COMMON FIXED POINTS OF GENERALIZED ALMOST NONEXPANSIVE MAPPINGS

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Abstract

The concept of a generalized almost nonexpansive mappings is introduced and the existence of common fixed points for this new class of mappings is proved. As an application, an invariant approximation result is obtained.

1 Introduction and preliminaries

In 1968, Kannan [12] proved a fixed point theorem for a map satisfying a contractive condition that did not require continuity at each point. This paper was a genesis for a multitude of fixed point papers over the next two decades. Sessa [13] coined the term weakly commuting maps. Jungck [8] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps [10].

The concept of almost contraction property was extended to a pair of selfmaps as follows:

Definition 1.1. Let T and f be two selfmaps of a metric space (X, d). A map T is called an almost f- contraction if there exist a constant $\delta \in]0,1[$ and some $L \geq 0$ such that

$$d(Tx, Ty) \le \delta \ d(fx, fy) + L \ d(fy, Tx), \tag{1}$$

for all $x, y \in X$. If we choose $f = I_X$, I_X is the identity map on X, we obtain the definition of almost contraction, the concept introduced by Berinde ([5], [6]). This concept was introduced by Berinde as 'weak contraction' in [5]. But in [6], Berinde renamed 'weak contraction' as 'almost contraction' which is appropriate. It was shown in [5] that any strict contraction, the Kannan [12] and Zamfirescu [14] mappings, as well as a large class of quasi-contractions, are all almost contractions.

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Let T and f be two selfmaps of a metric space (X, d). T is said to be f-contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(fx, fy)$ for all $x, y \in X$. This definition can be obtained directly from (1) if we take L = 0.

In 2006, Al-Thagafi and Shahzad [2] proved the following theorem which is a generalization of many known results.

Theorem 1.2 (Al-Thagafi and Shahzad ([2], Theorem 2.1)). Let E be a subset of a metric space (X,d) and f and T be selfmaps of E and $T(E) \subseteq f(E)$. Suppose that f and T are weakly compatible, T is f-contraction and T(E) is complete. Then f and T have a unique common fixed point in E.

Babu, Sandhya and Kameswari [3] considered the class of mappings that satisfy 'condition (B)'.

Let (X,d) be a metric space. A map $T:X\to X$ is said to satisfy 'condition (B)' if there exist a constant $\delta\in]0,1[$ and some $L\geq 0$ such that

$$d(Tx, Ty) \le \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},\$$

for all $x, y \in X$.

Recently, Berinde established the following fixed point result.

Theorem 1.3 (Berinde ([6], Theorem 3.4)). Let (X,d) be a complete metric space and $T: X \to X$ a mapping for which there exist $\alpha \in]0,1[$ and some $L \geq 0$ such that for all $x,y \in X$

$$d(Tx, Ty) \le \alpha M(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{2}$$

where, $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$. Then

- (1) T has a unique fixed point, i.e., $F(T) = \{x^*\}$;
- (2) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (1.1) converges to some $x^* \in F(T)$
- (3) the prior estimate

$$d(x_n, x^*) \le \frac{\alpha^n}{(1-\alpha)^2} d(x_0, x_1)$$

holds, for $n = 1, 2, \dots$,

(4) the rate of convergence of Picard iteration is given by

$$d(x_n, x^*) < \theta \ d(x_{n-1}, x^*)$$

for
$$n = 0, 1, 2, \cdots$$
.

The contractive condition (2) is termed as generalized condition B. We introduce the following definition as follows:

Definition 1.4. Let T and f be two selfmaps of a metric space (X, d). A map T is called *generalized almost f- contraction* if there exists $\delta \in]0,1[$ and $L \geq 0$ such that

$$d(Tx, Ty) \le \delta M(x, y) + L \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}$$
(3)

for all $x, y \in X$, where

$$M(x,y) = \max\{d(fx,fy),d(fx,Tx),d(fy,Ty),\frac{d(fx,Ty)+d(fy,Tx)}{2}\}.$$

If $f = I_X$, then we say that T satisfies 'generalized condition (B)'.

