ON ALMOST ESSENTIALLY RUSTON ELEMENTS OF A BANACH ALGEBRA

Snežana Č. Živković-Zlatanović* and Robin E. Harte

Abstract

We introduce the class of "almost essentially Ruston elements" with respect to a homomorphism between two Banach algebras, a class intermediate between Ruston and Fredholm elements.

1 Introduction

Let $\mathbb C$ denote the set of all complex numbers and let A and B denote complex Banach algebras, with identities denoted in both cases with 1, and invertible groups A^{-1} and B^{-1} , respectively. By $\sigma(x,A)$ we denote the spectrum of an element $x\in A$. The radical of A is the set

$$Rad(A) = \{x \in A : 1 - Ax \subset A^{-1}\} = \{x \in A : 1 - xA \subset A^{-1}\}.$$

The radical is unchanged if the invertible group A^{-1} is replaced by either the semi-group A^{-1}_{left} of left invertible elements, or the semigroup A^{-1}_{right} of right invertible elements, and can also be realised as the intersection of all maximal proper left ideals, similarly right ideals.

Let S be a subset of A. The *commutant* of S is defined by $comm(S) = \{x \in A : xs = sx \text{ for all } s \in S\}$. The perturbation class of S, denoted by P(S), is the set

$$P(S) = \{x \in A : x + s \in S \text{ for every } s \in S\}.$$

The quasinilpotents of A form the set

$$QN(A) = \{x \in A : \sigma(x, A) = \{0\}\} = \{x \in A : 1 - \mathbb{C}x \subset A^{-1}\}.$$
 (1)

Recall that [6, Theorem 2.11]: if $x, y \in A$, then

$$xy = yx \Longrightarrow \sigma(xy, A) \subset \sigma(x, A)\sigma(y, A)$$
 (2)

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and

$$xy = yx \Longrightarrow \sigma(x+y,A) \subset \sigma(x,A) + \sigma(y,A).$$
 (3)

From (2) it follows that

$$x \in QN(A) \Longrightarrow \operatorname{comm}(x)x \subset QN(A),$$
 (4)

which together with (1) implies

$$QN(A) = \{ x \in A : 1 - \text{comm}(x)x \subset A^{-1} \}.$$
 (5)

The radical can be recognised as the perturbation class of the invertible group, and the quasinilpotents as a sort of commutative analogue: there are the equivalencies

$$x \in \operatorname{Rad}(A) \iff (\forall g)(g \in A^{-1} \Longrightarrow x + g \in A^{-1})$$
 (6)

and

$$x \in QN(A) \iff (\forall g)(g \in A^{-1} \cap \text{comm}(x) \Longrightarrow x + g \in A^{-1}).$$
 (7)

Observe also that (6) holds separately for the left and right invertible semigroups and that (7) follows from (5) and (1).

Recall that [6, Theorem 1.43]

$$x \in A$$
 is invertible $\iff x + \operatorname{Rad}(A)$ is invertible in $A/\operatorname{Rad}(A)$. (8)

Consequently,

$$\sigma(x, A) = \sigma(x + \operatorname{Rad}(A), A/\operatorname{Rad}(A)) \tag{9}$$

and

$$x$$
 is quasinilpotent $\iff x + \operatorname{Rad}(A)$ is quasinilpotent. (10)

A map $T: A \to B$ is a homomorphism if T is linear and satisfies T(xy) = TxTy, $x, y \in A$, and T1 = 1. The homomorphism T has the Riesz property if 0 is the only one possible point of accumulation of $\sigma(x, A)$ for every $x \in T^{-1}(0)$, that is, if Tx = 0, then $\sigma(x, A)$ is either finite or a sequence converging to 0 [1].

If $T:A\to B$ is a homomorphism, then $T(A^{-1})\subset B^{-1}$ and hence

$$A^{-1} \subset A^{-1} + T^{-1}(0) \subset T^{-1}(B^{-1}).$$

Recall the following definitions from [1], [3]:

An element $a \in A$ is T-Fredholm if it has an invertible image,

$$a \in T^{-1}(B^{-1}),$$

and T-Weyl if it splits into the sum of an invertible and an essentially null element:

$$a \in A^{-1} + T^{-1}(0)$$
.

Thus, a is T-Weyl if

$$a = c + d$$
 with $c \in A^{-1}$, $Td = 0$.

