\varepsilon_B((\gamma(a_1))_r) \\
&\stackrel{(2.5)}{=} \gamma((S_A(a_5))_{r1})\gamma(a_2)S_B(\gamma(a_3)\gamma(S_A(a_4))) \\
&\quad \otimes (S_A(a_5))_{r2}\varepsilon_B((\gamma(a_1))_r) \\
&= \gamma((S_A(a_3))_{r1})\gamma(a_2) \otimes (S_A(a_3))_{r2}\varepsilon_B((\gamma(a_1))_r) \\
&\stackrel{(1.9)}{=} \gamma((S_A(a_5))_R)\gamma(a_3)\varepsilon_B((\gamma(a_1))_R) \otimes (S_A(a_4))_r\varepsilon_B((\gamma(a_2))_r) \\
&\stackrel{(1.8)}{=} \gamma((S_A(a_5))_R)\gamma(a_2)\varepsilon_B((\gamma(a_1))_R) \otimes (S_A(a_4))_r\varepsilon_B((\gamma(a_3))_r) \\
&\stackrel{(1.8)}{=} \gamma((S_A(a_4))_R)(\gamma(a_1))_R \otimes (S_A(a_3))_r\varepsilon_B((\gamma(a_2))_r) \\
&\stackrel{(2.3)}{=} \gamma(a_1)\gamma(S_A(a_4)) \otimes (S_A(a_3))_r\varepsilon_B((\gamma(a_2))_r) \\
&= (i_B \otimes \bar{S})\rho(a),
\end{aligned}$$

thus \bar{S} is a left B -comodule morphism.

Secondly, for all $a \in \bar{A}$, one can obtain

$$\begin{aligned}
(\bar{S} * i_{\bar{A}})(a) &= (S_A(a_2))_{R1}a_{3r}\varepsilon_B(((S_B(\gamma(a_2)))_{R2})_r)\varepsilon_B((\gamma(a_1))_R) \\
&\stackrel{(1.9)}{=} (S_A(a_4))_R\varepsilon_B((\gamma(a_1))_R)a_{5r}\varepsilon_B(((S_B(\gamma(a_3)))_{\bar{R}})_r)\varepsilon_B((\gamma(a_2))_{\bar{R}}) \\
&\stackrel{(2.4)}{=} (S_A(a_4))_R\varepsilon_B((\gamma(a_1))_R)a_{5r} \\
&\quad \varepsilon_B(((\gamma(a_2))_{\bar{R}1})\gamma((S_A(S_A(a_3)))_{\bar{R}\bar{r}}))(S_B((\gamma(a_2))_{\bar{R}1}))_{\bar{r}}) \\
&\stackrel{(1.10)}{=} (S_A(a_5))_R\varepsilon_B((\gamma(a_1))_R)a_{6r} \\
&\quad \varepsilon_B((\gamma(a_2)\gamma((S_A(S_A(a_4)))_{\bar{R}\bar{r}}))(S_B((\gamma(a_3))_{\bar{R}}))_{\bar{r}})_r) \\
&\stackrel{(2.3)}{=} (S_A(a_5))_R\varepsilon_B((\gamma(a_1))_R)a_{6r}\varepsilon_B((\gamma(a_2)S_B((\gamma(a_3))_{\bar{R}})\gamma((S_A(S_A(a_4)))_{\bar{R}}))_r) \\
&= (S_A(a_5))_R\varepsilon_B((\gamma(a_1))_R)a_{6r}\varepsilon_B((\gamma(a_2)S_B(\gamma(S_A(a_4))_{\bar{R}})(\gamma(a_3))_{\bar{R}})_r) \\
&\stackrel{(2.3)}{=} (S_A(a_5))_R\varepsilon_B((\gamma(a_1))_R)a_{6r}\varepsilon_B((\gamma(a_2)S_B(\gamma(a_3)\gamma(S_A(a_4))))_r) \\
&= (S_A(a_3))_Ra_{4r}\varepsilon_B((\gamma(a_1))_R)\varepsilon_B((\gamma(a_2))_r) \\
&\stackrel{(1.8)}{=} (S_A(a_2))_Ra_{3r}\varepsilon_B((\gamma(a_1))_{Rr})
\end{aligned}$$