Example 1.5. Let X = [0,1) with usual metric. Define $T, f: X \to X$ as

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x < \frac{2}{3} \\ \frac{2}{3} & \text{if } \frac{2}{3} \le x < 1 \end{cases}$$

and

$$f(x) = \begin{cases} \frac{5}{6} & \text{if } 0 \le x < \frac{2}{3} \\ \frac{4}{3} - x & \text{if } \frac{2}{3} \le x < 1. \end{cases}$$

Here T is generalized almost f – contraction with $\delta = \frac{1}{2}$ and L = 0.

But, when $x \in [0, \frac{2}{3})$ and $y = \frac{2}{3}$, we have $d(Tx, Ty) = \frac{1}{6}$; and $d(fx, fy) = \frac{1}{6}$ so that for any $\alpha \in [0, 1)$, T fails to be an f-contraction.

Let Y be a subset of a normed space $(X, \|.\|)$. The set $P_Y(u) = \{x \in Y : \|x - u\| = dist(u, Y)\}$ is called the set of best approximants to $u \in X$ out of Y, where $dist(u, Y) = inf\{\|y - u\| : y \in Y\}$. We shall use $\mathbb N$ to denote the set of positive integers, cl(Y) to denote the closure of a set Y and wcl(Y) to denote the weak closure of a set Y. Let $f: Y \to Y$ be a mapping. The set of fixed points of T(resp. f) is denoted by F(T)(resp. F(f)). A point $x \in Y$ is a coincidence point (common fixed point) of f and T if fx = Tx (x = fx = Tx). The set of coincidence points of f and T is denoted by C(f,T). A pair (f,T) of self-mappings on X is said to be weakly compatible if f and T commute at their coincidence point (i.e. fTx = Tfx, $x \in X$ whenever fx = Tx). A point $y \in X$ is called a point of coincidence of two self-mappings f and T on X if there exists a point $x \in X$ such that y = Tx = fx.

The set Y is called q-starshaped with $q \in Y$, if the segment $[q, x] = \{(1 - k)q + kx : 0 \le k \le 1\}$ joining q to x is contained in Y for all $x \in Y$. For further details we refer to [4], [7], [9],[11] and referenced mentioned therein.

Definition 1.6. Let X be a normed space and M be a q-starshaped subset of X. Then a selfmap T of X is said to be generalized almost f- nonexpansive if

$$d(Tx,Ty) \leq \max\{d(fx,fy), dist(fx,[q,Tx]), dist(fy,[q,Ty]), \frac{dist(fx,[q,Ty]) + dist(fy,[q,Tx])}{2}\} + L\min\{dist(fx,[q,Tx]), dist(fy,[q,Ty]), dist(fx,[q,Ty]), dist(fy,[q,Tx])\}$$

$$(4)$$

for all $x, y \in X$, $L \ge 0$.

Definition 1.7. Let (X, d) be a metric space, T and f be self-mappings on X, with $T(X) \subset f(X)$, and $x_0 \in X$. Choose a point x_1 in X such that $fx_1 = Tx_0$. This can be done since $T(X) \subset f(X)$. Continuing this process having chosen x_1, \dots, x_k , we choose x_{k+1} in X such that

$$fx_{k+1} = Tx_k, \quad k = 0, 1, 2, \cdots.$$

The sequence $\{fx_n\}$ is called a *T-sequence* with initial point x_0 .

2 Common fixed point theorems

First, we establish a result on the existence of points of coincidence and common fixed points for two weakly compatible maps. We then, apply this result to obtain common fixed point of generalized almost f – nonexpansive mapping.

Theorem 2.1. Let Y be a nonempty subset of a metric space (X,d), and f and T be weakly compatible self-maps of Y. Assume that $clT(Y) \subset f(Y)$, clT(Y) is complete, and T is generalized almost f- contraction. Then $Y \cap F(f) \cap F(T)$ is singleton.

