Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:4** (2010), 53–61 DOI: 10.2298/FIL1004053A

SOME PROPERTIES OF HYPERSPACES OF ČECH CLOSURE SPACES WITH VIETORIS-LIKE TOPOLOGIES

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Abstract

We study some topological properties of hyperspaces of Čech closure spaces endowed with Vietoris-like topologies. Some of these notions were introduced and considered in [9, 10] and [11], focussing on selection principles.

1 Introduction

In the first part of the book *Topological Spaces* [1] the theory of topological spaces is developed by considering the closure operator which need not be idempotent. We call such an operator a closure operator in the sense of Čech, or a Čech closure operator. In [2, 3, 4] different types of continuous-like functions between topological spaces were considered and topologies on sets of these function investigated. It was shown in [7] that these functions can be considered as continuous functions between closure spaces as well as that the corresponding results for function spaces hold in closure spaces, too. In [9, 10, 11] hyperspaces of closure spaces were introduced and some of their properties expressed by means of selection principles were proved generalizing the well-known topological results (see for example [5]). In the present paper we consider families of subset of a closure space equipped with different Vietoris-like topologies comparing the properties of the space and its hyperspaces.

2 Preliminaries

First we recall several definitions.

An operator $u : \mathcal{P}(X) \to \mathcal{P}(X)$ defined on the power set $\mathcal{P}(X)$ of a set X satisfying the axioms:

2010 *Mathematics Subject Classifications.* 54A05, 54B20, 54D05, 54D10, 54D30.
Key words and Phrases. Čech closure space, hyperspace, (generalized) Vietoris topology, strong compactness, topological modification.
Received: May 9, 2010
Communicated by Ljubiša Kočinac
* The third author was supported by the Ministry of Science, Technology and Development, Republic of Serbia.

(C1) $u(\emptyset) = \emptyset$,

(C2) $A \subset u(A)$ for every $A \subset X$,

(C3) $u(A \cup B) = u(A) \cup u(B)$ for all $A, B \subset X$,

is called a *Čech closure operator* and the pair (X, u) is a *Čech closure space*. For short, (X, u) will be denoted by X as well, and called a *closure space* or a *space*.

A subset A is *closed* in (X, u) if $u(A) = A$ holds. It is *open* if its complement is closed.

The *interior operator* $int_u : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by means of the closure operator in the usual way: $int_u = \mathrm{c} \circ u \circ \mathrm{c}$, where $\mathrm{c} : \mathcal{P}(X) \to \mathcal{P}(X)$ is the complement operator. A subset U is a *neighbourhood* of a point x in X if $x \in int_u U$ holds.

In (X, u), a point $x \in u(A)$ if and only if for each neighbourhood U of x, $U \cap A \neq \emptyset$ holds.

Separation axioms are defined in the usual way (see [1], Section 27). A space (X, u) is:

T_0 if for each two distinct points in X at least one has a neighbourhood which does not contain the other point.
T_1 if for each two distinct points in X the following holds: $(\{x\} \cap u(\{y\})) \cup (\{y\} \cap u(\{x\})) = \emptyset$ whenever $x \neq y$. It is equivalent to: every one-point subset of X is closed in (X, u).
T_2 (*Hausdorff*) if each two distinct points have disjoint neighbourhoods.
Regular if for each $x \notin u(A)$ there exist disjoint neighbourhoods U of x and V of A. It is equivalent to: for every point x and its neighbourhood U there is a neighbourhood V of x such that $x \in V \subset u(V) \subset U$ holds.

Since every completely regular (Tikhonov) Čech closure space is topological, we consider only spaces with lower separation axioms.

A collection $\{G_\alpha\}$ is an *interior cover* of a set A in (X, u) if the collection $\{int_u G_\alpha\}$ covers A. We suppose that the interior of every element of an interior cover is nonempty.

A subset A in a space (X, u) is *compact* (respectively *countably compact*) if every interior cover (respectively countable interior cover) of A has a finite subcover, not necessarily interior.

The following notations are used:
$\mathcal{H} = \{u(A) \mid A \subset X\}$, $\mathcal{H}^* = \mathcal{H} \setminus \{\emptyset\}$, $\mathcal{J} = \mathcal{H}^{\mathrm{c}} = \{\mathrm{int}_u(A) \mid A \subset X\}$,
$\mathbf{F}(X)$ is the family of all nonempty finite subsets of X,
$\mathbf{F}_n(X)$ is the family of all nonempty subsets of X that have at most n elements,
$\mathbf{K}(X)$ is the family of all nonempty compact subsets of X,
$\mathbf{2}^X$ is the family of all nonempty closed subsets of X,
$\mathbf{A}(X)$ is the family of all nonempty subsets of X.

