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LOCAL SPECTRAL PROPERTIES FOR THE HELTON CLASS OF AN OPERATOR

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Abstract

In this paper we study some local spectral properties of the Helton class of an operator. In particular, we show that the Helton class of an operator preserves the property (Q) under some condition. Also, we prove that if R has SVEP, $S \in \text{Helton}_k(R)$, and $T \prec S$ then a-Browder's theorem holds for $f(T)$ for each $f \in H(\sigma(T))$.

1 Introduction

Throughout this note let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote, respectively, the algebra of bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in B(\mathcal{H})$ we shall write $N(T)$ and $R(T)$ for the null space and range of T. Also, let $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, and let $\sigma(T)$, $\sigma_r(T)$, $\sigma_a(T)$, and $\sigma_p(T)$ denote the spectrum, right spectrum, approximate point spectrum, and point spectrum of T, respectively. An operator $T \in B(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and $\alpha(T) < \infty$ and an operator T is called *lower semi-Fredholm* if $\beta(T) < \infty$. An operator $T \in B(\mathcal{H})$ is called *Fredholm* if $\alpha(T) < \infty$ and $\beta(T) < \infty$. The *index* of a Fredholm operator $T \in B(\mathcal{H})$ is given by

$$i(T) := \alpha(T) - \beta(T).$$

$T \in B(\mathcal{H})$ is called *Weyl* if it is Fredholm of index zero, and *Browder* if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_B(T)$ of $T \in B(\mathcal{H})$ are defined by ([7])

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

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$$\sigma_B(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$
$$\sigma_{re}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not lower semi-Fredholm}\},$$

respectively. Evidently

$$\sigma_{re}(T) \subseteq \sigma_e(T) \subseteq \sigma_W(T) \subseteq \sigma_B(T) = \sigma_e(T) \cup \operatorname{acc}\sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$. If we write iso $K = K \setminus \operatorname{acc} K$ then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\ \};$$

and

$$p_{00}(T) := \sigma(T) \setminus \sigma_B(T).$$

We say that *Weyl's theorem holds for* $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T),$$

and that *Browder's theorem holds for* $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_W(T) = p_{00}(T).$$

We consider the sets

$$\Phi_+(\mathcal{H}) = \{T \in B(\mathcal{H}) : R(T) \text{ is closed and } \alpha(T) < \infty\},$$
$$\Phi_+^-(\mathcal{H}) = \{T \in B(\mathcal{H}) : T \in \Phi_+(\mathcal{H}) \text{ and } i(T) \leq 0\}.$$

By definition,

$$\sigma_{ea}(T) := \cap\{\sigma_a(T + K) : K \in K(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in K(\mathcal{H})\}$$

is the Browder essential approximate point spectrum.

We say that *a-Browder's theorem holds for* $T \in B(\mathcal{H})$ if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

We say that an operator T has the *single valued extension property at* λ (abbreviated SVEP at λ) if for every open set U containing λ the only analytic function $f : U \longrightarrow \mathcal{H}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

It is known ([6],[8]) that if $T \in B(\mathcal{H})$ then we have :

$$\text{Weyl's theorem} \Longrightarrow \text{Browder's theorem};$$

$$a\text{-Browder's theorem} \Longrightarrow \text{Browder's theorem}.$$

In [9] J. W. Helton initiated the study of operator T which satisfy an identity of the form

$$T^{*m} - \binom{m}{1} T^{*m-1}T + \cdots + (-1)^m T^m = 0. \tag{1}$$

We need further study for this class of operators based on (1). Let R and S be in $B(\mathcal{H})$ and let $C(R,S) : B(\mathcal{H}) \to B(\mathcal{H})$ be defined by $C(R,S)(A) = RA - AS$. Then

$$C(R,S)^k(I) = \sum_{j=0}^{k} (-1)^{k-j} \left(\begin{array}{c} k \\ j \end{array} \right) R^j S^{k-j}.$$

Definition 1.1. Let $R \in B(\mathcal{H})$. If there is an integer $k \geq 1$ such that an operator S satisfies $C(R,S)^k(I) = 0$, we say that S belongs to *Helton class* of R with order k. We denote this by $S \in \text{Helton}_k(R)$.

We remark that $C(R,S)^k(I) = 0$ does not imply that $C(S,R)^k(I) \neq 0$ in general.

