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ON STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS VIA NON-LINEAR INTEGRAL CONTRACTORS II

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Abstract

The present paper represents a continuation of paper [4], in which the existence and uniqueness problems for a general Ito–Volterra integrodifferential equation are investigated by using the concept of a non-linear random integral contractor. Since the Lipschitz condition and the random integral contractor for the coefficients of the considered equation, in general, cannot be compared, the notions of the modified Lipschitz condition and modified integral contractor are introduced on some function spaces, as well as the conditions of their equivalence. Some existence and uniqueness theorems are also given.

1 Introduction and preliminary results

In the present paper, we continue the investigation from paper [4] treating the existence and uniqueness of the solution to the following integrodifferential equation

$$\begin{aligned} dx(t) = F\Big(t, x(t), \int_0^t f_1(t,s,x(s))\,ds, \int_0^t f_2(t,s,x(s))\,dw(s)\Big)dt \qquad (1)\\ +G\Big(t, x(t), \int_0^t g_1(t,s,x(s))\,ds, \int_0^t g_2(t,s,x(s))\,dw(s)\Big)dw(t),\\ t\in[0,T],\ x(0)=x_0 \text{ a.s.,} \end{aligned}$$

where $w = (w(t), t \geq 0)$ is a scalar Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $(\mathcal{F}_t, t \geq 0)$ of non-decreasing sub-σ-algebras of $\mathcal{F}$ $(\mathcal{F}_t = \sigma\{w(s), 0 \leq s \leq t\})$, x_0 is a random variable independent of w, the functions $F : [0,T] \times R^3 \to R$, $G : [0,T] \times R^3 \to R$, $f_i : J \times R \to R$, $g_i : J \times R \to R$, $i = 1, 2$, where $J = \{(s,t) : 0 \leq s \leq t \leq T\}$, are assumed to be Borel measurable on their domains. Our study in [4] is based on the notion of

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a non-linear random integral contractor, which is a stochastic version of the well-known concept of integral contractors first introduced by Altman [1, 2] for studying some different classes of deterministic equations in Banach spaces. Kuo [5] was the first who applied the Altman's approach to stochastic differential equations of the Ito type and this concept was appropriately extended to various stochastic integral and integrodifferential equations (see references in [4], for instance). Let us also highlight that in all these papers, except partly in [6], random integral contractors are determined as the solutions to linear functional equations. In contrast to these cases, the notion of the non-linear random integral contractor is introduced in [4], to exceed the non-linearity of the Lebesgue and Ito integrals in Eq. (1) and to prove the existence and uniqueness of its solution.

In order to continue our study, we briefly present some notions and known results from paper [4] in the remainder of this section. Precisely, we introduce the concept of a bounded random integral contractor and we consider the appropriate existence and uniqueness problems for Eq. (1). In order to compare these results with the classical existence-and-uniqueness theorem based on the global Lipschitz condition, the notions of a modified Lipschitz condition and a modified random integral contractor are defined in Section 2. Some relations between them and conditions for their equivalence on some spaces of stochastic processes are investigated, as well as alternative existence-and-uniqueness theorems.

First, let $\mathcal{C}$ be a collection of scalar stochastic processes, defined on $[0,T]$, continuous a.s. and adapted to the filtration $(\mathcal{F}_t, t \geq 0)$. Let also $L_2([0,T]\times\Omega)$ be a collection of stochastic processes in $\mathcal{C}$ such that $P\big\{\int_0^T |x(t)|^2 dt < \infty\big\} = 1$.

For each $x \in \mathcal{C}$, let us denote that

$$(A_1x)(t) := \int_0^t f_1(t,s,x(s))\,ds, \quad (A_2x)(t) := \int_0^t f_2(t,s,x(s))\,dw(s),$$
$$(B_1x)(t) := \int_0^t g_1(t,s,x(s))\,ds, \quad (B_2x)(t) := \int_0^t g_2(t,s,x(s))\,dw(s),$$

$$F[x(t)] = F\big(t, x(t), (A_1x)(t), (A_2x)(t)\big),$$
$$G[x(t)] = G\big(t, x(t), (B_1x)(t), (B_2x)(t)\big),$$

and rewritten Eq. (1) in its integral form,

$$x(t) = x_0 + \int_0^t F[x(s)]\,ds + \int_0^t G[x(s)]\,dw(s), \quad t \in [0,T]. \tag{2}$$