$$\stackrel{(1.3)}{=} (S_A(a_2)a_3)_R \varepsilon_B((\gamma(a_1))_R)$$

$$\stackrel{(1.4)}{=} \varepsilon_{\bar{A}}(a)1_{\bar{A}}.$$

$(i_{\bar{A}} * \bar{S})(a) = \varepsilon_{\bar{A}}(a)1_{\bar{A}}$ is direct by Eq.(1.2) and Eq.(1.4).

Thus \bar{S} is an antipode of \bar{A} . The proof is completed. \blacksquare

The following result is direct by remark 2.6(2).

Theorem 3.4. Under the assumptions of Theorem 3.2, $(\bar{A}, \bar{m}, \bar{S})$ is given by Theorem 3.2 and Theorem 3.3, then $\bar{A} \#_{\times} B \cong A \#_R B$ as Hopf algebras.

4 Construct braided Hopf algebra from twisted tensor biproduct directly

The twisted tensor biproduct was studied in our recent work [6] which really generalizes the well-known Radford's biproduct [10]. While in this section, we will give a method to construct Radford's biproduct *directly* by defining the module action and comodule actions from the twisted tensor biproduct. Here a direct proof is provided.

Lemma 4.1. ([6]) Let B be a Hopf algebra. Let A be both an algebra and a coalgebra (but not necessarily bialgebra) such that there exists a linear map $S_A : A \rightarrow A$ satisfying $m_A(S_A \otimes i_A)\Delta_A = i_A$ and $m_A(i_A \otimes S_A)\Delta_A = i_A$. Then the following are equivalent ($\forall a, a' \in A, b, b' \in B$ and $T = t, R = r$):

1. The following conditions (B1) \sim (B8) hold:

(B1) $A \#_R B$ is a twisted tensor product.

(B2) $A \times_T B$ is a twisted tensor coproduct.

(B3) $T(1_A \otimes 1_B) = 1_B \otimes 1_A, \Delta_A(1_A) = 1_A \otimes 1_A$.

(B4) $(aa')_1 \otimes 1_{BT} \otimes (aa')_{2T} = a_1 a'_{1R} \otimes 1_{BTR} 1_{Bt} \otimes a_{2T} a'_{2t}$.

(B5) $b_T \otimes a_T = 1_{BT} b_t \otimes a_T 1_{At}$.

(B6) $(b_1 b'_1)_T \otimes 1_{AT} \otimes b_2 b'_2 = b_{1T} b'_{1t} \otimes 1_{AT} 1_{AtR} \otimes b_{2R} b'_2$.

(B7) $a_{R1} \otimes b_{R1T} \otimes a_{R2T} \otimes b_{R2} = a_{1R} \otimes b_{1TR} 1_{Bt} \otimes 1_{AT} a_{2tr} \otimes b_{2r}$.

(B8) $(\varepsilon_A \otimes \varepsilon_B)R = \varepsilon_B \otimes \varepsilon_A, \varepsilon_A$ is an algebra map.

2. $(A \boxtimes B, \bar{\Delta}, \bar{\varepsilon}, \bar{S})$ is a Hopf algebra, where the multiplication, $\bar{\Delta}, \bar{\varepsilon}$ and \bar{S} are given as

$$\begin{aligned} (a \boxtimes b)(a' \boxtimes b') &= (a \otimes 1)R(b \otimes a')(1 \otimes b'), \\ \bar{\Delta}(a \boxtimes b) &= (i_A \otimes T \otimes i_B)(\Delta_A(a) \otimes \Delta_B(b)), \\ \bar{\varepsilon} &= \varepsilon_A \otimes \varepsilon_B \quad \text{and} \quad \bar{S} = R(S_B \otimes S_A)T. \end{aligned}$$

In this case, we call $A \boxtimes B$ a *twisted tensor biproduct Hopf algebra*.