Example 1.2. Let $R \in B(l_2)$ be the unilateral weighted shift with the weight sequence $\{\alpha_n\}_{n=0}^{\infty} = \{\frac{1}{2}, \frac{2}{3}, \cdots\cdots, \frac{n+1}{n+2}, \cdots\}$ and let $S \in B(l_2)$ be the unilateral weighted shift with a weight sequence $\{\beta_n\}_{n=0}^{\infty} = \{2, \frac{7}{6}, \frac{15}{14}, \frac{26}{25} \cdots, \frac{(n+1)(3n+4)}{(n+2)(3n+1)}, \cdots\}$ which satisfies $\beta_{n+1}\beta_n - \frac{2(n+2)}{n+3}\beta_n + \frac{n+1}{n+3} = 0$. Then it is easy to show that $C(R,S)^k(I) = 0$, but $C(S,R)^k(I) \neq 0$ for some $k \geq 2$.

2 Spectral properties for Helton class

In this section we consider some relations between several spectral properties of an operator and its Helton class. We begin with the following lemma.

Lemma 2.1.([10]) Suppose R has SVEP and S belongs to Helton class of R with order k, i.e., $S \in \text{Helton}_k(R)$. Then S also has SVEP.

Recall that an operator T is called *hyponormal* if $T^*T \geq TT^*$. If T is a hyponormal operator then T has SVEP. In fact, if T is hyponormal then $N(T - \lambda) \subseteq N(T^* - \overline{\lambda})$ for each $\lambda \in \mathbb{C}$. Therefore T has finite ascent for each $\lambda \in \mathbb{C}$, and hence T has SVEP by [13, Proposition 1.8]. It is well known that if T is hyponormal then $\sigma_{re}(T) = \sigma_e(T)$ and $\sigma_r(T) = \sigma(T)$. We can extend this result as follows:

Lemma 2.2. Suppose T has SVEP. Then

$$\sigma_{re}(T) = \sigma_e(T) \text{ and } \sigma_r(T) = \sigma(T).$$

Proof. We first show that $\sigma_e(T) \subseteq \sigma_{re}(T)$. Suppose that $\lambda \notin \sigma_{re}(T)$. Then $T-\lambda$ is right-Fredholm, and hence $\beta(T-\lambda) < \infty$. Since T has SVEP at λ, it follows from [1, Corollary 3.19] that $\alpha(T-\lambda) \le \beta(T-\lambda)$. Therefore $T-\lambda$ is Fredholm, and so $\lambda \notin \sigma_e(T)$. Hence we have $\sigma_{re}(T) = \sigma_e(T)$. To prove the second equality it is sufficient to show that $\sigma(T) \subseteq \sigma_r(T)$. Suppose that $\lambda \notin \sigma_r(T)$. Then $T-\lambda$ is onto, and so $\beta(T-\lambda) = 0$. Since T has SVEP at λ, $\alpha(T-\lambda) \le \beta(T-\lambda) = 0$ by [1, Corollary 3.19]. Therefore $\alpha(T-\lambda) = 0$, and so $T-\lambda$ is invertible. So $\lambda \notin \sigma(T)$. Hence $\sigma_r(T) = \sigma(T)$. □

Proposition 2.3. Suppose that $S \in \text{Helton}_k(R)$. Then the following relations hold:
(i) $\sigma_r(R) \subseteq \sigma_r(S)$,
(ii) If R has SVEP then

$$\sigma_e(R) \subseteq \sigma_e(S),\ \sigma_W(R) \subseteq \sigma_W(S),\ \sigma_B(R) \subseteq \sigma_B(S), \text{ and } \sigma(R) \subseteq \sigma(S).$$