If the previous sum is commutative, then $a \in A$ is T-Browder:

$$a \in A^{-1} +_c T^{-1}(0)$$
.

Corresponding spectra of $a \in A$ are defined as:

$$\sigma_f^T(a) = \sigma(Ta, B)$$
-the Fredholm spectrum,

$$\sigma_w^T(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not } T - \text{Weyl}\}\text{-the Weyl spectrum.}$$

2 Almost essentially Ruston elements

Recall the following definitions from [4]:

An element $a \in A$ is T-Ruston if

$$a = c + d$$
 with $c \in A^{-1}, cd - dc \in \{0\}, Td \in QN(B),$

an element $a \in A$ is essentially T-Ruston if

$$a = c + d$$
 with $c \in A^{-1}$, $cd - dc \in T^{-1}(0)$, $Td \in QN(B)$.

Let us mention that essentially Ruston elements are called almost Ruston elements in [4]. We introduce the following elements which are intermediate between essentially Ruston and Fredholm elements:

An element $a \in A$ is almost essentially T-Ruston if

$$a = c + d$$
 with $c \in A^{-1}$, $cd - dc \in T^{-1}(\operatorname{Rad}(B))$, $Td \in QN(B)$.

2.1 Theorem. Let $T: A \to B$ be a homomorphism. If $a \in A$ is an almost essentially T-Ruston element, that is a = c + d where $c \in A^{-1}$, $Td \in QN(B)$ and $cd - dc \in T^{-1}(\text{Rad}(B))$, then $\sigma(Ta, B) = \sigma(Tc, B)$ and a is T-Fredholm.

Proof. Let $\pi: B \to B/\mathrm{Rad}(B)$ denote the quotient map and let a = c + d where $c \in A^{-1}$, $Td \in QN(B)$ and $cd - dc \in T^{-1}(\mathrm{Rad}(B))$. Hence $\pi(Tc)$ and $\pi(Td)$ commute. By (9) we have $\sigma(Ta, B) = \sigma(\pi(Ta), B/\mathrm{Rad}(B))$, $\sigma(\pi(Td), B/\mathrm{Rad}(B)) = \sigma(Td, B) = \{0\}$ and $\sigma(\pi(Tc), B/\mathrm{Rad}(B)) = \sigma(Tc, B)$. Then, according to (3), we have

$$\begin{split} \sigma(Ta,B) &= \sigma(\pi(Ta), B/\mathrm{Rad}(B)) \\ &= \sigma(\pi(Tc) + \pi(Td), B/\mathrm{Rad}(B)) \\ &\subset \sigma(\pi(Tc), B/\mathrm{Rad}(B)) + \sigma(\pi(Td), B/\mathrm{Rad}(B)) \\ &= \sigma(\pi(Tc), B/\mathrm{Rad}(B)) \\ &= \sigma(Tc, B). \end{split}$$

As c=a-d and $ad-da \in T^{-1}(\operatorname{Rad}(B))$, we get $\sigma(Tc,B) \subset \sigma(Ta,B)$. Hence $\sigma(Ta,B)=\sigma(Tc,B)$. Since $c\in A^{-1}$, it follows that $0\notin \sigma(Tc,B)$, which implies that $0\notin \sigma(Ta,B)$, i.e. $Ta\in B^{-1}$. Thus a is T-Fredholm. \square

Let us remark that the fact that every almost essentially T-Ruston element is T-Fredholm can be proved also by using (7):

Suppose that $a \in A$ is an almost essentially T-Ruston element, i.e. a = c + d where $c \in A^{-1}$, $Td \in QN(B)$ and $T(cd - dc) \in \text{Rad}(B)$. Then $\pi(Ta) = \pi(Tc) + \pi(Td)$, $\pi(Tc)$ is invertible, $\pi(Td)$ is quasinilpotent and $\pi(Tc)$ and $\pi(Td)$ commute. According to (7) we conclude that $\pi(Ta)$ is invertible which by (8) implies that Ta is invertible. Hence a is T-Fredholm.