Remark 4.2. Usually letting $R(b \otimes a) = b_1 \cdot a \otimes b_2$ and $T(a \otimes b) = a_{(-1)}b \otimes a_0$ for all $a \in A$ and $b \in B$ in Theorem 4.1, we can obtain the Radford's biproduct

Hopf algebra [10] $A \#_{\times} B$. But in the following, we will give a method to construct Radford's biproduct directly by defining the module action and comodule action from the twisted tensor biproduct.

Theorem 4.3. Let $A \boxtimes B$ be a twisted tensor biproduct Hopf algebra. For all $a \in A$ and $b \in B$, if we set

$$b \cdot a = \varepsilon_B(b_R)a_R \quad \text{and} \quad \rho(a) = 1_{BT} \otimes a_T$$

and assume that

$$\text{Eq.(1.7)} \quad \text{and} \quad 1_{AT} \otimes b_T = 1_A \otimes b \tag{4.1}$$

are satisfied. Then (A, B, \cdot, ρ) is exactly the Radford's biproduct.

Proof. Firstly, (B1) and (B2) in Part 1 of Lemma 4.1 are exactly the conditions Eq.(1.2) \sim (1.4) and the following Eq.(4.2) \sim (4.4), respectively.

$$b_{T1} \otimes b_{T2} \otimes a_T = b_{1T} \otimes b_{2t} \otimes a_{Tt} \tag{4.2}$$

$$b_T \otimes a_{T1} \otimes a_{T2} = b_{Tt} \otimes a_{1t} \otimes a_{2T} \tag{4.3}$$

$$b_T \varepsilon(a_T) = \varepsilon(a)b \quad \varepsilon(b_T)a_T = \varepsilon(b)a \tag{4.4}$$

Under the assumptions of Eq.(4.1), the conditions (B3) \sim (B8) will have the following forms:

$$\Delta(1_A) = 1_A \otimes 1_A \quad \varepsilon_A \text{ is an algebra map.} \tag{4.5}$$

$$(aa')_1 \otimes 1_{BT} \otimes (aa')_{2T} = a_1 a'_{1R} \otimes 1_{BTR} 1_{Bt} \otimes a_{2T} a'_{2t} \tag{4.6}$$

$$b_T \otimes a_T = 1_{BT} b \otimes a_T \tag{4.7}$$

$$a_{R1} \otimes b_{R1T} \otimes a_{R2T} \otimes b_{R2} = a_{1R} \otimes b_{1R} 1_{BT} \otimes a_{2Tr} \otimes b_{2r}. \tag{4.8}$$

Secondly, we use the above special forms to compute some formulas that will need later.

Applying $i_A \otimes \varepsilon_B \otimes \varepsilon_A \otimes i_B$ to Eq.(4.8), then by Eq.(4.1) one can get

$$a_R \otimes b_R = a_R \varepsilon_B(b_{1R}) \otimes b_2. \tag{4.9}$$

Using $\varepsilon_A \otimes i_B \otimes i_A \otimes \varepsilon_B$ to Eq.(4.8), then by Eq.(4.1) we have

$$b_{RT} \otimes a_{RT} = b_1 1_{BT} \otimes a_{TR} \varepsilon_B(b_{2R}). \tag{4.10}$$

Similarly applying $i_A \otimes \varepsilon_B \otimes \varepsilon_A \otimes i_B$ to Eq.(4.8), then by Eq.(1.8) and Eq.(4.9) we obtain

$$a_{RT} \varepsilon_B(b_{1R}) \otimes b_{2T} = a_{TR} \varepsilon_B(b_{2R}) \otimes b_1 1_{BT}. \tag{4.11}$$

Likewise applying $i_A \otimes \varepsilon_B \otimes i_A \otimes i_B$ to Eq.(4.8), then by Eq.(4.4) we get

$$a_{R1} \otimes a_{R2} \otimes b_R = a_{1R} \varepsilon_B(b_{1R}) \otimes a_{2r} \otimes b_{2r}. \tag{4.12}$$

Applying $i_A \otimes \varepsilon_B \otimes i_A$ to Eq.(4.6), then by Eq.(4.4) we have

$$(aa')_1 \otimes (aa')_2 = a_1 a'_{1R} \varepsilon_B(1_{BTR}) \otimes a_{2T} a'_2. \quad (4.13)$$

Finally we prove the conditions in subsection 1.2 such that (A, B, \cdot, ρ) is Radford's biproduct.