Proof As $T(Y) \subseteq f(Y)$, one can choose $\{fx_n\}$ which is a *T-sequence* with initial point x_0 . For each n, using (3), we have

$$d(Tx_n, Tx_{n+1}) \le \delta M(x_n, x_{n+1})$$

$$+L \min\{d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), d(fx_n, Tx_{n+1}), d(fx_{n+1}, Tx_n)\}$$
(5)

where

$$M(x_n, x_{n+1}) = \max\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{d(fx_n, Tx_{n+1}) + d(fx_{n+1}, Tx_n)}{2}\}.$$

Using $Tx_n = fx_{n+1}$ in (5), we obtain

$$d(Tx_n, Tx_{n+1}) \le \delta \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), \frac{d(Tx_{n-1}, Tx_{n+1})}{2}\}$$

$$= \delta \max\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\}.$$

If for some n, $\max\{d(Tx_{n-1},Tx_n),d(Tx_n,Tx_{n+1})\}=d(Tx_n,Tx_{n+1})$, then from above inequality we have

$$d(Tx_n, Tx_{n+1}) < \delta d(Tx_n, Tx_{n+1}),$$

a contradiction. Therefore

$$d(Tx_n, Tx_{n+1}) \le \delta d(Tx_{n-1}, Tx_n). \tag{6}$$

From (6), we obtain

$$d(Tx_n, Tx_{n+1}) \le \delta d_0$$

where $d_0 = d(Tx_0, Tx_1)$. Thus for $m, n \in N$ with m > n,

$$\begin{split} d(Tx_n, Tx_{m+n}) & \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \ldots + d(Tx_{n+m-1}, Tx_{m+n}) \\ & + (\delta)^n d_0 + (\delta)^{n+1} d_0 + \ldots + (\delta)^{n+m-1} d_0. \end{split}$$

So

$$d(Tx_n, Tx_{m+n}) \le \sum_{i=n}^{n+m-1} (\delta)^i d_0.$$

Therefore $\{Tx_n\}$ is a Cauchy sequences in T(Y). It follows from completeness of clT(Y) that $Tx_n \to w \in clT(Y)$ and hence $fx_n \to w$ as $n \to \infty$. Consequently, $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} Tx_n = w \in clT(Y)$. Thus w = fy for some $y \in Y$. Now we show that fy = Ty. If not, then for $n \ge 1$, we have

$$d(w,Ty) \le d(w,Tx_n) + d(Tx_n,Ty) \le d(w,Tx_n) + \delta M(x_n,y) + L\min\{d(fx_n,Tx_n), d(fy,Ty), d(fx_n,Ty), d(fy,Tx_n)\}, (7)$$

where

$$M(x_n, y) = \max\{d(fx_n, fy), d(fx_n, Tx_n), d(fy, Ty), \frac{d(fx_n, Ty) + d(fy, Tx_n)}{2}\}.$$

Letting $n \to \infty$, on both side of (7), we obtain

$$d(w, Ty) \le \delta d(w, Ty)$$

a contradiction. Hence Ty = w = fy. We now show the point of coincidence is unique. Suppose that for some $z \in Y$, fz = Tz. Then by inequality (3), we get

$$d(fy, fz) = d(Ty, Tz) \leq \delta M(y, z) + L \min\{d(fy, Ty), d(fz, Tz), d(fy, Tz), d(fz, Ty)\}, (8)$$

where

$$M(x,y) = \max\{d(fy, fz), d(fy, Ty), d(fz, Tz), \frac{d(fy, Tz) + d(fz, Ty)}{2}\}.$$

By (8), we have

$$d(fy, fz) \le \delta d(fy, fz).$$

Hence fz = fy = Ty as $\delta \in (0,1)$. This implies that the point of coincidence of f and T is unique. Since f and T are weakly compatible and fy = Ty, we obtain TTy = fTy = Tfy, thereby showing that TTy is a point of coincidence of f and T. By the uniqueness of point of coincidence, we have TTy = fTy = Ty; thus Ty is a common fixed point of f and T. Consequently Ty is unique common fixed point of f and T.