Proof. (i) Suppose first that $\lambda \notin \sigma_r(S)$. Then $S-\lambda$ is right invertible. So $(S-\lambda)^* = S^* - \overline{\lambda}$ is bounded below, and hence $\overline{\lambda} \notin \sigma_a(S^*)$. But $S \in \text{Helton}_k(R)$, hence $R^* \in \text{Helton}_k(S^*)$. So we have $\sigma_a(R^*) \subseteq \sigma_a(S^*)$ by [15, Theorem 3.6.1], and hence $\overline{\lambda} \notin \sigma_a(R^*)$. Therefore $R^* - \overline{\lambda}$ is bounded below, and so $R-\lambda$ is right invertible. Therefore $\lambda \notin \sigma_r(R)$. Hence we have $\sigma_r(R) \subseteq \sigma_r(S)$.
(ii) Suppose now that $\lambda \notin \sigma_e(S)$. By Lemma 2.2, $\sigma_{re}(S) = \sigma_e(S)$ when S has SVEP. Since $S \in \text{Helton}_k(R)$, it follows from [12, Theorem 3.1] that $\lambda \notin \sigma_{re}(R)$. But R has SVEP, hence $\sigma_{re}(R) = \sigma_e(R)$ by Lemma 2.2. Therefore $\lambda \notin \sigma_e(R)$, and hence $\sigma_e(R) \subseteq \sigma_e(S)$. Next, we show that $\sigma(R) \subseteq \sigma(S)$. Since R has SVEP and $S \in \text{Helton}_k(R)$, it follows from Lemma 2.1 that S has SVEP. Therefore $\sigma_r(R) = \sigma(R)$ and $\sigma_r(S) = \sigma(S)$ by Lemma 2.2, and hence $\sigma(R) \subseteq \sigma(S)$. Now we show that $\sigma_B(R) \subseteq \sigma_B(S)$. Observe that $\sigma_B(T) = \sigma_e(T) \cup \operatorname{acc}\sigma(T)$ for any $T \in B(\mathcal{H})$. So we have $\sigma_B(R) = \sigma_e(R) \cup \operatorname{acc}\sigma(R) \subseteq \sigma_e(S) \cup \operatorname{acc}\sigma(S) = \sigma_B(S)$, and hence $\sigma_B(R) \subseteq \sigma_B(S)$. Finally, we show that $\sigma_W(R) \subseteq \sigma_W(S)$. Since R and S have SVEP, Browder's theorem holds for R and S, respectively. Therefore $\sigma_W(R) = \sigma_B(R)$ and $\sigma_W(S) = \sigma_B(S)$, and hence $\sigma_W(R) \subseteq \sigma_W(S)$. □

Theorem 2.4. Suppose that Browder's theorem holds for R and let $S \in \text{Helton}_k(R)$. Then S satisfies Browder's theorem when $\sigma_W(R) \subseteq \sigma_W(S)$.

Proof. Suppose that Browder's theorem holds for R. Then $\sigma_W(R) = \sigma_B(R)$, and so R has SVEP at each $\lambda \in \mathbb{C} \setminus \sigma_W(R)$. Now we first show that S has SVEP at each $\lambda \in \mathbb{C} \setminus \sigma_W(R)$. Let U be any open set containing λ and let $f : U \to \mathcal{H}$ be any analytic function such that $(\mu - S)f(\mu) = 0$ for all $\mu \in U$. Since $S \in \text{Helton}_k(R)$ and the terms of the equation below are equal to zero when $j + s \neq r$, it suffices to consider only the case of $j + s = r$. Then we have the following equations:

$$\begin{aligned}
&\sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (R-\mu)^j (\mu - S)^{k-j} \\
= \; & \sum_{j=0}^{k} \sum_{r=0}^{j} \sum_{s=0}^{k-j} (-1)^{k-(s+r)} \begin{pmatrix} k \\ j \end{pmatrix} \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} k-j \\ s \end{pmatrix} R^r \mu^{j+s-r} S^{k-(j+s)} \\
= \; & \sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j} f(\mu) - (R-\mu)^k f(\mu) \\
= \; & \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (R-\mu)^j (\mu - S)^{k-j} f(\mu) - (R-\mu)^k f(\mu) \\
= \; & \sum_{j=0}^{k-1} \begin{pmatrix} k \\ j \end{pmatrix} (R-\mu)^j (\mu - S)^{k-j} f(\mu) \\
= \; & [\sum_{j=0}^{k-1} \begin{pmatrix} k \\ j \end{pmatrix} (R-\mu)^j (\mu - S)^{k-j-1}](\mu - S) f(\mu) = 0.
\end{aligned}$$

Since $\sum_{j=0}^{k} (-1)^{k-j} \begin{pmatrix} k \\ j \end{pmatrix} R^j S^{k-j} = 0$, we get that $(R-\mu)^k f(\mu) = 0$ for all $\mu \in U$. Since R has SVEP at each $\lambda \in \mathbb{C} \setminus \sigma_W(R)$ and U is any open neighborhood of λ, $(R-\mu)^{k-1} f(\mu) = 0$ for all $\mu \in U$. By induction, we have $f(\mu) = 0$ for all $\mu \in U$. Therefore S has SVEP at each $\lambda \in \mathbb{C} \setminus \sigma_W(R)$. But $\sigma_W(R) \subseteq \sigma_W(S)$; hence S has SVEP at each $\lambda \in \mathbb{C} \setminus \sigma_W(S)$. Therefore Browder's Theorem holds for S. □

The following example shows that although R has SVEP and $S \in \text{Helton}_k(R)$, Weyl's theorem may not hold for S.