Applying $i_A \otimes \varepsilon_B$ to Eq.(1.2) \sim (1.4) respectively and by Eq.(1.8), then we get (A, \cdot) is left B -module algebra.

Letting $b = 1_B$ in Eq.(4.2) \sim (4.4) respectively and by Eq.(4.7), then one can obtain that (A, ρ) is left B -comodule coalgebra.

Using $i_A \otimes i_A \otimes \varepsilon_B$ to Eq.(1.9) and by Eq.(4.1), then we have (A, \cdot) is left B -module coalgebra.

(A, ρ) is left B -comodule algebra by applying $\varepsilon_A \otimes i_B \otimes i_A$ to Eq.(4.6) and by Eq.(4.1) and Eq.(4.5).

By Eq.(4.7), the equation Eq.(4.10) is exactly $(b_1 \cdot a)_{(-1)} b_2 \otimes (b_1 \cdot a)_0 = b_1 a_{(-1)} \otimes b_2 \cdot a_0$. While Eq.(4.11) is Eq.(1.12).

Thus (A, B, \cdot, ρ) is a Radford's biproduct. ■

References

- [1] N. Andruskiewitsch and H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3 . *J. Algebra* 209(1998): 658-691.
- [2] M. Beattie and D. Bulacu, Braided Hopf algebras obtained from coquasitriangular Hopf algebras. *Comm. Math. Phys.* 282(2008): 115-160.
- [3] S. Caenepeel, B. Ion, G. Militaru and S. L. Zhu, The factorization problem and the smash biproduct of algebras and coalgebras. *Algebra. Represent. Theory* 3 (2000): 19-42.
- [4] Y. Doi and M. Takeuchi, Multiplication alteration by two-cocycles-The quantum version. *Comm. Algebra* 22(1994): 5715-5732.
- [5] V. G. Drinfel'd, On almost cocommutative Hopf algebras. *Leningrad Math. J.* 1(2)(1990): 321-342.
- [6] T. S. Ma and S. H. Wang, General double quantum groups. *Comm. Algebra* 38(2)(2010): 645-672.
- [7] T. S. Ma and S. H. Wang, Bitwistor and quasitriangular structures of bialgebras. *Comm. Algebra* (2010), to appear.
- [8] S. Majid, Doubles of quasitriangular Hopf algebras. *Comm. Algebra* 19(1991): 3061-3073.
- [9] S. Montgomery, Hopf algebras and their actions on rings. *CBMS Lectures in Math.* Vol. 82(1993), Providence, RI: AMS.

- [10] D. E. Radford, The structure of Hopf algebra with a projection. *J. Algebra* 92(1985): 322-347.
- [11] M. E. Sweedler, "Hopf Algebras". Benjamin, New York, 1969.
- [12] S. H. Wang, A construction of braided Hopf algebras. *Tsukuba J. Math.* 26(2002): 269-289.
- [13] S. H. Wang and T. S. Ma, Singular solutions to the quantum Yang-Baxter equations. *Comm. Algebra* 37 (2009), 296-316.
- [14] W. Z. Zhao, (f, τ) -compatible Hopf algebra pair (B, H) . *Acta Math. Sci. Ser. A Chin. Ed.* 27(1)(2007): 155-165.

Tianshui Ma

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, CHINA

E-mail: matianshui@yahoo.com

Shuanhong Wang

Department of Mathematics, Southeast University, Nanjing 211189, CHINA

E-mail: shuanhwang2002@yahoo.com

Shaoxian Xu

School of Mathematics and Statistics, Nanyang Normal University, Nanyang 473061, CHINA

E-mail: nytcxsx@163.com