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ON THE EXISTENCE OF SOLUTIONS FOR NON-LINEAR FUNCTIONAL INTEGRAL EQUATION

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Abstract

We have proved the existence of monotonic solutions of a nonlinear functional integeral equation by using Darbo fixed point theorem associated with a measure of noncompactness.

1 Introduction

Integral equations of various types create the important subject of several mathematical investigations and appear often in many applications.

The main objective of the present paper is to study the solvability of a nonlinear functional integral equation. The theory of equations of such a type is very developed. Nevertheless, there are a lot of problems concerning the solvability of such equations in some classes of functions which are not satisfactory and not completely solved till now.

In this paper, we introduce some rather simply and convenient conditions that ensure the existence of solutions of the equations in the space of all Lebesgue integrable on set $(0,1)$. In our considerations we will use the technique of measures of noncompactness and the modified version of the fixed point theorem of Darbo [6].

2 Notions and Some Auxiliary Facts

Let E be a measurable set and $f\ (x)$ is a real function defined on E. We say that $f\ (x)$ is Lebesgue measurable or briefly, measurable on E . if for each real number k the values $x \in E$ for which $f\ (x) \succ k$ is measurable. Let $I = (0,1)$ be the

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Lebesgue measurable subset of R_+ and $L^1(I)$ be the space of Lebesgue measurable functions on a measurable subset I with the norm

$$\|y\|_{L^1(I)} = \int_I |y(t)|\, dt.$$

For further purposes we shall write L^1 instead of $L^1(I)$. Moreover, the norm in the space L^1 will be denoting by $\|.\|$.

Now, let us assume that a function $f(t,y) = f : I \times R \longrightarrow R_+$ satisfies Caratheodory conditions i.e. it is measurable in t for any $y \in R$ and continuous in y for almost all $t \in I$. Then to every function $y(t)$ which is measurable on I we may assign the function

$$(Fy)(t) = f, \quad (t, y(t)), t \in I.$$

It is well known that the function (Fy) is also measurable on I. The operator F defined in such a way is said to be the Superposition (or Nemytskii) operator generated by the function f.

Although the Superposition operator is very simple, it turns out to be one of the most important operators studied in a non-linear functional analysis [1]. We have the following theorem due to Appell and Zabrejko [3].

Theorem 1. The superposition operator F maps continuously the space $L_1(I)$ into itself if and only if

$|f(t, y)| \preceq a(t) + b|y|,$
for all $t \in I$ and $y \in R$, where $a(t) \in L_1(I)$ and $b \geq 0$.

Next, we will mention a desired theorem concerning the compactness in measure of a subset X of $L_1(I)$ [8].

Theorem 2. Let X be a bounded subset of $L_1(I)$ consisting of functions which are a.e. nondecreasing (or nonincreasing) on the interval I. Then X is compact in measure.

Further, we recall a few facts about the linear integral operator. Let $k(t,s) : I \times I \longrightarrow R_+$ be a measurable with respect to its variable, then for any function $y \in L_1(I)$ the integral

$$(Ky)(t) = \int_0^1 k(t,s)\, y(s)\, ds,$$

exists for every $t \in I$.

Moreover, the function $(Ky)(t)$ belongs to the space L_1. Therefore K is a linear operator which maps L_1 into L_1 and K is also bounded since $\|Ky\| \leq \|K\|_{L_1(I)} \|y\|$.

In the sequel, we have the following theorem due to Krzyz [13].

Theorem 3. Assume that $k(t,s) : I \times I \longrightarrow R_+$ is a measurable function, and the linear integral operator

$$(Ky)(t) = \int_0^1 k(t,s)\, y(s)\, ds, \qquad t \in I$$

maps L_1 into itself. Then K transforms the set of nonincreasing functions from L_1 into itself if and only if for any $p \succ 0$ the following implication is true

$$t_1 \prec t_2 \Longrightarrow \int_0^p k(t_1, s)\, ds \succeq \int_0^b k(t_2, s)\, ds .$$

Finally, we give a note on measure of noncompactness and fixed point theorem. Let E be an arbitrary Banach space and X be a nonempty and bounded subset of E . Let B_r be a closed ball in E centered at θ and radius r .

Let us recall the notion of the measure of weak and strong noncompactness defined by De Blasi [9] and Hausdorff [5] respectively in the following way :

$$\beta(X) = \inf . \left\{ \begin{array}{c} r \succ 0 \; there\ exists\ aweakly\ compact\ subset\ Y \\ of\ \ E\ such\ \ that\ X \subset \ Y + B_r \end{array} \right\},$$

$$\chi(X) = \inf \left\{ \begin{array}{c} r \succ 0 \; there\ exists\ compact\ subset \\ Y\ of\ \ E\ such\ \ that\ X \subset \ Y + B_r \end{array} \right\}.$$

The functions $\beta(X)$ and $\chi(X)$ possess several useful properties whichmay be found in [9] and [4] .

