Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:3** (2010), 103–109 DOI: 10.2298/FIL1003103M

MATRIX TRANSFORMATIONS OF STRONGLY CONVERGENT SEQUENCES INTO V_σ

S.A. Mohiuddine and M. Aiyub

Abstract

In this paper, we define the spaces $\omega(p,s)$ and $\omega_p(s)$, where

$$\omega(p,s)=\{x:\frac{1}{n}\sum_{k=1}^{n}K^{-s}|x_k-\ell|^{p_k}\to 0 \ \text{ for some } \ \ell,\ s\geq 0\}$$

and if $p_k=p$ for each k, we have $\omega(p,s)=\omega_p(s)$. We further characterize the matrix classes $(\omega(p,s),V_\sigma)$, $(\omega_p(s),V_\sigma)$ and $(\omega_p(s),V_\sigma)_{reg}$, where V_σ denotes the set of bounded sequences all of whose σ-mean are equal.

1 Introduction

In [11], Schaefer has defined the concept of σ-conservative, σ-regular and σ-coercive matrices and characterized matrix classes (c,V_σ), $(c,V_\sigma)_{reg}$ and (ℓ_∞,V_σ), where ℓ_∞ and c are the Banach spaces of bounded and convergent sequences $x=(x_{jk})$ with the usual norm $\|x\|=\sup_k|x_k|$, and V_σ denote the set of all bounded sequences all of whose invariant means (or σ-means) are equal. In [9], Mursaleen characterized the class $(c(p),V_\sigma)$, $(c(p),V_\sigma)_{reg}$ and $(\ell_\infty(p),V_\sigma)$ matrices which generalized the results due to Schaefer [11]. In [9], the author has determined the matrices of classes $(\ell(p),V_\sigma)$ and $(M_0(p),V_\sigma)$.

In this paper, we define some sequence spaces for more general sequence $s=(s_k)$. We further characterize the matrix classes from this spaces to the space V_σ of invariant mean, i.e. we obtain necessary and sufficient conditions to characterize the matrices of classes $(\omega(p,s),V_\sigma)$, $(\omega_p(s),V_\sigma)$ and $(\omega_p(s),V_\sigma)_{reg}$.

2010 *Mathematics Subject Classifications.* 40C05, 40H05.
Key words and Phrases. Invariant mean; matrix transformations; sequence spaces.
Received: August 13, 2009
Communicated by Dragan Djordjevic

2 Preliminaries

Let σ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional φ on ℓ_∞ is said to be an *invariant mean* or a σ-*mean* [11] if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k;

(ii) $\varphi(e) = 1$;

(iii) $\varphi(x) = \varphi(x_{\sigma(k)})$ for all $x \in \ell_\infty$.

By V_σ, we denote the set of bounded sequences all of whose σ-means are equal. We say that a sequence $x = (x_k)$ is σ-*convergent* if and only if $x \in V_\sigma$. For $\sigma(n) = n + 1$, the set V_σ is reduced to the set f of almost convergent sequences [2,10]. Note that $c \subset V_\sigma \subset \ell_\infty$.

The class V_2^σ and matrix transformations of double sequences, we refer to Çakan, Altay and Mursaleen [1], Mursaleen and Mohiuddine [5,6,7,8].

If $x = (x_n)$, write $Tx = (x_{\sigma(n)})$. It is easy to show that

$$V_\sigma = \{x \in \ell_\infty : \lim_m t_{mn}(x) = Le,\ L = \sigma\text{-}\lim x\},$$

where

$$t_{mn}(x) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n$$

and $\sigma^m(n)$ denotes the m-*th* iterate of σ at n.

