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A REMARKABLE EQUALITY REFERRING TO SPLINE FUNCTIONS IN HILBERT SPACES

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Abstract

In the introduction of this paper is presented the definition of the generalized spline functions as solutions of a variational problem and are shown some theorems regarding to the existence and uniqueness. The main result of this article consists in a remarkable equality verified by the generalized spline elements, based on the properties of the spaces, operator and interpolatory set involved, which can be used as a characterization theorem of the generalized spline functions in Hilbert spaces.

1 Introduction

Definition 1. *Let E_1 be a real linear space, $(E_2, \|.\|_2)$ a normed real linear space, $T : E_1 \to E_2$ an operator and $U \subseteq E_1$ a non-empty set. The problem of finding the elements $s \in U$ which satisfy*

$$\|T(s)\|_2 = \inf_{u \in U} \|T(u)\|_2, \tag{1}$$

is called the general spline interpolation problem, corresponding to the set U.

A solution of this problem, provided that it exists, is named general spline interpolation element, corresponding to the set U.

The set U is called interpolatory set.

In the sequel we assume that E_1 is a real linear space, $(E_2, (.\,,.)_2, \|.\|_2)$ is a real Hilbert space, $T : E_1 \to E_2$ is a linear operator and $U \subseteq E_1$ is a non-empty set.

Theorem 1. (Existence Theorem) *If U is a convex set and $T(U)$ is a closed set, then the general spline interpolation problem (1) (corresponding to U) has at least one solution.*

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The proof is shown in the papers [1, 3].

For every element $s \in U$ we define the set

$$U(s) := U - s. \tag{2}$$

Lemma 1. *For every element $s \in U$ the set $U(s)$ is non-empty ($0_{E_1} \in U(s)$).*

The result follows directly from the relation (2).

Theorem 2. (Uniqueness Theorem) *If U is a convex set, $T(U)$ is a closed set and there exists a solution $s \in U$ of the general spline interpolation problem (1) (corresponding to U), such that $U(s)$ is a linear subspace of E_1, then the following statements are true*

i) *For any solutions $s_1, s_2 \in U$ of the general spline interpolation problem (1) (corresponding to U) we have*

$$s_1 - s_2 \in Ker(T) \cap U(s); \tag{3}$$

ii) *The element $s \in U$ is the unique solution of the general spline interpolation problem (1) (corresponding to U) if and only if*

$$Ker(T) \cap U(s) = \{0_{E_1}\}. \tag{4}$$

A proof is presented in the papers [1, 2].

Lemma 2. *For every element $s \in U$ the following statements are true*

i) *$T(U(s))$ is a non-empty set ($0_{E_2} \in T(U(s))$);*

ii) *$T(U) = T(s) + T(U(s))$;*

iii) *If $U(s)$ is a linear subspace of E_1, then $T(U(s))$ is a linear subspace of E_2.*

For a proof see the paper [1].

Lemma 3. *For every element $s \in U$ the set $(T(U(s)))^{\perp}$ has the following properties*

i) *$(T(U(s)))^{\perp}$ is a non-empty set ($0_{E_2} \in (T(U(s)))^{\perp}$);*

ii) *$(T(U(s)))^{\perp}$ is a linear subspace of E_2;*

iii) *$(T(U(s)))^{\perp}$ is a closed set;*

iv) *$(T(U(s))) \cap (T(U(s)))^{\perp} = \{0_{E_2}\}$.*

A proof is shown in the paper [1].

2 Main result

Theorem 3. *An element $s \in U$, such that $U(s)$ is a linear subspace of E_1, is a solution of the general spline interpolation problem (1) (corresponding to U) if and only if the following equality is true*

$$\|T(u) - \widetilde{w}\|_2^2 = \tag{5}$$

$$= \|T(u) - T(s)\|_2^2 + \|T(s) - \widetilde{w}\|_2^2, \quad (\forall)\ u \in U,\ (\forall)\ \widetilde{w} \in (T(U(s)))^{\perp}.$$

Proof. Let $s \in U$ be an element, such that $U(s)$ is a linear subspace of E_1.
1) Suppose that s is a solution of the general spline interpolation problem (1) (corresponding to U) and show that the equality (5) is true.