Let $\Phi : [0,T]\times R^3 \to R$, $\Gamma : [0,T]\times R^3 \to R$, $\Phi_i : J\times R \to R$, $\Gamma_i : J \times R \to R$, $i = 1,2$, be measurable mappings, bounded in the sense that there exist positive constants $\alpha, \beta, \alpha_i, \beta_i$, $i = 1,2$, such that for every $(t,x,u,v) \in [0,T]\times R^3$, $(t,s,x) \in J\times R$, $y \in R$,

$$\begin{aligned} &|\Phi(t,x,u,v)\cdot y| \leq \alpha\,|y|, \quad |\Gamma(t,x,u,v)\cdot y| \leq \beta\,|y|, \\ &|\Phi_i(t,s,x)\cdot y| \leq \alpha_i\,|y|, \quad\ |\Gamma_i(t,s,x)\cdot y| \leq \beta_i\,|y|, \ i=1,2. \end{aligned} \tag{3}$$

Then, the following operators and notations are introduced: For every $x, y \in \mathcal{C}$,

$$\begin{aligned}
((\tilde{\Phi}_1 x)y)(t) &:= \int_0^t \Phi_1(t,s,x(s))\, y(s)\, ds, \\
((\tilde{\Phi}_2 x)y)(t) &:= \int_0^t \Phi_2(t,s,x(s))\, y(s)\, dw(s), \\
((\tilde{\Gamma}_1 x)y)(t) &:= \int_0^t \Gamma_1(t,s,x(s))\, y(s)\, ds, \\
((\tilde{\Gamma}_2 x)y)(t) &:= \int_0^t \Gamma_2(t,s,x(s))\, y(s)\, dw(s).
\end{aligned} \tag{4}$$

$$\begin{aligned}
\Phi[x(t), y(t)] &= \Phi\big(t, x(t), ((\tilde{\Phi}_1 x)y)(t), ((\tilde{\Phi}_2 x)y)(t)\big), \\
\Gamma[x(t), y(t)] &= \Gamma\big(t, x(t), ((\tilde{\Gamma}_1 x)y)(t), ((\tilde{\Gamma}_2 x)y)(t)\big).
\end{aligned}$$

Likewise, we can define the non-linear operator A: For every $x, y \in \mathcal{C}$,

$$\begin{aligned}
((Ax)y)(t) := y(t) &+ \int_0^t \Phi[x(s), y(s)]\, y(s)\, ds \qquad (5)\\
&+ \int_0^t \Gamma[x(s), y(s)]\, y(s)\, dw(s), \quad t \in [0,T].
\end{aligned}$$

Since $(Ax)y \in \mathcal{C}$, the following notions are introduced in [4].

Definition 1. *Let there exist a positive constant K such that for every $x, y \in \mathcal{C}$ the following inequalities hold almost surely:*

$$\begin{aligned}
&\big|F[x(t) - ((Ax)y)(t)] - F[x(t)] + \Phi[x(t), y(t)] \cdot y(t)\big| \\
&\quad \le K\big[||y||_t + \big|(A_1(x - (Ax)y))(t) - (A_1 x)(t) + ((\tilde{\Phi}_1 x)y)(t)\big| \\
&\qquad + \big|(A_2(x - (Ax)y))(t) - (A_2 x)(t) + ((\tilde{\Phi}_2 x)y)(t)\big|\big] \\
&\big|f_i(t,s,x(s) - ((Ax)y)(s)) - f_i(t,s,x(s)) + \Phi_i(t,s,x(s)) \cdot y(s)\big| \\
&\quad \le K\, ||y||_s, \ \ i = 1,2, \qquad (6)\\
&\big|G[x(t) - ((Ax)y)(t)] - G[x(t)] + \Gamma[x(t), y(t)] \cdot y(t)\big| \\
&\quad \le K\big[||y||_t + \big|(B_1(x - (Ax)y))(t) - (B_1 x)(t) + ((\tilde{\Gamma}_1 x)y)(t)\big| \\
&\qquad + \big|(B_2(x - (Ax)y))(t) - (B_2 x)(t) + ((\tilde{\Gamma}_2 x)y)(t)\big|\big], \\
&\big|g_i(t,s,x(s) - ((Ax)y)(s)) - g_i(t,s,x(s)) + \Gamma_i(t,s,x(s)) \cdot y(s)\big| \\
&\quad \le K\, ||y||_s, \ \ i = 1,2,
\end{aligned}$$

where $||y||_t = \sup_{0 \le s \le t} |y(s)|$. Then, the set of functions $\{F, f_1, f_2, G, g_1, g_2\}$ has a bounded random integral contractor

$$\begin{aligned}
\Big\{ I + \int_0^t \Phi\Big(s, x, \int_0^s \Phi_1\, dr, \int_0^s \Phi_2\, dw(r)\Big) ds \qquad (7)\\
+ \int_0^t \Gamma\Big(s, x, \int_0^s \Gamma_1\, dr, \int_0^s \Gamma_2\, dw(r)\Big) dw(s) \Big\}.
\end{aligned}$$