Example 2.5. Let $R \in B(l_2)$ be defined by

$$R(x_0, x_1, x_2, \cdots, x_n, \cdots) := (\frac{1}{3}x_1, \frac{3}{10}x_2, \frac{5}{21}x_3, \cdots, \frac{2n+1}{(n+1)(2n+3)}x_n, \cdots)$$

for all $\{x_n\} \in l_2$. Then R has SVEP. Suppose that S is the weighted shift operator which satisfy $Sx_n = \beta_n x_{n+1}$ for all $\{x_n\} \in l_2$ and an operator S belongs to Helton class of R with order 2. We can find that for all nonnegative integer n

$$S(x_0, x_1, x_2, \cdots, x_n, \cdots) := (\frac{1}{2}x_1, \frac{1}{3}x_2, \frac{1}{4}x_3, \cdots, \frac{1}{(n+1)}x_n, \cdots).$$

Indeed, set $\alpha_n := \frac{2n+1}{(n+1)(2n+3)}$ and $\beta_n := \frac{1}{n+1}$. Then

$$\begin{aligned}
& \beta_{n+1}\beta_n - 2\alpha_{n+1}\beta_n + \alpha_n\alpha_{n+1} \\
= & \frac{1}{n+2}\frac{1}{n+1} - 2\frac{2n+3}{(n+2)(2n+5)}\frac{1}{n+1} + \frac{2n+1}{(n+1)(2n+3)}\frac{2n+3}{(n+2)(2n+5)} \\
= & \frac{1}{n+2}\{\frac{1}{n+1} - \frac{2(2n+3)}{(2n+5)(n+1)} + \frac{2n+1}{(n+1)(2n+5)}\} \\
= & \frac{1}{n+2}\{\frac{2n+5}{(n+1)(2n+5)} - \frac{2(2n+3)}{(2n+5)(n+1)} + \frac{2n+1}{(n+1)(2n+5)}\} \\
= & 0 \ \text{ for every nonnegative integer } \ n,
\end{aligned}$$

and so

$$\|S^k\| = \frac{1}{(k+1)!}$$

for all $k = 0, 1, 2, \cdots$. Therefore S is quasinilpotent, and hence it has SVEP. Hence a-Browder's theorem holds for S. However, S does not satisfy Weyl's theorem because $\sigma(S) = \sigma_W(S) = \{0\}$ and $\Pi_{00}(S) = \{0\}$.

However, a-Browder's theorem performs better. Recall that an operator $X \in B(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. $S \in B(\mathcal{H})$ is said to be a *quasiaffine transform of* $T \in B(\mathcal{H})$ (notation: $S \prec T$) if there is a quasiaffinity $X \in B(\mathcal{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are *quasisimilar.* In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 2.6. Let R have SVEP and $S \in \text{Helton}_k(R)$. Suppose that $T \prec S$. Then a-Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. Since R has SVEP and $S \in \text{Helton}_k(R)$, it follows from Lemma 2.1 that S has SVEP. Now we show that T has SVEP. Let U be any open set and let $f : U \longrightarrow \mathcal{H}$ be any analytic function such that $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $T \prec S$, there exists a quasiaffinity X such that $XT = SX$. So $X(T - \lambda) = (S - \lambda)X$ for all $\lambda \in U$. Since $(T - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, $0 = X(T - \lambda)f(\lambda) = (S - \lambda)Xf(\lambda)$ for all $\lambda \in U$. But S has SVEP; hence $Xf(\lambda) = 0$ for all $\lambda \in U$. Since X is a quasiaffinity, $f(\lambda) = 0$ for all $\lambda \in U$. Therefore T has SVEP. Since T has SVEP,

a-Browder's theorem holds for T. Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$, and so we have by [5, Theorem 3.1]

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)) \text{ for all } f \in H(\sigma(T)).$$

Therefore a-Browder's theorem holds for $f(T)$ for each $f \in H(\sigma(T))$. □

From the proof of Theorem 2.6, we obtain the following corollary.

Corollary 2.7. Let R have SVEP and $S \in \text{Helton}_k(R)$. Then a-Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$.

Lemma 2.8. Suppose that $S \in \text{Helton}_k(R)$. Then $S^n \in \text{Helton}_k(R^n)$ for any positive integer n and any integer $k \geq 2$.