If p_k is real such that $p_k > 0$ and $\sup_k p_k < \infty$ (see Maddox [4] and Simons [12])

$$\ell(p) = \{x : \sum_k |x_k|^{p_k} < \infty\},$$

$$\ell_\infty(p) = \{x : \sup_k |x_k|^{p_k} < \infty\},$$

$$c(p) = \{x : |x_k - \ell|^{p_k} \to 0 \ \text{ for some } \ \ell\},$$

$$\omega(p) = \{x : \frac{1}{n} \sum_{k=1}^{n} |x_k - \ell|^{p_k} \to 0 \ \text{ for some } \ \ell\}.$$

We define

$$\omega(p, s) = \{x : \frac{1}{n} \sum_{k=1}^{n} K^{-s} |x_k - \ell|^{p_k} \to 0 \ \text{ for some } \ \ell,\ s \geq 0\},$$

where $s = (s_k)$ is an arbitrary sequence with $s_k \neq 0,\ (k = 1, 2, \cdots)$. If $p_k = p$ for each k, we have $\ell(p) = \ell_p,\ \ell_\infty(p) = \ell_\infty,\ c(p) = c,\ \omega(p) = \omega_p$ and $\omega(p, s) = \omega_p(s)$.

If E is a subset of ω, the space of complex sequences, then we shall write E^+ for generalized Köthe-Toeplitz dual of E, i.e.

$$E^+ = \{a : \sum_k a_k x_k \ \text{ converges for every } \ x \in E\}.$$

If $0 < p_k \leq 1$ then $\omega^+(p) = \mathbb{M}$, where

$$\mathbb{M} = \left\{a : \sum_{r=0}^{\infty} \max_r\{(2^r N^{-1})^{1/p_k}|a_k|\} < \infty, \text{ for some integer } N > 1\right\},$$

and $\max_r$ is the maximum taken over $2^r \leq k < 2^{r+1}$ (see Theorem 4 [3]).

If X is a topological linear space, we shall denote X^* the continuous dual of X, i.e. the set of all continuous linear functional on X. Obviously,

$$[\omega(p,s)]^* = \left\{a : \sum_{r=0}^{\infty} \max_r\left\{(2^r N^{-1})^{1/p_k}\left|\frac{a_k}{s_k}\right|\right\} < \infty, \text{ for some integer } N > 1\right\}.$$

3 Main results

We shall use the notation $a(n,k)$ to denote the element a_{nk} of matrix A and we write for all integers $n, m \geq 1$

$$\begin{aligned} t_{mn}(Ax) &= (Ax_n + TAx_n) + \cdots + T^m Ax_n)/(m+1) \\ &= \sum_k t(n,k,m) x_k \end{aligned}$$

where

$$t(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{m} a(\sigma^j(n),k).$$

Theorem 3.1. Let $0 < p_k \leq 1$, then $A \in (\omega(p,s), V_\sigma)$ if and only if

(i) there exists an integer $B > 1$ such that for every n

$$D_n = \sup_m \sum_{r=0}^{\infty} \max_r (2^r B^{-1})^{1/p_k}\left|\frac{t(n,k,m)}{s_k}\right| < \infty,$$

(ii) $a_{(k)} = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma$ for each k;

(iii) $a = \{\sum_k a_{nk}\}_{n=1}^{\infty} \in V_\sigma$.

In this case the σ-limit of Ax is $(\lim x)[u - \sum_k u_k] + \sum_k u_k x_k$ for every $x \in \omega(p,s)$, where $u = \sigma\text{-}\lim a$ and $u_k = \sigma\text{-}\lim a_{(k)}$, $k = 1, 2, \cdots$.

Proof. Suppose that $A \in (\omega(p,s), V_\sigma)$. Define $e^k = (0, 0, \cdots 0, 1, 0, \cdots)$ having 1 in the kth entry. Since e and e^k are in $\omega(p,s)$, necessity of (ii) and (iii) is obvious. Now we know that $\sum_k t(n,k,m)x_k$ converges for each m and $x \in \omega(p,s)$ therefore $(t(n,k,m))_k \in \omega^+(p,s)$ and

$$\sum_{r=0}^{\infty} \max_r (2^r B^{-1})^{1/p_k} \left| \frac{t(n,k,m)}{s_k} \right| < \infty$$

for each m (see [3]). Furthermore, if $f_{mn}(x) = t_{mn}(Ax)$ then $\{f_{mn}\}_m$ is a sequence of continuous linear functional on $\omega(p,s)$ such that $\lim_{m\to\infty} t_{mn}(Ax)$ exists. Therefore by Banach-Steinhaus theorem, necessity of (i) is follows immediately.