Lemma 2.2. Let f and T be self-maps on a nonempty q-starshaped subset Y of a normed space X, f and T are weakly compatible, and T is generalized almost f-nonexpansive with $clT(Y) \subset f(Y)$, define a mapping T_n on Y by

$$T_n x = (1 - \mu_n)q + \mu_n T x,$$

where $\{\mu_n\}$ is a sequence of numbers in (0,1) such that $\lim_{n\to\infty} \mu_n = 1$. Then for each $n \geq 1$, T_n and f have exactly one common fixed point x_n in Y such that

$$fx_n = x_n = (1 - \mu_n)q + \mu_n Tx_n,$$

provided one of the following conditions hold;

- (i) $cl(T_n(Y))$ is complete for each n,
- (ii) for each n, $wcl(T_n(Y))$ is complete.

Proof. By definition,

$$T_n x = (1 - \mu_n)q + \mu_n T x.$$

Note that T_n is a self mapping on Y and $clT_n(Y) \subset f(Y)$. Also by (4),

$$\begin{split} \|T_{n}x - T_{n}y\| &= \mu_{n} \|Tx - Ty\| \\ &\leq \mu_{n} max\{\|fx - fy\|, dist(fx, [q, Tx]), dist(fy, [q, Ty]), \\ &\frac{dist(fx, [q, Ty]) + dist(fy, [q, Tx])}{2}\} \\ &+ \mu_{n} L \min\{dist(fx, [q, Tx]), dist(fy, [q, Ty]), dist(fx, [q, Ty]), \\ &dist(fy, [q, Tx])\} \\ &\leq \mu_{n} max\{\|fx - fy\|, \|fx - T_{n}x\|, \|fy - T_{n}y\|, \\ &\frac{\|fx - T_{n}y\| + \|fy - T_{n}x\|}{2}\} + \mu_{n} L\{\|fx - T_{n}x\|, \|fy - T_{n}y\|, \\ &\|fx - T_{n}y\|, \|fy - T_{n}x\|\} \end{split}$$

for each $x, y \in Y$. By Theorem 2.1, for each $n \ge 1$, there exists a unique $x_n \in Y$ such that $x_n = fx_n = T_n x_n$. Thus for each $n \ge 1$, $F(T_n) \cap F(f) \ne \phi$.

(ii) Conclusion follows from Theorem 2.1.

Theorem 2.3. Let f and T be self-maps on a q-starshaped subset Y of a normed space X. Assume that f and T are weakly compatible, , T is a generalized almost f-nonexpansive mapping with $clT(Y) \subset f(Y)$. Then $F(T) \cap F(f) \neq \phi$, provided one of the following conditions holds;

- (i) cl(T(Y)) is compact and T is continuous;
- (ii) X is complete, f is weakly continuous, wcl(T(Y)) is weakly compact and f-T is demiclosed at 0.

Proof.

(i) Define a mapping T_n on Y by

$$T_n x = (1 - \mu_n)q + \mu_n T x,$$

where $\{\mu_n\}$ is a sequence of numbers in (0,1) such that $\lim_{n\to\infty}\mu_n=1$. Notice that compactness of cl(T(Y)) implies that $clT_n(Y)$ is compact and thus complete. From Lemma 2.2, for each $n\geq 1$, there exists $x_n\in Y$ such that $x_n=fx_n=(1-\mu_n)q+\mu_nTx_n$. Also,