Definition 2. *A bounded random integral contractor (7) is said to be regular if the equation*

$$(Ax)\,y = z \tag{8}$$

has a solution y in $\mathcal{C}$ for any x and z in $\mathcal{C}$.

Definition 3. *The functions F and G in Eq. (2) are said to be stochastically closed if for any x and x_n in $\mathcal{C}$, such that $x_n \to x$ and $F[x_n] \to y$, $G[x_n] \to z$ in $L_2([0,T] \times \Omega)$, it follows that $y = F[x]$ and $z = G[x]$ almost surely, for every $t \in [0,T]$.*

The main results of paper [4] are the following existence-and-uniqueness theorems.

Theorem 1. *Let F and G be stochastically closed and $\int_0^T |F[x_0]|^2 dt < \infty$, $\int_0^T |G[x_0]|^2 dt < \infty$ a.s. Let also the set of functions $\{F, f_1, f_2, G, g_1, g_2\}$ has a bounded random integral contractor (7). Then, Eq. (2) has a solution x in $\mathcal{C}$.*

Theorem 2. *Let the functions F, f_1, f_2, G, g_1, g_2 satisfy the assumptions of Theorem 1 and the bounded random integral contractor (7) is regular. Then, the solution x to Eq. (2) in $\mathcal{C}$ is unique.*

Remember that the classical existence-and-uniqueness theorem (see Murge and Pachpatte [7, 8]) requires that $E|x_0|^2 < \infty$ and that the coefficients of Eq. (2) satisfy the global Lipschitz and linear growth conditions: Let there exist a constant $L > 0$ such that for all $(t,s) \in J$ and $(x,y,z), (x',y',z') \in R^3$,

$$\begin{aligned} &|F(t,x,y,z) - F(t,x',y',z')| \le L\big(|x-x'| + |y-y'| + |z-z'|\big), \\ &|f_i(t,s,x) - f_i(t,s,x')| \le L|x-x'|,\ i=1,2 \end{aligned} \tag{9}$$

$$\begin{aligned} &|F(t,x,y,z)| \le L\big(1 + |x| + |y| + |z|\big), \\ &|f_i(t,s,x)| \le L\big(1+|x|\big),\ i=1,2, \end{aligned} \tag{10}$$

and analogously for G, g_1, g_2. Then, Eq. (2) has a unique a.s. continuous and $\mathcal{F}_t$-adapted solution $x(t)$ satisfying $E\sup_{t\in[0,T]} |x(t)|^2 < \infty$.

If F, f_1, f_2, G, g_1, g_2 satisfy the global Lipschitz condition (9), then F and G are stochastically closed and the set $\{F, f_1, f_2, G, g_1, g_2\}$ has a trivial integral contractor (7) with $\Phi = \Gamma = \Phi_i = \Gamma_i \equiv 0, i = 1,2$, and vice versa. In [4] is shown that the global Lipschitz condition (9) implies the existence of a class of non-trivial bounded integral contractors $\Big\{I + \int_0^t \Phi\Big(s, x, \int_0^s \Phi_1 x\, dr, 0\Big)\, ds\Big\}$, as well as that Eq. (2) could have a regular bounded random integral contractor, although the Lipschitz condition, in general, did not have to be valid.

2 Main results

Since we saw in Section 1 that the Lipschitz condition and the regular bounded random integral contractor cannot, generally, be compared, it is reasonable to focus our analysis on conditions and function spaces which give some alternative existence and uniqueness assertions, as well as to estabilish relations between them. To do this, we will follow only partly the ideas from papers [3] and [9].

First, let us denote that $L_2(\mathcal{C})$ is a class of stochastic processes $x \in \mathcal{C}$ with the norm

$$||\, x||_*^2 \, := E||\, x||_T^2 < \infty.$$

Certainly, $\big(L_2(\mathcal{C}), ||\cdot||_*\big)$ is a Banach space.