Conversely, suppose that the conditions (i), (ii) and (iii) hold and $x \in \omega(p,s)$. We know that $(t(n,k,m))_k$ and u_k are in $\omega^+(p,s)$ the series $\sum_k t(n,k,m)x_k$ and $\sum_k u_k x_k$ converges for each m. We put

$$c(n,k,m) = t(n,k,m) - u_k$$

then

$$\sum_k t(n,k,m)x_k = \sum_k u_k x_k + \ell \sum_k c(n,k,m) + \sum_k c(n,k,m)(x_k - \ell)$$

by (ii) for an integer $k_0 > 0$, we have

$$\lim_m \sum_{k \le k_0} c(n,k,m)(x_k - \ell) = 0,$$

where ℓ being the limit of x for $x \in \omega(p,s)$. Now since

$$\sup_m \sum_r \max_r (2^r B^{-1})^{1/p_k} |c(n,k,m)| \le 2D_n,$$

$$\lim_m \sum_{k \le k_0} \left| \frac{t(n,k,m) - u_k}{s_k} \right| |s_k(x_k - \ell)| = 0,$$

whence

$$\lim_m \sum_k t(n,k,m)x_k = \ell u + \sum_k u_k(x_k - \ell).$$

This completes the proof of the theorem.

Theorem 3.2. Let $1 \le p_k < \infty$, then $A \in (\omega_p(s), V_\sigma)$ if and only if

(i) for every n,

$$M(A) = \sup_m \sum_r 2^{r/p} \left(\sum_r \left| \frac{t(n,k,m)}{s_k} \right|^q \right)^{1/q} < \infty,$$

where $p^{-1} + q_{-1} = 1$;

(ii) $a_{(k)} \in V_\sigma$ for each k;

(iii) $a \in V_\sigma$.

In this case the σ-limit is same as in Theorem 3.1.

Proof. Let the conditions are satisfied and $x \in \omega_p(s)$. Now

$$\begin{aligned} |t_{mn}(Ax)| &\le \sum_{r=0}^{\infty} \sum_r \left| \frac{t(n,k,m) s_k x_k}{s_k} \right| \\ &\le \sum_{r=0}^{\infty} \left(\sum_r \left| \frac{t(n,k,m)}{s_k} \right|^q \right)^{1/q} \left(\sum_r |x_k|^p \right)^{1/p} \\ &\le M(A)\|x\| < \infty, \end{aligned}$$

therefore $t_{mn}(Ax)$ is absolutely and uniformly convergent for each m. Note that (i) and (ii) imply that

$$\sum_{r=0}^{\infty} 2^{r/p} \left(\sum_r |s_k u_k| \right)^{1/q} \le M(A) < \infty$$

by Hölder's inequality $\sum_k u_k x_k < \infty$. Now as in the converse part of Theorem 3.1; it follows that $A \in (\omega_p(s), V_\sigma)$.

Conversely, suppose that $A \in (\omega_p(s), V_\sigma)$. Since e^k and e are in $\omega_p(s)$, necessity of (ii) and (iii) is obvious. For the necessity of (i), suppose that

$$t_{mn}(Ax) = \sum_k t(n,k,m) x_k$$

exits for each m whenever $x \in \omega_p(s)$. Then for each m and $r \ge 0$, define

$$f_{mr}(x) = \sum_r t(n,k,m) x_k.$$

Then $\{f_{mn}\}_m$ is a sequence of continuous linear functional on $\omega_p(s)$, since

$$\begin{aligned} |f_{mr}(x)| &\le \left(\sum_r \left| \frac{t(n,k,m)}{s_k} \right|^q \right)^{1/q} \left(\sum_r |s_k x_k|^p \right)^{1/p} \\ &\le 2^{r/p} \left(\sum_r \left| \frac{t(n,k,m)}{s_k} \right|^q \right)^{1/q} \|x\|, \end{aligned}$$

it follows ([4], corollary on pp. 114), that for each m

$$\lim_j \sum_{r=0}^{j} f_{mr}(x) = t_{mn}(Ax)$$

is in the dual space ω_p^*, hence there exists K_{mn} such that

$$\left|\frac{t(n,k,m)}{s_k}\right| \leq K_{mn}\|x\|. \tag{3.2.1}$$

For each m, we take any integer $j > 0$ and define $x \in \omega_p(s)$ as in ([4] Theorem 7 p. 173), we get

$$\sum_{r=0}^{j} 2^{r/p}\left(\sum_{r}\left|\frac{t(n,k,m)}{s_k}\right|^q\right)^{1/q} \leq K_{mn},$$

whence for each m

$$\sum_{r=0}^{\infty} 2^{r/p}\left(\sum_{r}\left|\frac{t(n,k,m)}{s_k}\right|^q\right)^{1/q} \leq K_{mn} < \infty. \tag{3.2.2}$$