$$||x_n - Tx_n|| = ||(1 - \mu_n)q + \mu_n Tx_n - Tx_n||$$

= $(1 - \mu_n)||q - Tx_n|| \to 0$

- as $n \to \infty$. Since cl(T(Y)) is compact, there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to y$ as $m \to \infty$. Now, $x_m = (1 \mu_m)q + \mu_n Tx_m$ implies that $x_m \to y$ as $m \to \infty$. By the continuity of f and T and the fact $||x_m Tx_m|| \to 0$, we have $y \in F(T) \cap F(f)$. Thus $F(T) \cap F(f) \neq \phi$.
- (ii) The weak compactness of wclT(Y) implies that $wclT_n(Y)$ is weakly compact and hence complete due to completeness of X. From Lemma 2.2, for each $n \geq 1$, there exists $x_n \in Y$ such that $x_n = fx_n = T_nx_n = (1 \mu_n)q + \mu_nTx_n$. The analysis in (i), implies that $||x_n Tx_n|| \to 0$ as $n \to \infty$. The weak compactness of wclT(Y) implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $y \in Y$ as $m \to \infty$. Weak continuity of f implies that fy = y. Also we have, $fx_m Tx_m = x_m Tx_m \to 0$ as $m \to \infty$. If f T is demiclosed at 0, then fy = Ty and hence $F(T) \cap F(f) \neq \phi$.

Following is an invariant approximation result.

Theorem 2.4. Let Y be a subset of a normed space X and $f, T : X \to X$ be mappings such that $u \in F(f) \cap F(T)$ for some $u \in X$ and $T(\partial Y \cap Y) \subseteq Y$. Suppose that $P_Y(u)$ is nonempty and q-starshaped, f is continuous on $P_Y(u)$, $||Tx - Tu|| \le ||fx - fu||$ for each $x \in P_Y(u)$ and $f(P_Y(u)) \subseteq P_Y(u)$. If T and f are weakly compatible, F(f) is nonempty and q-starshaped for $q \in F(f)$, T is almost generalized f-nonexpansive type then $P_Y(u) \cap F(f) \cap F(T) \neq \phi$, provided one of the following conditions is satisfied;

- (i) T is continuous and $cl(T(P_Y(u)))$ is compact;
- (ii) X is complete, $wcl(T(P_Y(u)))$ is weakly compact, f is weakly continuous and either f T is demiclosed at **0**.

Proof. Let $x \in P_Y(u)$. Then for any $h \in (0,1)$, ||hu+(1-h)x-u|| = (1-h)||x-u|| < dist(u,C). It follows that the line segment $\{hu+(1-h)x: 0 < h < 1\}$ and the set Y are disjoint. Thus x is not in the interior of Y and so $x \in \partial Y \cap Y$. Since $T(\partial Y \cap Y) \subseteq Y$, Tx must be in Y. Also $fx \in P_Y(u)$, $u \in F(f) \cap F(T)$ and f and T satisfy $||Tx - Tu|| \le ||fx - fu||$, thus we have

$$||Tx - u|| = ||Tx - Tu|| \le ||fx - fu|| = ||fx - u|| = dist(u, Y).$$

It further implies that $Tx \in P_Y(u)$. Therefore T is a self map of $P_Y(u)$. The result now follows from Theorem 2.3.

References

- [1] M. A. Al-Thagafi, Common fixed points and best approximation, J. Approx. Theory 85(1996), 318-323.
- [2] M. A. Al-Thagafi and N. Shahzad, *Noncommuting selfmaps and invariant approximations*, Nonlinear Anal., 64(2006), 2778-2786.

- [3] G. V. R. Babu, M. L. Sandhya and M. V. R. Kameswari, A note on a fixed point theorem of Berinde on weak contractions, Carpathian J. Math., 24(1) (2008), 08–12.
- [4] I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory and Applications, vol. 2006, Article ID 74503, 7 pages, 2006.
- [5] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum, 9(1) (2004), 43–53.
- [6] V. Berinde, General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces, Carpathian J. Math., 24(2) (2008), 10–19.
- [7] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9(4) (1986), 771-779.
- [8] G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Amer. Math. Soc., 103(1988), 977-983.
- [9] G. Jungck and S. Sessa, Fixed point theorems in best approximation theory, Math. Japon., 42(1995), 249-252.
- [10] G. Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East J. Math. Sci., 4(1996), 199-215.
- [11] G. Jungck and N. Hussain, Compatible maps and invariant approximations, J. Math. Anal. Appl., 325(2007), 1003-1012.
- [12] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 10 (1968), 71–76.
- [13] S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math., 32(1982), 149-153.
- [14] T. Zamfirescu, Fix point theorems in metric spaces, Arch. Mat. (Basel), 23 (1972), 292–298.

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