The following assertion, closely connected with Theorem 1 and Theorem 2, has an important role in our investigation.

Theorem 3. *Let the conditions of Theorem 2 be valid and* $E|x_0|^2 < \infty$. *Then, Eq. (2) has a unique solution* $x \in L_2(\mathcal{C})$.

Proof. Since from Theorem 1 and Theorem 2 it follows that Eq. (2) has a unique solution $x \in \mathcal{C}$, it remains to prove that $x \in L_2(\mathcal{C})$.

Because of the regularity of the bounded random integral contractor, the operator equation

$$((Ax)y)(t) = x(t) - x_0, \quad t \in [0,T], \tag{11}$$

has a solution $y \in \mathcal{C}$. Then, from (5) we find that

$$y(t) + \int_0^t \Phi[x(s), y(s)]\, y(s)\, ds + \int_0^t \Gamma[x(s), y(s)]\, y(s)\, dw(s) = x(t) - x_0. \tag{12}$$

Substituting (2) to (12), it follows that

$$\begin{aligned} y(t) = &\int_0^t \big(F[x(s)] - \Phi[x(s), y(s)]\, y(s)\big)\, ds \\ &+ \int_0^t \big(G[x(s)] - \Gamma[x(s), y(s)]\, y(s)\big)\, dw(s), \end{aligned}$$

so that

$$\begin{aligned} E \sup_{0\le s\le t} |y(s)|^2 \le\, & 2E \sup_{0\le s\le t} \Big| \int_0^s \big(F[x(r)] - \Phi[x(r), y(r)]\, y(r)\big)\, dr\Big|^2 \\ &+ 2E \sup_{0\le s\le t} \Big| \int_0^s \big(G[x(r)] - \Gamma[x(r), y(r)]\, y(r)\big)\, dw(r)\Big|^2. \end{aligned} \tag{13}$$

However, (11) yields $F[(x-(Ax)y)(t)] = F[x_0]$ a.s. and $[G(x-(Ax)y)(t)] = G[x_0]$ a.s., so that we come to the following estimate by using (6), the Hölder inequality, the Schwarz inequality and integration by parts,

$$
\begin{aligned}
& E \sup_{0\le s\le t} \Big| \int_0^s \big(F[x(r)] - \Phi[x(r), y(r)]\, y(r)\big)\, dr \Big|^2 \\
& = E \sup_{0\le s\le t} \Big| \int_0^s \big(F[x(r)] - \Phi[x(r), y(r)]\, y(r) - F[(x-(Ax)y)(r)] + F[x_0]\big)\, dr \Big|^2 \\
& \le 2TK^2 \int_0^t E\big[||y||_s + |A_1(x-(Ax)y)(s) - A_1 x(s) + ((\tilde{\Phi}_1 x)y)(s)| \\
& \quad + |A_2(x-(Ax)y)(s) - A_2 x(s) + ((\tilde{\Phi}_2 x)y)(s)|\big]^2\, ds + 2T\int_0^T E|F[x_0]|^2\, ds \\
& \le 6TK^2 \Big[\int_0^t E||y||_s^2\, ds + K^2 \int_0^t s \int_0^s E||y||_r^2\, dr\, ds + K^2 \int_0^t \int_0^s E||y||_r^2\, dr\, ds \Big] \\
& \quad + 2T\int_0^T E|F[x_0]|^2\, ds \\
& \le 6TK^2\big[1 + K^2(T^2/2 + T)\big] \int_0^t E||y||_s^2\, ds + 2T\int_0^T E|F[x_0]|^2\, ds.
\end{aligned}
$$

Similarly, by applying Doob inequality to the Ito integral in (13), we observe that

$$
\begin{aligned}
& E \sup_{0\le s\le t} \Big| \int_0^s \big(G[x(r)] - \Gamma[x(r), y(r)]\, y(r)\big)\, dw(r) \Big|^2 \\
& \quad \le 24K^2\big[1 + K^2(T^2/2+T)\big] \int_0^t E||y||_s^2\, ds + 8\int_0^T E|G[x_0]|^2\, ds.
\end{aligned}
$$