Proof. Let $S \in \text{Helton}_k(R)$. Then we obtain the inclusion:

$$\text{Helton}_k(R) \subseteq \text{Helton}_{k+1}(R). \tag{2}$$

Indeed, we have the following relation:

$$\begin{aligned}
& RC(R,S)^k(I) - C(R,S)^k(I)S \\
= \; & \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}R^{i+1}S^{k-i} - \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}R^jS^{k+1-j} \\
= \; & R^{k+1} + \sum_{j=1}^{k}(-1)^{k+1-j}\left[\binom{k}{j-1} + \binom{k}{j}\right]R^jS^{k+1-j} + (-1)^{k+1}S^{k+1} \\
= \; & R^{k+1} + \sum_{j=1}^{k}(-1)^{k+1-j}\binom{k+1}{j}R^jS^{k+1-j} + (-1)^{k+1}S^{k+1} \\
= \; & \sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}R^jS^{k+1-j} \\
= \; & C(R,S)^{k+1}(I).
\end{aligned}$$

So we obtain that $C(R,S)^{k+1}(I) = 0$ when $C(R,S)^k(I) = 0$. Hence $S \in \text{Helton}_{k+1}(R)$. To see this lemma, we will use the induction. If $n = 2$, it is easy to check that $S^2 \in \text{Helton}_2(R^2)$. Assume that $S^m \in \text{Helton}_2(R^m)$ holds when $n = m$. By induction, we want to show that $S^{m+1} \in \text{Helton}_2(R^{m+1})$. From [15, Lemma 3.5.3] we get that

$$C(R^{m+1}, S^{m+1})^2(I) = 2RC(R^m, S^m)^2(I)S - R^2C(R^{m-1}, S^{m-1})^2(I)S^2.$$

Since $C(R^m, S^m)^2(I) = C(R^{m-1}, S^{m-1})^2(I) = 0$ by induction hypotheses,

$$C(R^{m+1}, S^{m+1})^2(I) = 0.$$

Hence $S^{m+1} \in \text{Helton}_2(R^{m+1})$. Thus $S^n \in \text{Helton}_2(R^n)$ for any positive integer n. Since $\text{Helton}_2(R^n) \subseteq \text{Helton}_k(R^n)$ for any positive integer $k \geq 2$ by (2), $S^n \in \text{Helton}_k(R^n)$ for any positive integer n. Now we will prove that $S^n \in \text{Helton}_k(R^n)$ for fix n. If $k = 2$, then the proof follows from the above results. Assume that $S^n \in \text{Helton}_m(R^n)$ holds for $k = m$. By (2), $S^n \in \text{Helton}_{m+1}(R^n)$ holds for $k = m + 1$. By induction, $S^n \in \text{Helton}_k(R^n)$ for any positive integer n. □

An operator T is said to have the *property* (H) if $H_0(T - \lambda) = N(T - \lambda)$ for all $\lambda \in \mathbb{C}$, where $H_0(T - \lambda) = \{x \in H : \lim_{n\to\infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\}$. Although the property H seems to be strong, the class of operators having property (H) is large ([1]).

Theorem 2.9. Suppose that $S \in \text{Helton}_k(R)$ with $k \geq 2$. Then $H_0(S - \lambda) \subseteq H_0(R - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. Suppose that $x \in H_0(S - \lambda)$. Since $S \in \text{Helton}_k(R)$, we have $S - \lambda \in \text{Helton}_k(R - \lambda)$. It follows from Lemma 2.8 that $(S - \lambda)^n \in \text{Helton}_k((R - \lambda)^n)$ for any positive integer n and any integer $k \geq 2$. Hence

$$\sum_{j=0}^{k}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}((R - \lambda)^n)^j((S - \lambda)^n)^{k-j} = 0.$$

Therefore for any $x \in H_0(S - \lambda)$ we have

$$[\sum_{j=0}^{k-1}(-1)^{k-j}\begin{pmatrix} k \\ j \end{pmatrix}((R - \lambda)^n)^j((S - \lambda)^n)^{k-j-1}](S - \lambda)^n x = -(R - \lambda)^{nk}x.$$