The convenient and handly formula for the function $\beta(X)$ in the space L^1 was gives by J. Appelland E. De pascale [2]:

$$\beta(X) = \lim_{c \longrightarrow 0} \left\{ \sup_{x \in \mathrm{X}} \left\{ \sup \left[\begin{array}{c} \int_D \; | \, y\ (t)\ | \, dt : \\ D \sqsubset \ (0,1)\, ,\ meas.\ D \ \ \leq \varepsilon \end{array} \right] \right\} \right\},$$

where the symbol $meas.D$ stands for Lebesgue measure of a subset D.

The two measures $\beta(X)$ and $\chi(X)$ are connected in the case when X is compact in measure as in the following theorem.

Theorem 4. Assume that X be an arbitrary nonempty and bounded subset of $L^1(I)$. if X is compact in measure then

$\beta(X) = \chi(X)$ [4] .

As an application of measures of noncompactness, we recall the fixed point theorem due to Darbo.

Theorem 5. Let Q be a nonempty, bounded, closed

and convex subset of E and let $T : Q \longrightarrow Q$ be a continuous transformation which is a contraction with respect to the measure of non compact χ i.e. there exists $\theta \in I$ such that $\chi(TX) \leq \theta\, \chi(X)$ for any nonemptysubset X of Q . Then T has at least one fixed point in the set Q.

Main Results

Now, we will be investigated the following functional integral equation

$$y(t) = g(t) + f\ (t,\ \lambda(t) \int_0^1 k(t,s)\, y(\phi(s))\, ds, \quad t \in (0,1) \quad (1)$$

In this paper, we prove the existence of a monotone solution for equation (1). For convenience the operator

$$(Ty)(t) = g(t) + f\ (t,\ \lambda(t) \int_0^1 k(t,s)\, y(\phi(s))\, ds,$$

which can be writing as the product

$$T\ y = g + \lambda\ F\ Ky\ (\phi),$$

where F is the superposition operator generated by the function

$$(F\ y)(t) = f\ (t, y\ (t)).$$

Thus equation (1) becomes

$$T\ y = y(t) = g\ (t) + \lambda\ F\ Ky\ (\phi). \qquad (2)$$

We treat equation (1) under the following assumptions:

(i) The function $g \in L^1$ is a.e. non-increasing on the interval I ,
(ii) $f\ : I \times R \longrightarrow R_+$ satisfies Caratheodory conditions and there exist a function $a(t) \in L^1$anda non-negative constant b such that

$$|\ f\ (t,y)| \le a(t) + b|\ y\ |,$$

for all $t \in I$ and $y \in R$. Moreover, $f\ (t,y)$ is assumed to be non-increasing on the set $I \times R \longrightarrow R_+$ with respect to t and non-decreasing with respect to y,

(iii) $k\ : I \times I \longrightarrow R_+$ is measurable with respect to both its variables and such that the integral operator K maps L^1 into itself,

(iv) For every $p \succ 0$ and for all $t_1, t_2 \in I$ the following condition is satisfied

$$t_1 \prec t_2 \Longrightarrow \int_0^p k(t_1, s)\, ds \succeq \int_0^p k(t_2, s)\, ds\ ,$$

(v) $\phi : I \longrightarrow I$ is an increasing absolutely continuous and there exists constant $M \succ 0$ such that $\varphi(t) \succeq M$ for almost all $t \in I$,

(vi) $\lambda : I \longrightarrow R_+$ is bounded non-increasing function i.e.$|\lambda(t)| \prec B$, for all $t \in I$

$(vii)\qquad \frac{b\ B\ \|\ k\ \|}{M} \prec 1.$

Then we can prove the following theorem;

Theorem 6. If the assumptions formulated above are satisfied, then the equation (1) has at least one solution $y \in L^1$ which is a.e.non-increasing on the interval I.

Proof. First of all observe that for a given $y \in L^1$ the function $T\,y$ belong to L^1 which is a consequence of the assumption $(ii), (iii), (v), (vii)$.