These estimates together with (13) yield

$$
E||y||_t^2 \le c_1 \int_0^t E||y||_s^2\, ds + c_2, \quad t\in[0,T],
$$

where c_1 and c_2 are generic constants. From now on, by applying the Gronwall-Bellman lemma, we deduce that

$$
E||y||_t^2 < \infty, \quad t \in [0,T].
$$

To prove that $x \in L_2(\mathcal{C})$, we will start form (12), then apply the boundedness (3) of the mappings $\Phi, \Gamma, \Gamma_i, \Phi_i, i = 1,2$ and use the foregoing estimate. Hence,

$$
\begin{aligned}
||x||_*^2 &= E \sup_{0\le t\le T} |x(t)|^2 \\
&\le 4\Bigg[E|x_0|^2 + E||y||_T^2 + E \sup_{0\le s\le T}\Big|\int_0^t \Phi[x(s),y(s)]\,y(s)\,ds\Big|^2 \\
&\quad + E\sup_{0\le s\le T}\Big|\int_0^t \Gamma[x(s),y(s)]\,y(s)\,dw(s)\Big|^2\Bigg] \\
&\le 4\Bigg[E|x_0|^2 + E||y||_T^2 + (\alpha^2 T + 4\beta^2)\int_0^T E||y||_t^2\,dt\Bigg] \\
&< \infty,
\end{aligned}
$$

which completes the proof. □

Theorem 3 gives us a motivation to study, under various conditions, the existence and uniqueness of the solution to Eq. (2) which belongs to $L_2(\mathcal{C})$. Because of that, we introduce the following norm on the space $L_2(\mathcal{C})$: For a fixed number $\lambda > 0$ and for every $x \in L_2(\mathcal{C})$,

$$|||\, x|||^2 := \sup_{0\le t\le T} E\big\{||\, x||_t^2 \cdot e^{-2\lambda t}\big\}.$$

Since

$$||\, x||_*^2 \cdot e^{-2\lambda T} \le |||\, x|||^2 \le ||\, x||_*^2,$$

the norms $|||\cdot|||$ and $||\cdot||_*$ are equivalent and, therefore, $(L_2(\mathcal{C}), |||\cdot|||)$ is also a Banach space. This norm enables us to prove the following assertion which is an important tool to be used in the sequel.

Proposition 1. *Let the mappings $\Phi, \Gamma, \Phi_i, \Gamma_i, i = 1, 2$, satisfy the conditions (3). Then, Eq. (8) has a unique solution $y \in L_2(\mathcal{C})$ for every $x, z \in L_2(\mathcal{C})$. Moreover, there exists a constant $\mu > 0$, independent of x and z such that*

$$E||y||_t^2 \le \mu\, E||z||_t^2, \quad t \in [0, T]. \tag{14}$$

Proof. Let us define an operator $\mathcal{S} : L_2(\mathcal{C}) \to \mathcal{C}$ in the following way: For fixed $x, z \in L_2(\mathcal{C})$ and arbitrary $y \in L_2(\mathcal{C})$, let

$$
\begin{aligned}
(\mathcal{S}y)(t) := z(t) &- \int_0^t \Phi[x(s),y(s)]\,y(s)\,ds \\
&- \int_0^t \Gamma[x(s),y(s)]\,y(s)\,dw(s), \quad t\in[0,T].
\end{aligned} \tag{15}
$$

By using (3) we find for $0 \le t \le T$ that

$$E||\mathcal{S}y||_t^2 \le 3\left[E||z(s)||_t^2 + B\int_0^t E||y||_s^2\,ds\right], \tag{16}$$

where $B = \alpha^2 T + 4\beta^2$. Since $||\mathcal{S}y||_*^2 \le 3\big[||z||_*^2 + BT||y||_*^2\big] < \infty$, we conclude that $\mathcal{S}y \in L_2(\mathcal{C})$ and, therefore, $\mathcal{S} : L_2(\mathcal{C}) \to L_2(\mathcal{C})$.

The next step is to prove that there exists a constant $\lambda > 0$ such that the operator $\mathcal{S}$ is a contraction.

Starting from (15), it follows for arbitrary $y_1, y_2 \in L_2(\mathcal{C})$ that

$$E||\mathcal{S}y_1 - \mathcal{S}y_2||_t^2 \le 2B \int_0^t E||y_1 - y_2||_s^2 \, ds, \quad t \in [0,T].$$

By using the norm $|||\cdot|||$, we observe for a number $\lambda > 0$ that

$$\begin{aligned} E||\mathcal{S}y_1 - \mathcal{S}y_2||_t^2 &\le 2B \int_0^t E \sup_{0\le r\le s} |y_1(r) - y_2(r)|^2 \cdot e^{-2\lambda s} \cdot e^{2\lambda s} \, ds \\ &\le \frac{B}{\lambda} |||y_1 - y_2|||^2 \cdot e^{2\lambda t}, \quad t \in [0,T]. \end{aligned}$$

