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G_δ-BLUMBERG SPACES

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Abstract

A topological space X is called a G_δ-Blumberg space If for every real-valued function f on X, there exists a dense G_δ-set D in X such that the restriction of f to D is continuous. In this paper, the behaviour of this space under taking subspaces and superspaces, images and preimages are studied, and a G_δ-Blumberg space which is a generalization of an almost P-space is characterized. Some unsolved problems are posed.

1 Introduction

Let X be a topological space and $A \subseteq X$, $int_X A$ denotes the interior of A in X, $cl_X A$ denotes the closure of A in X. Where no ambiguity can arise, the interior of A in X is denoted by A° and the closure of A in X is denoted by $\overline{A}$. In this paper $I(X)$ denotes the set of all isolated points of topological space X. A topological space X is said to be almost discrete if $\overline{I(X)} = X$, and if $I(X) = \emptyset$, then X is called crowded or dense-in-itself.

Recall that a topological space X is Baire if the intersection of any sequence of dense open sets of X is dense.

Let X be crowded, if $\overline{D} = \overline{(X \backslash D)} = X$ for some subset D of X, then X is called resolvable, otherwise X is called irresolvable.

Let X and Y be topological spaces and let $F(X,Y)$ be the set of functions on X into Y. In this paper $F(X,\mathbb{R})$ is denoted by $F(X)$. It is clear that $F(X)$ with addition and multiplication defined pointwise, is a commutative ring. The collection of continuous members of $F(X)$ is denoted by $C(X)$. The zero-set of $f \in F(X)$ is denoted by $Z(f)$ and is defined by $Z(f) = \{x \in X : f(x) = 0\}$. The complement of $Z(f)$ in X, is called cozero-set of f and it is denoted by $Coz(f)$. Let X be a topological space. $\mathcal{D}$(X), $\mathcal{DO}(X)$ and $\mathcal{DG}$(X) denote the set of dense, dense open and dense G_δ subspaces of X, respectively. Let X be a topological space. If for every $f \in F(X)$ there exists a $D \in \mathcal{DG}$(X) such that $f|D \in C(D)$, then X is

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called a G_δ-Blumberg space. In Section 2, we introduce G_δ-Blumberg spaces and we give examples. In Theorem 2.6 and Corollary 2.7 we give characterizations of G_δ-Blumberg spaces. In Section 3, we characterize some subspaces and superspaces of a G_δ-Blumberg space, which are G_δ-Blumberg spaces. In Theorem 3.7 we show that a preimage of a G_δ-Blumberg space under irreducible mapping is also a G_δ-Blumberg space. Section 4 of our paper is devoted to a generalization of almost P-spaces which are G_δ-Blumberg spaces.

As usual, we let **c** denote the cardinality of the continuum.

2 A generalized S-Z function

Let X and Y be a topological spaces. Let $T(X,Y)$ denote the set of all f in $F(X,Y)$ such that there exists a D in $\mathcal{D}(X)$ and $f|D$ is continuous. In this paper $T(X,\mathbb{R})$ is denoted by $T(X)$.

In 1922, Blumberg [1] proved that if X is a separable complete metric space then for every real valued function f defined on X, there is a dense subset D of X such that $f|D \in C(D)$. ,i.e., $T(X) = F(X)$. A topological space X is called a Blumberg space if $T(X) = F(X)$. In 1960, Bradford and Goffman [2] showed that if X is metric, then X is a Blumberg space if and only if X is a Baire space. In 1974, White proved in [3] that if X is a Baire space having a σ-disjoint pseudo-base, then X is a Blumberg space. In 1976, Alas [4] improved White's result by showing that, if X is a Baire space having a σ-disjoint pseudo-base and Y is a second countable Hausdorff space, then $F(X,Y) = T(X,Y)$. In 1984, Piotrowski and Szymanski [5] proved that if X is a Baire space having a σ-disjoint pseudo-base and Y is a second countable space then $F(X,Y) = T(X,Y)$. They also showed that $T(X) = F(X)$ if and only if $T(X,Y) = F(X,Y)$ for every second countable space Y.

Definition 1. *Let X be a topological space and let*

$$T^{'}(X) = \{f \in F(X) | \exists D \in \mathcal{D}O(X) \ \text{ such that } \ \mathrm{f|D \in C(D)}\}.$$

If $T^{'}(X) = F(X)$, then X is called a strongly Blumberg space, abbreviated as S.B. space.

Strongly Blumberg spaces are introduced and studied in [6]. It was shown in [6] that under $V = L$, a topological space X is a strongly Blumberg space if and only if it is almost discrete.