Therefore we have:

$$Browder \Rightarrow \frac{Ruston}{Weyl} \Rightarrow essentially Ruston \Rightarrow almost essentially Ruston \Rightarrow Fredholm$$
(11)

The essentially Ruston spectrum of $a \in A$ is defined as:

$$\sigma_{er}^T(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not essentially } T - \text{Ruston}\},$$

and the almost essentially Ruston spectrum of $a \in A$ is defined as:

$$\sigma_{aer}^T(a) = \{ \lambda \in \mathbb{C} : a - \lambda \text{ is not almost essentially } T - \text{Ruston} \}.$$

Let us remark that

$$\sigma_{aer}^T(a) = \bigcap \{ \sigma(a-d, A) : Td \in QN(B), ad - da \in T^{-1}(Rad(B)) \},$$

and consequently, this spectrum is compact. Clearly,

$$\sigma_f^T(a) \subset \sigma_{aer}^T(a) \subset \sigma_{er}^T(a) \subset \sigma_w^T(a),$$
 (12)

and we conclude that $\sigma_{aer}^T(a)$ is nonempty.

In [4, Theorem 6.6] it is proved that if $T:A\to B$ is a homomorphism with closed range which satisfies the Riesz property, then Ruston elements are Browder and essentially Ruston elements are Weyl. We can improve the second assertion:

2.2 Theorem. If $T:A\to B$ is a homomorphism with closed range which satisfies the Riesz property, then every almost essentially T-Ruston element is T-Weyl. **Proof.** Let a=c+d, where $c\in A^{-1}$, $d\in T^{-1}(QN(B))$ and $cd-dc\in T^{-1}(\operatorname{Rad}(B))$. As $TcTd-TdTc\in\operatorname{Rad}(B)$, it follows that $Tc^{-1}Td-TdTc^{-1}\in\operatorname{Rad}(B)$, since $\operatorname{Rad}(B)$ is a two-sided ideal. If $\pi:B\to B/\operatorname{Rad}(B)$ denotes the quotient map, then $\pi(Tc^{-1})$ and $\pi(Td)$ commute, and by (10), $\pi Td\in QN(B/\operatorname{Rad}(B))$. From

(4) it follows that $\pi(T(c^{-1}d)) \in QN(B/\text{Rad}(B))$, and so $T(c^{-1}d) \in QN(B)$ by (10). Therefore, $c^{-1}a = 1 + c^{-1}d$ is T-Ruston and by [4, Theorem 6.6.1], $c^{-1}a$ is T-Browder. Thus $a = c(c^{-1}a)$ is T-Weyl.

2.3 Corollary. If $T: A \to B$ is a homomorphism with closed range and satisfies the Riesz property, then the set of T-Weyl elements, the set of essentially T-Ruston elements and the set of almost essentially T-Ruston elements are equal.

Proof. Follows from Theorem 2.2 and the diagram (11).

2.4 Corollary. If $T:A\to B$ is a homomorphism with closed range and satisfies the Riesz property, then for $a\in A$

$$\sigma_{aer}^T(a) = \sigma_{er}^T(a) = \sigma_w^T(a).$$

If $K \subset \mathbb{C}$ is compact, then ηK denotes the connected hull of K, i.e. ηK is the union of K and the bounded components of the complement of K.

2.5 Corollary. If $T: A \to B$ is an onto homomorphism and $a \in A$, then $\sigma_{aer}^T(a) \subset \eta \sigma_f^T(a)$ and hence $\eta \sigma_{aer}^T(a) = \eta \sigma_f^T(a)$.

Proof. Follows from the first and the second inclusion in (12), and [4, Corollary 7.4].

2.6 Corollary. If $T:A\to B$ is a homomorphism with closed range and $a\in A$, then $\sigma_{aer}^T(a)\subset \eta\sigma_f^T(a)$ and hence $\eta\sigma_{aer}^T(a)=\eta\sigma_f^T(a)$.

Proof. Follows from the first and the second inclusion in (12), and [4, Corollary 7.5].

V. Müller [5, Definition 1] introduced the following concepts:

A subset R of a Banach algebra A is an *upper semiregularity* if

- (i) $a \in R$, $n \in \mathbb{N} \Rightarrow a^n \in R$,
- (ii) if a, b, c, d are mutually commuting elements of A satisfying ac + bd = 1 and $a, b \in R$, then $ab \in R$,
 - (iii) R contains a neighbourhood of 1,

and an lower semiregularity if

- (i) $a \in R$, $n \in \mathbb{N}$, $a^n \in R \Rightarrow a \in R$,
- (ii) if a, b, c, d are mutually commuting elements of A satisfying ac + bd = 1 and $ab \in R$, then $a, b \in R$.
- **2.7 Corollary.** Let $T: A \to B$ be a homomorphism. If either B is commutative or T has closed range and satisfies the Riesz property, then the set of almost essentially T-Ruston elements is an upper semiregularity.