If we choose $\lambda > 4B + 1$, we find that

$$|||\mathcal{S}y_1 - \mathcal{S}y_2|||^2 \le \frac{B}{\lambda} |||y_1 - y_2|||^2 < \frac{1}{4} |||y_1 - y_2|||^2$$

and, therefore, $\mathcal{S}$ is a contraction on $L_2(\mathcal{S})$. Obviously, what remains is to apply the Banach fixed point theorem to conclude that Eq. (8) has a unique solution $y \in L_2(\mathcal{C})$, which completes the proof of the first part of this lemma.

To prove the second part, we take $\mathcal{S}y = y$ in (16) and obtain

$$E||y||_t^2 \le 3 \left[E||z||_t^2 + B \int_0^t E||y||_s^2 \, ds \right], \quad t \in [0,T].$$

It is now easy to arrive at the desired relation (14) by applying the Gronwall–Bellman lemma. □

The following assertion holds straightforwardly.

Theorem 4. *Let the conditions of Theorem 1 be valid and* $E|x_0|^2 < \infty$. *Then, Eq.* (2) *has a unique solution* x *in* $L_2(\mathcal{C})$.

Proof. Since the conditions of Proposition 1 are satisfied, it follows that there exists a regular bounded random integral contractor (7) defined on $L_2(\mathcal{C}) \subset \mathcal{C}$, so the proof immediately holds by virtue of Theorem 3. □

Since the Lipschitz condition and the bounded random integral contractor cannot be compared on the space $\mathcal{C}$, we introduce some modifications to these notions on $L_2(\mathcal{C})$ and state conditions for their equivalence.

First, let us introduce the following version of the Lipschitz condition.

Definition 4. *Let there exist a constant* $L_1 > 0$ *such that for all* $(t,s) \in J$ *and* $x, y, z, x', y', z' \in L_2(\mathcal{C})$, $i = 1, 2$,

$$E|F(t,x,y,z) - F(t,x',y',z')|^2 \le L_1\big[E||x-x'||_t^2 + E||y-y'||_t^2 + E||z-z'||_t^2\big],$$
$$E|f_i(t,s,x) - f_i(t,s,x')|^2 \le L_1 E||x-x'||_t^2, \quad i = 1,2, \tag{17}$$

and analogously for G, g_1, g_2. *Then, we say that the functions* F, f_1, f_2, G, g_1, g_2 *satisfy the modified Lipschitz condition on the space* $L_2(\mathcal{C})$.

Obviously, if the functions F, f_1, f_2, G, g_1, g_2 satisfy the Lipschitz condition (9) on $L_2(\mathcal{C})$, then they satisfy the modified Lipschitz condition (17), while the opposite assertion is not valid. Moreover, by following the proofs of the classical existence-and-uniqueness theorems, it is not difficult to conclude from paper [7], for example, that they are valid under conditions (17) instead of (9).

The following relation between the bounded random integral contractor (7) and the modified Lipschitz condition (17) holds on $L_2(\mathcal{C})$.

Proposition 2. *Let the functions* F, f_1, f_2, G, g_1, g_2 *from Eq. (2) have a bounded random integral contractor (7). Then, they satisfy the modified Lipcshitz condition (17) on the space* $L_2(\mathcal{C})$.

Proof. Let (7) be a bounded random integral contractor for the functions F, f_1, f_2, G, g_1, g_2. Then, from Proposition 1 it follows for fixed $x, z \in L_2(\mathcal{C})$ that there exists a unique solution $y \in L_2(\mathcal{C})$ of Eq. (8). On the basis of (6) and from the boundedness (3) of the mappings Φ, Φ_1, Φ_2, we see that, almost surely,

$$\begin{aligned}
&\big|F[(x-z)(t)] - F[x(t)]\big|^2 \\
&\le 2\big|F[(x-z)(t)] - F[x(t)] + \Phi[x(t),y(t)]\,y(t)|^2 + 2|\Phi[x(t),y(t)]\,y(t)\big|^2 \\
&\le 2K^2\big[||y||_t + |A_1(x-z)(t) - A_1x(t) + ((\tilde{\Phi}_1 x)y)(t)| \\
&\quad + |A_2(x-z)(t) - A_2x(t) + ((\tilde{\Phi}_2 x)y)(t)|\big]^2 + 2\alpha^2||y||_t^2.
\end{aligned} \tag{18}$$