Let $\lambda \in [0,1]$ be an arbitrary number and $T(u_1), T(u_2) \in T(U)$ be arbitrary elements ($u_1, u_2 \in U$). From Lemma 2 ii) results that there are the elements $T(\widetilde{u}_1), T(\widetilde{u}_2) \in T(U(s))$ ($\widetilde{u}_1, \widetilde{u}_2 \in U(s)$) so that $T(u_1) = T(s) + T(\widetilde{u}_1), T(u_2) = T(s) + T(\widetilde{u}_2)$. Consequently, we have

$$(1-\lambda)T(u_1) + \lambda T(u_2) = (1-\lambda)(T(s) + T(\widetilde{u}_1)) + \lambda(T(s) + T(\widetilde{u}_2)) =$$

$$= T(s) + ((1-\lambda)T(\widetilde{u}_1) + \lambda T(\widetilde{u}_2)).$$

Because $U(s)$ is a linear subspace of E_1, applying Lemma 2 iii), results that $T(U(s))$ is a linear subspace of E_2, hence $(1-\lambda)T(\widetilde{u}_1) + \lambda T(\widetilde{u}_2) \in T(U(s))$. Therefore, we have $(1-\lambda)T(u_1) + \lambda T(u_2) \in T(s) + T(U(s))$ and using Lemma 2 ii) we obtain

$$(1-\lambda)T(u_1) + \lambda T(u_2) \in T(U),$$

i.e. $T(U)$ is a convex set.

Since $s \in U$ is a solution of the general spline interpolation problem (1) (corresponding to U) it follows that

$$\|T(s)\|_2 = \inf_{u \in U} \|T(u)\|_2$$

and seeing the equality $\{T(u) \mid u \in U\} = \{t \mid t \in T(U)\}$ it obtains

$$\|T(s)\|_2 = \inf_{t \in T(U)} \|t\|_2. \tag{6}$$

Let $t \in T(U)$ be an arbitrary element ($u \in U$).

We consider a certain $\alpha \in (0,1)$ and define the element

$$t' = (1-\alpha)T(s) + \alpha t. \tag{7}$$

Because $\alpha \in (0,1)$, $T(s), t \in T(U)$ and taking into account that $T(U)$ is a convex set, from the relation (7) results

$$t' \in T(U). \tag{8}$$

Therefore, from the relations (6), (8) we deduce

$$\|T(s)\|_2 \leq \|t'\|_2$$

and considering the equality (7) we find

$$\|T(s)\|_2 \leq \|(1-\alpha)T(s) + \alpha t\|_2,$$

which is equivalent to

$$\|T(s)\|_2^2 \leq \|(1-\alpha)T(s) + \alpha t\|_2^2. \tag{9}$$

Using the properties of the inner product it obtains

$$\|(1-\alpha)T(s) + \alpha t\|_2^2 = \|T(s) + \alpha(t - T(s))\|_2^2 = \tag{10}$$

$$= \|T(s)\|_2^2 + 2\alpha(T(s), t - T(s))_2 + \alpha^2\|t - T(s)\|_2^2.$$

Substituting the equality (10) in the relation (9) it follows that

$$\|T(s)\|_2^2 \leq \|T(s)\|_2^2 + 2\alpha(T(s), t - T(s))_2 + \alpha^2\|t - T(s)\|_2^2,$$

i.e.

$$2\alpha(T(s), t - T(s))_2 + \alpha^2\|t - T(s)\|_2^2 \geq 0$$

and dividing by $2\alpha \in (0, 2)$ we obtain

$$(T(s), t - T(s))_2 + \frac{\alpha}{2}\|t - T(s)\|_2^2 \geq 0. \tag{11}$$

Because $\alpha \in (0, 1)$ was chosen arbitrarily it follows that the inequality (11) holds $(\forall)\ \alpha \in (0, 1)$ and passing to the limit for $\alpha \to 0$ it obtains

$$(T(s), t - T(s))_2 \geq 0.$$

As the element $t \in T(U)$ was chosen arbitrarily we deduce that the previous relation is true $(\forall)\ t \in T(U)$, i.e.

$$(T(s), t - T(s))_2 \geq 0, \quad (\forall)\ t \in T(U). \tag{12}$$

Let show that in the relation (12) we have only equality. Suppose that $(\exists)\ t_0 \in T(U)$ such that

$$(T(s), t_0 - T(s))_2 > 0. \tag{13}$$

Using the properties of the inner product, from the relation (13) we find

$$(T(s), T(s) - t_0)_2 < 0. \tag{14}$$

Because $t_0 \in T(U)$ it results that $T(s) - t_0 \in T(s) - T(U)$ and considering Lemma 2 ii) it obtains $T(s) - t_0 \in -T(U(s))$. But, $U(s)$ being a linear subspace of E_1, applying Lemma 2 iii) we deduce that $T(U(s))$ is a linear subspace of E_2, hence

$-T(U(s)) = T(U(s))$. Consequently, $T(s) - t_0 \in T(U(s))$ and using Lemma 2 ii) we find $T(s) - t_0 \in T(U) - T(s)$, i.e.

$$(\exists)\ t_1 \in T(U) \text{ such that } T(s) - t_0 = t_1 - T(s). \tag{15}$$

From the relations (14) and (15) it follows that there is an element $t_1 \in T(U)$ so that $(T(s), t_1 - T(s))_2 < 0$, which is in contradiction with the relation (12).

Therefore, the relation (12) is equivalent to

$$(T(s), t - T(s))_2 = 0, \quad (\forall)\ t \in T(U). \tag{16}$$

Let $\widetilde{t} \in T(U(s))$ be an arbitrary element.