Proof. If B is commutative, then from the definitions it is obvious that the set of almost essentially T-Ruston elements is equal to the set of essentially T-Ruston

elements. If T has closed range and satisfies the Riesz property, then, by Corollary 2.3, these sets are equal again. Now the assertion follows from [4, Corollary 8.4]. \square

2.8 Corollary. Let $T: A \to B$ be a homomorphism. If either B is commutative or T has closed range and satisfies the Riesz property, then

$$\sigma_{aer}^T(f(a)) \subset f(\sigma_{aer}^T(a))$$

for all $a \in A$ and all functions f analytic on a neighbourhood of $\sigma(a)$ and non-constant on each component of its domain of definition.

Proof. Follows from Corollary 2.7 and [4, Corollary 8.5].

In [4] it is remarked that from [2, Example 4.4] it follows that the set of T-Weyl elements is not a lower semiregularity even if the homomorphism T has the Riesz property and closed range. From Corollary 2.3 and the same example it follows that, in general, the set of almost essentially Ruston elements is not a lower semiregularity.

In [4] it is proved that if $a \in A$ is a Ruston (essentially Ruston) element, then a^n is a Ruston (essentially Ruston) element for every $n \in \mathbb{N}$. We prove:

2.9 Theorem. Let $T: A \to B$ be a homomorphism and $a \in A$. If a is almost essentially T-Ruston, then a^n is almost essentially T-Ruston for every $n \in \mathbb{N}$.

Proof. Suppose that a=c+d, where $c\in A^{-1}$, $Td\in QN(B)$ and $cd-dc\in T^{-1}(\operatorname{Rad}(B))$. Then $\pi(Tc)$ and $\pi(Td)$ commute, and by (9), $\sigma(\pi(Td), B/\operatorname{Rad}(B))=\sigma(Td,B)=\{0\}$, where $\pi:B\to B/\operatorname{Rad}(B)$ denotes the quotient map. For $n\in\mathbb{N}$, $a^n=c_1+d_1$, with $c_1=c^n\in A^{-1}$ and $d_1=\sum_{k=1}^n\binom{n}{k}c^{n-k}d^k$. Hence, according to (2) and (3), we get

$$\sigma(\pi(Td_1), B/\operatorname{Rad}(B)) = \sigma(\sum_{k=1}^n \binom{n}{k} (\pi(Tc))^{n-k} (\pi(Td))^k, B/\operatorname{Rad}(B))$$

$$\subset \sum_{k=1}^n \binom{n}{k} (\sigma(\pi(Tc), B/\operatorname{Rad}(B)))^{n-k} (\sigma(\pi(Td), B/\operatorname{Rad}(B)))^k$$

$$= \{0\}.$$

By (9), $\sigma(Td_1, B) = \sigma(\pi(Td_1), B/\text{Rad}(B))$, and so $Td_1 \in QN(B)$. Since $\pi(Tc)$ and $\pi(Td)$ commute, it follows that $\pi(Tc_1)$ and $\pi(Td_1)$ commute, i.e. $c_1d_1 - d_1c_1 \in T^{-1}(\text{Rad}(B))$. Therefore, a^n is almost essentially T-Ruston.

In [4] it is noticed that product of two Ruston elements does not have to be Ruston. We remark that also product of two essentially Ruston element is not always essentially Ruston:

Suppose that a = c + d where $c \in A^{-1}$, $Td \in QN(B)$ and $TcTd - TdTc \in Rad(B)\setminus\{0\}$, in other words a is almost essentially T-Ruston but not essentially

T-Ruston. As in the proof of Theorem 2.2 we conclude that $T(c^{-1}d) \in QN(B)$. Thus $1+c^{-1}d$ is T-Ruston and so it is an essentially T-Ruston element. Since $a=c(1+c^{-1}d)$, it follows that a is a product of two essentially T-Ruston elements which is not essentially T-Ruston.

Moreover, we see that every almost essentially T-Ruston element is the product of two T-Ruston elements.

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Snežana Č. Živković-Zlatanović Department of Mathematics, University of Niš *E-mail*: mladvlad@ptt.rs Robin E. Harte

School of Mathemetics, Trinity College Dublin

E-mail: rharte@maths.tcd.ie