Definition 2. *Let X and Y be topological spaces and*

$$TG(X,Y) = \{f \in F(X,Y) | \exists D \in \mathcal{DG}(X) \ \text{ such that } \ \mathrm{f|D \in C(D,Y)}\},$$

where $C(D,Y)$ denotes the collection of all continuous functions from D into Y. $TG(X,\mathbb{R})$ is denoted by $TG(X)$. If $TG(X) = F(X)$ then X is called a G_δ-Blumberg space, abbreviated as G_δ-B. space.

The fact that the class of G_δ-B. spaces properly contained in the class of Blumberg spaces follows from the next example.

Example 1. *(CH) Sierpiński and Zigmund in [7] showed that there exists an* $f \in F(\mathbb{R}) = T(\mathbb{R})$*, called S-Z function, such that for every* $M \in \mathcal{D}(\mathbb{R})$ *of cardinality* $\mathbf{c}$*,* $f|M \notin C(M)$*. Since* $\mathbb{R}$ *has no countable dense* G_δ*-set [8],* f *has no continuous restriction to any dense* G_δ*-set. So the Blumberg space* $\mathbb{R}$ *is not a* G_δ*-B. space.*

In the following definition we give a generalization of a $S - Z$ function to some topological spaces:

Definition 3. *Let* X *be a Blumberg space which is not a* G_δ*-B. space and* $f \in F(X) \setminus TG(X)$*. Then* f *is called a generalized S-Z function.*

We now show that the class of G_δ-B. spaces properly contains the class of strongly Blumberg spaces.

Example 2. *Let* $X = \{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$*, and define* $\tau_X = \{\{0, \frac{1}{n}, \frac{1}{n+1}, \cdots\} : n \in \mathbb{N}\}$*, as a topology on* X*. Since* $cl_X\{0\} = X$*, and* $\{0\}$ *is* G_δ*,* X *is a* G_δ*-B. space, but* X *is not an almost discrete space.*

Recall that a commutative ring R is called (von Neumann) regular if for each $r \in R$ there exists an $s \in R$ such that $r = r^2 s$. Clearly $F(X)$ is a regular ring.

Proposition 1. *If* $TG(X)$ *is a subring of* $F(X)$*, then* $TG(X)$ *is a regular subring.*

Proof. $f \in TG(X)$ implies that there exists a $D \in \mathcal{DG}$(X) such that $f|D \in C(D)$. Let $g(x) = \frac{1}{f(x)}$ if x is in cozero-set of f, and $g(x) = 0$ if $x \in Z(f)$. It is easily seen that $D_1 = Coz(f|D) \cup int_D Z(f|D)$ is dense G_δ in X and $g|D_1 \in C(D_1)$. So $g \in TG(X)$, and $f^2 g = f$ and $g^2 f = g$. Thus $TG(X)$ is a regular subring of $F(X)$. □

If X is Baire, then $TG(X)$ is a subring of $F(X)$, and so by the above proposition $TG(X)$ is its regular subring. In [9] some characterizations of a space X that $TG(X)$ is a subring of $F(X)$ is given.

The following theorem and corollary are proved in the same ways as Theorem 1 and Corollary 2 in [5], respectively.

Theorem 1. *A space* X *is a* G_δ*-B. space, if and only if for every countable cover* $(P_n)_{n \in \mathbb{N}}$ *of* X*, there exists a* $D \in \mathcal{DG}(X)$ *such that* $P_n \bigcap D$ *is a* G_δ *in* X *for every* $n \in \mathbb{N}$*.*□

Corollary 1. *Let* X *be a topological space. Then the following conditions are equivalent.*

1. For every real-valued function f *on* X *there exists a* $D \in \mathcal{DG}(X)$ *such that* $f|D$ *is continuous.*

2. Let Y *be a second countable space. Then for every function* f *on* X *into* Y *there exists a* $D \in \mathcal{DG}(X)$ *such that* $f|D$ *is continuous.*□

3 Subspaces and pre-images of G_δ-B. spaces

Theorem 2. *Let U be a G_δ-set in a space X, $U \subseteq A \subseteq \overline{U}$ and let A be Baire. Then U is a G_δ-B. space if and only if A is so.*

Proof. $\Rightarrow$:Let U be a G_δ-B. space. Suppose $g \in F(A)$. Since U is a G_δ-B. space and $g|U \in F(U)$, there exists a $B \in \mathcal{DG}(\mathrm{U})$ such that $g|B \in C(B)$. Since U is a G_δ in X, B is a G_δ and dense subset of A. Therefore A is a G_δ-B. space.
$\Leftarrow$: Suppose A is a G_δ-B. space and $g \in F(U)$. We can extend g to a function $h \in F(A)$, then by hypothesis there exists a $B \in \mathcal{DG}(A)$ such that $h|B \in C(B)$. Since U is a G_δ set in X, and U is Baire, $B \cap U \in \mathcal{DG}(U)$. So $g|(B \cap U) \in C(B \cap U)$. Therefore U is a G_δ-B. space. □

Corollary 2. *a.) Let W be a G_δ dense subset of X. Let X be Baire. Then W is a G_δ-B. space if and only if X is a G_δ-B. space.*
b.) Every open subset of a G_δ-B. space is a G_δ-B. space.
c.) Every regular closed subset of a G_δ-B. space is a G_δ-B. space.