Applying Lemma 2 ii) we obtain that $\widetilde{t} \in T(U) - T(s)$, so there is an element $t \in T(U)$ such that $\widetilde{t} = t - T(s)$. Using the relation (16) we deduce

$$(T(s), \widetilde{t})_2 = 0.$$

As the element $\widetilde{t} \in T(U(s))$ was chosen arbitrarily we find that the previous relation is true $(\forall)\ \widetilde{t} \in T(U(s))$, hence

$$T(s) \in (T(U(s)))^{\perp}. \tag{17}$$

Let $u \in U$, $\widetilde{w} \in (T(U(s)))^{\perp}$ be arbitrary elements.

Using the properties of the inner product we find

$$\|T(u) - \widetilde{w}\|_2^2 = \|(T(u) - T(s)) + (T(s) - \widetilde{w})\|_2^2 = \tag{18}$$

$$= \|T(u) - T(s)\|_2^2 + 2(T(u) - T(s), T(s) - \widetilde{w})_2 + \|T(s) - \widetilde{w}\|_2^2.$$

Since $u \in U$ it is obvious that

$$T(u) \in T(U),$$

therefore

$$T(u) - T(s) \in T(U) - T(s)$$

and applying Lemma 2 ii) it follows that

$$T(u) - T(s) \in T(U(s)). \tag{19}$$

As $T(s) \in (T(U(s)))^{\perp}$, $\widetilde{w} \in (T(U(s)))^{\perp}$ and taking into account Lemma 3 ii) we obtain

$$T(s) - \widetilde{w} \in (T(U(s)))^{\perp}. \tag{20}$$

Consequently, from the relations (19) and (20) we deduce

$$(T(u) - T(s), T(s) - \widetilde{w})_2 = 0. \tag{21}$$

Substituting the equality (21) in the relation (18) we find

$$\|T(u) - \widetilde{w}\|_2^2 = \|T(u) - T(s)\|_2^2 + \|T(s) - \widetilde{w}\|_2^2. \tag{22}$$

As the elements $u \in U$, $\widetilde{w} \in (T(U(s)))^{\perp}$ were chosen arbitrarily it follows that the previous equality is true $(\forall)\ u \in U$, $(\forall)\ \widetilde{w} \in (T(U(s)))^{\perp}$, i.e.

$$\|T(u) - \widetilde{w}\|_2^2 =$$

$$= \|T(u) - T(s)\|_2^2 + \|T(s) - \widetilde{w}\|_2^2, \quad (\forall)\ u \in U,\ (\forall)\ \widetilde{w} \in (T(U(s)))^{\perp}.$$

2) Suppose that the equality (5) is true and show that s is a solution of the general spline interpolation problem (1) (corresponding to U).

Let $u \in U$ be an arbitrary element.

According to Lemma 3 i) we have $0_{E_2} \in (T(U(s)))^{\perp}$ and considering $\widetilde{w} = 0_{E_2}$ in the equality (5) and taking into account the properties of the norm, we obtain

$$\|T(u)\|_2^2 = \|T(u) - T(s)\|_2^2 + \|T(s)\|_2^2,$$

hence

$$\|T(s)\|_2^2 = \|T(u)\|_2^2 - \|T(u) - T(s)\|_2^2 \leq \|T(u)\|_2^2,$$

with equality if and only if $\|T(u) - T(s)\|_2^2 = 0$, i.e. $T(u) = T(s)$.

The previous relation implies

$$\|T(s)\|_2 \leq \|T(u)\|_2,$$

with equality if and only if $T(u) = T(s)$.

As the element $u \in U$ was chosen arbitrarily we obtain that the previous inequality is true $(\forall)\ u \in U$, i.e.

$$\|T(s)\|_2 \leq \|T(u)\|_2, \quad (\forall)\ u \in U, \tag{23}$$

and the equality is attained in the element $T(u) = T(s)$, which is equivalent to

$$\|T(s)\|_2 = \inf_{u \in U} \|T(u)\|_2.$$

Consequently, s is a solution of the general spline interpolation problem (1) (corresponding to U). □

Remark 1. *The equality presented in Theorem 3 is remarkable because it represents a necessary and sufficient condition to characterize the solution of the general spline interpolation problem in Hilbert spaces. Also, from this theorem we obtain a few interesting inequalities and some optimal approximation properties satisfied by the general spline interpolation functions, like* $\|T(u) - T(s)\|_2 \leq \|T(u) - \widetilde{w}\|_2$, $(\forall)\ u \in U$, $(\forall)\ \widetilde{w} \in (T(U(s)))^{\perp}$, *respectively* $\|T(s) - \widetilde{w}\|_2 = \inf_{u \in U} \|T(u) - \widetilde{w}\|_2$, $(\forall)\ \widetilde{w} \in (T(U(s)))^{\perp}$.

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