Since $\lim_{n\to\infty} \|(S - \lambda)^n x\|^{\frac{1}{n}} = 0$ for all $x \in H_0(S - \lambda)$, it follows that

$$\lim_{n\to\infty} \|(R - \lambda)^{nk}x\|^{\frac{1}{n}} = 0$$

for all $x \in H_0(S - \lambda)$ and any integer $k \geq 2$. Hence $\lim_{n\to\infty} \|(R - \lambda)^{nk}x\|^{\frac{1}{nk}} = 0$ for all $x \in H_0(S - \lambda)$ and any integer $k \geq 2$. Thus $\lim_{N\to\infty} \|(R - \lambda)^N x\|^{\frac{1}{N}} = 0$ for all $x \in H_0(S - \lambda)$. Therefore $x \in H_0(R - \lambda)$, and hence $H_0(S - \lambda) \subseteq H_0(R - \lambda)$. □

Corollary 2.10. Suppose that R has the property (H) and $S \in \text{Helton}_k(R)$ with $k \geq 2$. Then S has the property (H) when $H_0(R - \lambda) \subseteq N(S - \lambda)$ for all $\lambda \in \mathbb{C}$.

Proof. Since R has the property (H), we have $H_0(R-\lambda) = N(R-\lambda)$ for all $\lambda \in \mathbb{C}$. Since $S \in \text{Helton}_k(R)$ with $k \geq 2$, it follows from Theorem 2.9 that $H_0(S-\lambda) \subseteq H_0(R-\lambda)$ for all $\lambda \in \mathbb{C}$. Therefore we obtain $H_0(S-\lambda) \subseteq H_0(R-\lambda) = N(S-\lambda)$. The converse inclusion is trivial. Hence $H_0(S-\lambda) = N(S-\lambda)$ for all $\lambda \in \mathbb{C}$, and so S has the property (H). □

An operator T is said to have the *property* (Q) if $H_0(T-\lambda)$ is closed for every $\lambda \in \mathbb{C}$. If T is a quasinilpotent operator then it has the property (Q) because $H_0(T-\lambda) = \{0\}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and $H_0(T) = \mathcal{H}$. It is well known that the following implication holds:

$$\text{Property (Q)} \Longrightarrow \text{SVEP}.$$

The following theorem shows that the Helton class of an operator preserves the property (Q) under some condition.

Theorem 2.11. Suppose that R has the property (Q) and $S \in \text{Helton}_k(R)$ with $k \geq 2$. If $\sigma_S(x)$ is a singleton set for each nonzero $x \in \overline{H_0(S-\lambda)}$ and each $\lambda \in \mathbb{C}$ then S has the property (Q), where $\sigma_S(x)$ is the local spectrum of S at the point x.

Proof. We need only to prove that $\overline{H_0(S-\lambda)} \subseteq H_0(S-\lambda)$. Since $S \in \text{Helton}_k(R)$ with $k \geq 2$, it follows from Theorem 2.9 that $H_0(S-\lambda) \subseteq H_0(R-\lambda)$ for each $\lambda \in \mathbb{C}$. Since R has the property (Q), $H_0(R-\lambda)$ is closed. Therefore $\overline{H_0(S-\lambda)} \subseteq H_0(R-\lambda)$. Now we shall show that the inclusion $H_0(R-\lambda) \subseteq H_0(S-\lambda)$ holds. Let x be a nonzero element in $H_0(R-\lambda)$. Since R has the property (Q), it has SVEP. Since $S \in \text{Helton}_k(R)$, it follows from Lemma 2.1 that S has SVEP. It is well known that if T has SVEP then $H_0(T-\lambda) = X_T(\{\lambda\})$, where $X_T(\Omega) = \{x \in \mathcal{H} \mid \sigma_T(x) \subseteq \Omega\}$. So we obtain that $H_0(R-\lambda) = X_R(\{\lambda\})$ and $H_0(S-\lambda) = X_S(\{\lambda\})$, respectively. Therefore $x \in X_R(\{\lambda\})$, and hence $\sigma_R(x) \subseteq \{\lambda\}$. Since R has SVEP and since $x \neq 0$, $\sigma_R(x) = \{\lambda\}$. On the other hand, it follows from [15, Theorem 3.6.4] that $\sigma_R(x) \subseteq \sigma_S(x)$ when $S \in \text{Helton}_k(R)$ and R has SVEP. Since $\sigma_S(x)$ is a singleton set for each nonzero $x \in \overline{H_0(S-\lambda)}$, $\sigma_S(x) = \{\lambda\}$. Hence $x \in X_S(\{\lambda\})$, and so $x \in H_0(S-\lambda)$. Thus $H_0(R-\lambda) \subseteq H_0(S-\lambda)$. Therefore the proof is complete. □

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