Hence, for all $t \in [0,T]$,

$$\begin{aligned}
&\big|F[(x-z)(t)] - F[x(t)]\big|^2 \\
&\le 10K^2\Bigg[E||y||_t^2 + E|A_1(x-z)(t) - A_1x(t)|^2 + \alpha_1^2 t\int_0^t E||y||_s^2\,ds \\
&\quad + E|A_2(x-z)(t) - A_2x(t)|^2 + \alpha_2^2\int_0^t E||y||_s^2\,ds\Bigg] + 2\alpha^2||y||_t^2.
\end{aligned}$$

The application of the property (14) from Proposition 1 implies that

$$\begin{aligned}
&\big|F(x-z)(t) - Fx(t)\big|^2 \\
&\le \bar{K}\big[E||z||_t^2 + E|A_1(x-z)(t) - A_1x(t)|^2 + E|A_2(x-z)(t) - A_2x(t)|^2\big],
\end{aligned}$$

where $\bar{K}$ is a generic constant, which confirms that F satisfies the modified Lipschitz condition (17). This fact can be analogously proved for other functions. □

In order to formulate the opposite assertion with respect to Proposition 2, we introduce the notion of a modified bounded random integral contractor on $L_2(\mathcal{C})$.

Definition 5. *Let there exist a constant $K_1 > 0$ such that for all $(t,s) \in J$ and $x, y \in L_2(\mathcal{C})$,*

$$\begin{aligned}
&E\big|F[(x-(Ax)y)(t)] - F[x(t)] + \Phi[x(t), y(t)]\, y(t)\big|^2 \qquad (19)\\
&\quad \le K_1\big[E||y||_t^2 + E|A_1(x-(Ax)y)(t) - A_1x(t) + ((\tilde{\Phi}_1 x)y)(t)|^2\\
&\qquad + E|A_2(x-(Ax)y)(t) - A_2x(t) + ((\tilde{\Phi}_2 x)y)(t)|^2\big],\\
&E\big|f_i(t,s,x(s) - ((Ax)y)(s)) - f_i(t,s,x(s)) + \Phi_i(t,s,x(s))y(s)\big|^2\\
&\quad \le K_1 E||y||_s^2, \quad i = 1,2,
\end{aligned}$$

and analogously for G, g_1, g_2. Then, we say that the set of functions $\{F, f_1, f_2, G, g_1, g_2\}$ has a modified bounded random integral contractor on the space $L_2(\mathcal{C})$,

$$\begin{aligned}
&\Big\{I + \int_0^t \Phi\Big(s, x, \int_0^s \Phi_1 x\, dr, \int_0^s \Phi_2 x\, dw(r)\Big) ds \qquad (20)\\
&\qquad + \int_0^t \Gamma\Big(s, x, \int_0^s \Gamma_1 x\, dr, \int_0^s \Gamma_2 x\, dw(r)\Big) dw(s)\Big\}_E .
\end{aligned}$$

Following the proofs of Theorem 1 and Theorem 2 (see paper [4]), it is easy to see that they will be valid and, moreover, they will be shorter, if the set of functions $\{F, f_1, f_2, G, g_1, g_2\}$ has the modified bounded random integral contractor (20) instead of the bounded random integral contractor. Likewise, the equivalence between the modified Lipschitz condition (17) and the modified bounded random integral contractor (20) can be proved.

Proposition 3. *The functions F, f_1, f_2, G, g_1, g_2 satisfy the modified Lipschitz condition (17) if and only if they have the modified bounded random integral contractor (20).*

Proof. Since $x, y \in L_2(\mathcal{C})$ implies that $(Ax)y \in L_2(\mathcal{C})$, we can start from (5) and apply the same reasoning as for the operator $\mathcal{S}$ defined in (15). So, we observe that

$$E||(Ax)y||_t^2 \le 3\left[E||y||_t^2 + B\int_0^t E||y||_s^2\, ds\right]$$

and, therefore,

$$E||(Ax)y||_t^2 \le c\, E||y||_t^2, \quad t \in [0,T], \qquad (21)$$

where c is a generic constant.