Now, since $t_{mn}(x)(Ax)$ is absolutely convergent, we have

$$|t_{mn}(x)| \leq \sum_{r=0}^{\infty} 2^{r/p}\left(\sum_{r}\left|\frac{t(n,k,m)}{s_k}\right|^q\right)^{1/q}\|x\|$$

so that

$$K_{mn}(x) \leq \sum_{r=0}^{\infty} 2^{r/p}\left(\sum_{r}\left|\frac{t(n,k,m)}{s_k}\right|^q\right)^{1/q}. \tag{3.2.3}$$

By virtue of (3.2.2) and (3.2.3),

$$K_{mn} = \sum_{r=0}^{\infty} 2^{r/p}\left(\sum_{r}\left|\frac{t(n,k,m)}{s_k}\right|^q\right)^{1/q}.$$

Finally, by (Theorem 11 [4], p. 114) for every n, the existence of $\lim_m t_{mn}(Ax)$ on $\omega_p(s)$ implies that

$$\sup_m K_{mn} = \sup_m \sum_{r=0}^{\infty} 2^{r/p}\left(\sum_{r}\left|\frac{t(n,k,m)}{s_k}\right|^q\right)^{1/q} < \infty$$

which is (i).

This completes the proof of the theorem.

Theorem 3.3. Let $0 < p_k < \infty$, then $A \in (\omega_p(s), V_\sigma)_{reg}$ if and only if condition (i), (ii) with σ- $\lim = 0$ and (iii) with σ- $\lim = +1$ of Theorem 3.2 hold.

4 Acknowledgment

Research of the first author was supported by the Department of Atomic Energy, Government of India under the NBHM-Post Doctoral Fellowship programme number 40/10/2008-R&D II/892.

References

[1] C. Çakan, B. Altay and Mursaleen, *σ-convergence and σ-core of double sequences*, Appl. Math. Lett. 19 (2006), 1122-1128.

[2] G.G. Lorentz, *A contribution to theory of divergent sequences*, Acta Math. 80 (1948), 167-190.

[3] C.G. Lascarides and I.J. Maddox, *Matrix transformations between some classes of sequences*, Proc. Camb. Phil. Soc. 68 (1970), 99-104.

[4] I.J. Maddox, *Elements of Functional Analysis*, Univ. Press, Cambridge, 1970.

[5] Mursaleen and S.A. Mohiuddine, *Double σ-Multiplicative Matrices*, J. Math. Anal. Appl. 327 (2007) 991-996.

[6] M. Mursaleen and S.A. Mohiuddine, *Regularly σ-Conservative and σ-Coercive Four Dimensional Matrices*, Comp. Math. Appl. 56 (2008) 1580-1586.

[7] M. Mursaleen and S.A. Mohiuddine, *Invariant Mean and some Core Theorems for Double Sequences*, Taiwanese J. Math. 14 (1) (2010) 21-33.

[8] M. Mursaleen and S.A. Mohiuddine, *On σ-conservative and boundedly σ-conservative four-dimensional matrices*, Comp. Math. Appl. 59 (2010) 880-885.

[9] Mursaleen, *On infinite matrices and invariant means*, Indian J. Pure Appl. Math. 10 (4) (1979), 457-460.

[10] Qamaruddin and S.A. Mohiuddine, *Almost Convergence and some matrix transformations*, Filomat 21 (2) (2007), 261-266.

[11] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. 36 (1972), 104-110.

[12] S. Simons, *The sequence spaces $\ell(p)$ and $m(p)$*, Proc. London Math. Soc. 3 (15) (1965), 422-436.

Addresses:

S.A. Mohiuddine
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail: mohiuddine@gmail.com

M. Aiyub
Department of Mathematics, University of Bahrain, P.O. Box-32038, Kingdom of Bahrain
E-mail: maiyub2002@yahoo.com