Additionally, using (2) we get

$$\|Ty\| = \int_0^1 \left(\left| g\,(t) + f \left(t\ ,\ \lambda\,(t) \int_0^1 k\,(t,s)\ y\,(\phi\,(s))\right) ds\right|\right) dt$$

$$\leq \int_0^1 |\ g\ |\,d\,t + \int_0^1 \left| f\ (t, \lambda\,(t))\ \int_0^1 k\,(t,s)\ y\,(\phi\,(s))\ ds \right| dt$$

$$\leq \|\ g\ \| + \int_0^1 \left[a(t) + b\ \left|\ \lambda(t)\ \int_0^1 k\,(t,s)\ y\,(\phi\,(s))\ ds\ \right|\right] dt$$

$$\leq \|\ g\ \| + \|\ a\ \| + b\ B\ \|k\ y\ (\varphi)\|$$

$$\leq \|\ g\ \| + \|\ a\ \| + b\ B\ \|k\ \| \int_0^1 |y\,(\phi\,(s))\ |\,ds$$

$$\leq \|\ g\ \| + \|\ a\ \| + \frac{b\ B\ \|\ k\ \|}{M} \int_0^1 |y\,(\phi\,(s))|\,\varphi'(s)ds$$

$$\leq \|\ g\ \| + \|\ a\ \| + \frac{b\ B\ \|\ k\ \|}{M}\ \|y\|\,.$$

From this estimation and (v), (vii) we infer that the operator T maps the ball B_r into itself, where

$$r\ = \frac{\|\ g\ \| + \|\ a\ \|}{1 - b\ \ B\ \|\ k\ \|M^{-1}}.$$

Further, let Q_r stand for the subset of B_r consisting of all functions which are a.e. positive and nonincreasing on I . Note that Q_r is nonempty, bounded, closed and convex subset of L^1. Moreover, in view of Theorem 2 the set Q_r is compact in measure. Next, take $y \in Q_r$, then $y\ (\varphi)$ is a.e. positive and nonincreasing on I and consequently $K\ y\ (\varphi)$ is also of the same type in virtue of the assumption (iv) and Theorem 3.

Further, the assumption (ii) permits us to deduce that

$$T\ \ y\ \ == g\ (t\ ) + \lambda(t)\ F\ Ky\ (\phi)$$

is also a.e. positive and nonincreasing on I . This fact, together with the assertion $T\ : B_r \longrightarrow B_r$ gives that T is a self mapping of the set Q_r .

From now on assume that Y is a nonempty subset of Q_r and $\epsilon \succ 0$ is fixed, then for an arbitrary $y \in Y$ and for a set $D \subset I$, $D \leq \varepsilon$ we obtain

$$\int_D (Ty)(t)\,dt = \int_D |g(t)|\,dt + \int_D \left| f\left(t, \lambda(t) \int_0^1 k(t,s)\,y(s)\,ds\right| dt\right.$$

$$\preceq \|g\|_{L^1(D)} + \int_D |\lambda(t)| \left[a(t) + b \left|\int_0^1 k(t,s)\,y(s)\,ds\right|\right] dt$$

$$\preceq \|g\|_{L^1(D)} + B\left[a + b\ \|Ky\|_{L^1(D)}\right].$$

Further, keeping in mind that the operator K transforms the space $L^1(D)$ into itself and is continuous we derive

$$\int_D (Ty)(t)\,dt \preceq \|g\|_{L^1(D)} + B\left[a + b\ \|K\|_D\,\|y\|_{L^1(D)}\right],$$

where the symbol $\|K\|_D$ denotes the norm of the operator $K:\ L^1(D) \longrightarrow L^1(D)$.

Consequently, we get

$$\int_D (Ty)(t)\,dt \preceq \|g\|_{L^1(D)} + B\left[a + b\ \|K\|_D \int_D |y(\phi(t))|\,dt\right]$$

$$\preceq \|g\|_{L^1(D)} + B\left[a + \frac{b\,\|K\|_D}{M} \int_D |y(\phi(t))|\,\varphi'(t)dt\right].$$

Now, applying the theorem on integration by substitution for Lebesgue integral we may write the last estimation as

$$\int_D (Ty)(t)\,dt \preceq \|g\|_{L^1(D)} + a\,B + \frac{b\,B\,\|K\|_D}{M} \int_{\varphi(D)} |y(t)|\,dt.$$

Hence, taking into account the equality

$$\lim_{\varepsilon \longrightarrow 0} \left\{ \sup \left[\int_D g(t)\,dt + a\,B : D \sqsubset I,\ meas.\ D \leq \varepsilon\right]\right\} = 0$$

Consequently, taking into account that the function φ is assumed to be absolutely continuous we have

$$\beta(T\,Y) \preceq \frac{b\,\|B\|\|K\|_D}{M}\beta(Y),$$

where β is the De Blasi measure of weak noncompactness.

In view of the properties of the set Q_r established before and Theorem 4 we can rewrite the last inequality in the following form

$$\chi(T\,Y) \preceq \frac{b\,\|B\|_{L^1(D)}\|K\|_D}{M}\chi(Y),$$

where χ is the Hausdorf measure of noncompact.

Thus, in virtue of the assumption (v) we can apply Theorem 5 which guarantees that the equation (1) has at least one solution. These complete the proof.

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