Let F, f_1, f_2, G, g_1, g_2 satisfy the modified Lipschitz condition (17). By using

(21) we find for every $t \in [0,T]$ and $x, y \in L_2(\mathcal{C})$ that

$$\begin{aligned} &E\big|F[(x-(Ax)y)(t)] - F[x(t)] + \Phi[x(t),y(t)]\,y(t)\big|^2 \\ &\quad \le 2L_1\big[E||(Ax)y||_t^2 + E|A_1(x-(Ax)y)(t) - A_1x(t)|^2 \\ &\qquad + E|A_2(x-(Ax)y)(t) - A_2x(t)|^2\big] + 2\alpha^2 E||y||_t^2 \\ &\quad \le K_1\big[E||y||_t^2 + E|A_1(x-(Ax)y)(t) - A_1x(t) + ((\tilde{\Phi}_1x)y)(t)|^2 \\ &\qquad + E|A_2(x-(Ax)y)(t) - A_2x(t) + ((\tilde{\Phi}_2x)y)(t)|^2\big], \end{aligned}$$

where K_1 is a constant. Hence, the proof of this part of Proposition 3 holds since the other relations in (19) can be proved analogously.

Conversely, let us suppose that the set of functions $\{F, f_1, f_2, G, g_1, g_2\}$ has the modified bounded random integral contractor (20). From Proposition 1, it follows for every $x, z \in L_2(\mathcal{C})$ that there exists $y \in L_2(\mathcal{C})$ which is a solution to the equation $(Ax)y = z$, satisfying $E||y||_t^2 \le \mu\, E||z||_t^2$ for all $t \in [0,T]$. These facts and the procedure applied in the proof of Proposition 2 yield

$$\begin{aligned} &\big|F[(x-z)(t)] - F[x(t)]\big|^2 \\ &\quad \le L_1\big[E||z||_t^2 + E|A_1(x-z)(t) - A_1x(t)|^2 + E|A_2(x-z)(t) - A_2x(t)|^2\big], \end{aligned}$$

and similarly for the other functions. Therefore, the modified Lipschitz condition (17) holds, which completes the proof. □

We close our discussion by the convenient version of the classical existence-and-uniqueness theorem discussed in Section 1. More precisely, the next theorem summarizes the foregoing assertions.

Theorem 5. *Let the functions F, f_1, f_2, G, g_1, g_2 satisfy the modified Lipschitz condition (17) and $E|x_0|^2 < \infty$, $\int_0^T |F[x_0]|^2\, ds < \infty$ a.s., $\int_0^T |G[x_0]|^2\, ds < \infty$ a.s. Then, Eq. (2) has a unique solution x in $L_2(\mathcal{C})$.*

Proof. Since the functions F, f_1, f_2, G, g_1, g_2 satisfy the modified Lipcshitz condition (17), then F and G are stochastically closed in the sense of Definition 3. Hence, the proof holds straightforwardly by virtue of Proposition 3 and Theorem 4. □

Remark. The problems considered in this paper and in [4] could be consequently extended to more general stochastic equations and stochastic integrodifferential equations involving martingales and martingale measures instead of the Brownian motion.

References

[1] M. Altman, *Inverse differentiability contractors and equations in Banach space*, Studia Mathematica **46** (1973), 1–15.

[2] M. Altman, *Contractors and contractor directions*, Marcel Dekker, New York, 1978.

[3] M. Jovanović, S. Janković, *Existence and uniqueness problems for nonlinear stochastic hereditary integrodifferental equations*, Indian J. Pure Appl. Math., **32** (5) (2001), 695-710.

[4] M. Jovanović, S. Janković, *On stochastic integrodifferential equations via nonlinear integral contractors I*, Filomat, 23:3 (2009), 167–180.

[5] H.H. Kuo, *On integral contractors*, J. of Integral Equations **1** (1979), 35–46.

[6] M.G. Murge, and B.G. Pachpatte, *Existence and uniqueness of solutions of nonlinear Ito type stochastic integral equations*, Math. Acta Scientia, **7** 2 (1987), 207–216.

[7] M.G. Murge, B.G. Pachpatte, *On generalized Itô type stochastic integral equation, Yokohama Mathematical Journal*, **34** (1986), 23–33.

[8] M.G. Murge, B.G. Pachpatte, *Explosion and asumptotic behavior of nonlinear Ito type stochastic integrodifferential equations*, Kodai Math. J., **9** (1986), 1–18.

[9] V. Popescu, A. Rascanu, *On bounded deterministic and stochastic integral contractors*, Anale stiintifice ale Universitatii "Al. I. Cuza", Iasi, Tomui XXXIV. s. I a. Mathematica, **1** (1988), 37–51.

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