Proof. *a.*). It is an immediate consequence of the above theorem. *b.*) It is clear. For proof *c.*), let A be a regular closed in a G_δ-B. space X. So by b.) A^o is a G_δ-B. space, and so by Theorem 2 $A = \overline{A^o}$ is a G_δ-B. space. □

Corollary 3. *Let X be a topological space, then the following statements are equivalent.*
a.) X is a G_δ-B. space.
b.) Every dense G_δ subset D of X is a G_δ-B. space.
c.) There exists a dense G_δ subset D of X which is a G_δ-B. space.

Proof. *a.*) $\Rightarrow$ *b.*) Let X be a G_δ-B. space, and let D be a dense G_δ in X. Then X is Baire, and since every dense G_δ in a Baire space is Baire, D is Baire. So by Theorem 2, D is a G_δ-B. space.
b) $\Rightarrow$ *c*) It is obvious.
c) $\Rightarrow$ *a*) It follows from Corollary 2. □

Example 3. *(CH) Let $\beta = \{\{r\} |\ r \in \mathbb{Q}\} \cup \tau$, where τ is the natural topology on the real line. Then β is a base for a topology on $X = \mathbb{R}$ and X is a G_δ-B. space. $G = \mathbb{R} \setminus \mathbb{Q}$ is a closed G_δ set in X. G is not a G_δ-B. space. Otherwise by Corollary 3, the real line, with ordinary topology, is a G_δ-B. space. But by Example 1 there is a real-valued function f defined on X, such that for every G_δ and dense subset D of the real line $f|D \notin C(D)$ and this is a contradiction.*

Example 3 shows that the property of being G_δ-B. space need not be inherited by arbitrary subspaces. This example shows that a closed G_δ in a G_δ-B. space, need not be a G_δ-B. space.

Remark 1. *If $(X, \tau X)$ is not a G_δ-B. space, where τX is the topology on X. Then when X is retopologised with the discrete topology, X becomes a G_δ-B. space. So* *the identity function from X with the discrete topology to $(X, \tau X)$ is a continuous mapping from a G_δ-B. space to a space which is not a G_δ-B. space. Thus the image of a G_δ-B. space under a continuous mapping need not be a G_δ-B. space.*

Recall that a continuous mapping $f : X \to Y$ of X onto Y is irreducible, if $f(F) \neq Y$ for every proper closed subset F in X. In the light of Theorem 1 we show that a preimage of a G_δ-B. space under irreducible mapping is also a G_δ-B. space.

Theorem 3. *Let Y be a G_δ-B. space and let $f : X \to Y$ be an irreducible mapping. Then X is a G_δ-B. space.*

Proof. Let $\mathcal{P} = (P_n)_{n\in\mathbb{N}}$ be an arbitrary countable cover of X. Then $f(\mathcal{P}) = \{f(P) \mid P \in \mathcal{P}\}$ is a countable cover of Y. Since Y is a G_δ-B. space, Theorem 1 implies that there exists a dense G_δ subset D' of Y such that for every $P \in \mathcal{P}$, $f(P) \bigcap D'$ is a G_δ-set. For every $y \in D'$ we select one member in $f^{-1}(y)$, and let D be the set of these selected members. Since f is an irreducible mapping and D' is dense in Y we conclude that D is dense in X. If $P \in \mathcal{P}$, then $P \bigcap D = f^{-1}(f(P) \bigcap D')$, and since f is a continuous mapping, $P \bigcap D$ is a G_δ-set. Thus by Theorem 1, X is a G_δ-B. space.

□

4 When almost GP-spaces are G_δ-B. spaces

Recall that a completely regular space in which every non-empty G_δ-set has non-empty interior is called an almost P-space [10].

Almost P-spaces are generalized in [9] as follows:

Definition 4. *Let X be a topological space. If every dense G_δ subset of X has nonempty interior, then X is called an almost GP-space.*

Proposition 2. *Let X be crowded. If X is an almost GP-space and X is a G_δ-B. space, then X is an irresolvable space.*

Proof. Suppose to the contrary that D and $D^c = X \setminus D$ are dense dense in X. Let f be the characteristic function of D^c. Then by hypothesis there exists a G_δ and dense subset W of X such that $f|W \in C(W)$. Since D and D^c are dense, $f|int_X W \notin C(int_X W)$, and this is a contradiction. Thus X is an irresolvable space. □

Theorem 4. *The following conditions are equivalent in ZFC:*
(1) there exists an irresolvable Blumberg space X.
(2) there exists a crowded almost GP-space which is a G_δ-B. space.

Proof. (1) ⇒ (2). Let X be an irresolvable Blumberg space. By [[11], Fact 3.1] X has a non-empty open hereditarily irresolvable subspace Y. So Y is an almost GP-space and by [12] $T'(Y) = T(Y)$. It is clear that Y will be a Blumberg space as well, so $T(Y) = F(Y)$. Since $T^{'}(Y) \subseteq TG(Y) \subseteq F(Y)$, we have $TG(Y) = F(Y)$, i.e., Y is a G_δ-B. space. (2) $\Rightarrow$ (1). Suppose that X is a crowded almost GP-space which is a G_δ-B. space. Then by Proposition 2 X is irresolvable. Since every G_δ-B. space is Blumberg we are done. □

Theorem 5. *Let X be an almost P-space. Then X is a G_δ-B. space if and only if X is an S.B. space.*

Proof. If X is a G_δ-B. space and $f \in F(X, \mathbb{R})$, then there exists a dense and G_δ subset D of X such that $f|D \in C(D)$. Since X is a completely regular space, $int_X D$ is dense in D [10]. Since D is dense in X, $int_X D$ is dense in X and $f|int_X D \in C(int_X D)$. So X is a $S.B.$ space. The converse is obvious. □

An almost GP-space X is called a GID-space if every dense G_δ-set of X has dense interior in X [9]. With slight changes in the proof of the above theorem, we note that Theorem 5 is true for a GID-space.

Lemma 1. *If $V = L$, then there is no crowded irresolvable G_δ-S.B. space (respectively Blumberg space and strongly Blumberg space).*

Proof. To the contrary, suppose that X is a crowded G_δ-B. space. Since every G_δ-B. space (respectively Blumberg space and strongly Blumberg space) is a Baire space [3], we have a Baire irresolvable space under V=L, and by [13] this is a contradiction. □

Let X be a crowded topological space, then it was shown in [12] that if $T(X) = T^{'}(X)$, then X is an irresolvable space.

Proposition 3. *Let X be a G_δ-B. space. If X is a crowded almost P-space, then X is an open-hereditarily irresolvable space.*

Proof. Let U be a nonempty open set, by Corollary 3 U is a G_δ-B. space and Theorem 5 imply that U is a $S.B.$ space. So by [12] U is an irresolvable space. Therefore X is an open-hereditarily irresolvable space. □

Corollary 4. *Under V=L, every almost P-space X which is a G_δ-B. space is almost discrete.*

Proof. Let U be a nonempty subset of X. Then by Corollary 2 U is a G_δ-B. space, so U is Baire. By [10], U is an almost P-space. Thus by [13] and Proposition 3 U has an isolated point and so X is an almost discrete space. □

Remark 2. *Not every almost P-subspace of a G_δ-B. space need be a G_δ-B. space. For example, $X = \beta\mathbb{N}$, the Stone Čech compactification of natural numbers, is a G_δ-B. space since the set of all isolated points of $\beta\mathbb{N}$ is dense in $\beta\mathbb{N}$. $Y = \beta\mathbb{N}\backslash\mathbb{N}$ is a closed subset of $\beta\mathbb{N}, I(Y) = \emptyset$ and Y is a compact almost P-space [14]. By [12] the closed subset $Y = \beta\mathbb{N}\backslash\mathbb{N}$ is not a S.B. space, and so by Theorem 5 Y is not a G_δ-B. space. We note that by [3] Y is a Blumberg space, and so there exists a generalized S-Z function in $F(Y)$.*

5 Open problems

In the following we list a number of questions which we could not answer.

Problem 1. *Is there a Hausdorff G_δ-B. space which is not a strongly Blumberg space?*

By the first paragraph after proof of Theorem 5, we know that in the class of GID-spaces every G_δ-B. space is a strongly Blumberg space. This gives a partial answer to Problem 1. We note that if the answer of the Problem 1 is negative, then under $V = L$, for every crowded Blumberg space X there exists a generalized $S - Z$ function in $F(X)$.

Problem 2. *Are the following conditions equivalent in ZFC?*
(1) There exists a Baire irresolvable space.
(2) There exists a crowded almost GP-space which is a G_δ-B. space.

Problem 3. *Let X be crowded and let X be an almost GP-space. Are the following conditions equivalent in ZFC?*
(1) X is a G_δ-B. space.
(2) X is a S.B. space.

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