The *topological modification* $\hat{u}$ of the operator u is the finest Kuratowski closure operator coarser than u. The corresponding topology $\mathcal{T}(\hat{u})$ consists of all open sets in (X, u).

We also consider the topology $\mathcal{T}(\tilde{u})$ on (X, u) having for a basis the collection $\mathcal{J}$. Its (Kuratowski) closure operator will be denoted by $\tilde{u}$. The collection $\mathcal{H}$ is a base for closed subsets in $(X, \mathcal{T}(\tilde{u}))$.

In general $\hat{u}$ is coarser and $\tilde{u}$ is finer than u. Namely, for every $A \subset X$, $\tilde{u}(A) \subset u(A) \subset \hat{u}(A)$ holds.

Indeed, $x \in \tilde{u}(A)$ if and only if for each $U \subset X$, $x \in \mathrm{int}_u U$ implies $\mathrm{int}_u U \cap A \neq \emptyset$, hence $U \cap A \neq \emptyset$, which is equivalent to $x \in u(A)$. It follows next that for every $U = \mathrm{int}_u U$, $x \in U$ implies $U \cap A \neq \emptyset$ which is equivalent to $x \in \hat{u}(A)$.

Note that u, $\hat{u}$ and $\tilde{u}$ coincide when u is a Kuratowski closure operator.

A subset S of a closure space (X, u) is *connected* ([1], Definition 20 B.1.) if $S = A \cup B$ and $u(A) \cap B = \emptyset = A \cap u(B)$ implies $A = \emptyset$ or $B = \emptyset$.

By Theorem 20. B.2. of [1], a closure space (X, u) is connected if and only if it is not the union of two disjoint nonempty open (open-and-closed) subsets. Hence

Lemma 1. *(X, u) is connected if and only if $(X, \hat{u})$ is connected.*

Recall that a closure space (X, u) is *locally connected at* a point x if connected neighbourhoods of x form a local base at x and (X, u) is *feebly locally connected at* $x \in X$ if there exists a connected neighbourhood of x. (See [1], Definition 21 A.1. and Section 21 B.)

Lemma 2. *(See [1], 21 B.9.) If a closure space is (feebly) locally connected, then its topological modification possesses the corresponding property.*

We introduce the following

Definition A space (X, u) (respectively a subset A) is *strongly compact* if every interior cover of X (respectively A) has a finite interior subcover. In other words, X (respectively A) is strongly compact in (X, u) if and only if it is $\mathcal{T}(\tilde{u})$-compact.

For a subset $A \subset X$ the usual notations are $A^+ = \{H \in \mathcal{H} \mid H \subset A\}$ and $A^- = \{H \in \mathcal{H} \mid H \cap A \neq \emptyset\}$.

Open sets in (X, u) define the Vietoris topology $\mathbf{V}$ on $\mathbf{A}(X)$ and its subcollections, while the elements of $\mathcal{T}(\tilde{u})$ define the Vietoris topology that will be denoted by $\mathbf{V}^{\#}$.

In the setting of Čech closure spaces the generalized upper and lower Vietoris topologies $\mathbf{W}^+$ and $\mathbf{W}^-$ on the family $\mathcal{H}^*$ were introduced in [11] in the following way.

The collection $\mathcal{J}^+ = \{G^+ \mid G \in \mathcal{J}\}$ is a basis for the topology $\mathbf{W}^+$ on $\mathcal{H}^*$, while $\mathbf{W}^-$ is defined by the collection $\mathcal{J}^- = \{G^- \mid G \in \mathcal{J}\}$. Its basis elements are of the form $G_1^- \cap \cdots \cap G_n^-$ where $G_i \in \mathcal{J}$ for $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$.

We introduce the *generalized Vietoris topology* on $\mathcal{H}^*$ (and in the same way on $\mathbf{A}(X)$) as $\mathbf{W} = \mathbf{W}^+ \vee \mathbf{W}^-$. Its basis elements are of the form
$G^+ \cap (\cap_{i=1}^n G_i^-)$ where $G, G_1, \ldots, G_n \in \mathcal{J}$,
for which we use the notation
$< G; G_1, \ldots, G_n >= \{H \in \mathcal{H}^* \mid H \subset G \text{ and } H \cap G_i \neq \emptyset \text{ for } i = 1, \ldots, n\}$.
Without loss of generality we suppose that $G_i \subset G$ for $i = 1, \ldots, n$.

On $\mathbf{A}(X)$ (and its subcollection $\mathcal{H}^*$) the following holds
$\mathbf{V} \subset \mathbf{W} \subset \mathbf{V}^{\#}$.

When (X, u) is a topological space $\mathbf{V} = \mathbf{W} = \mathbf{V}^{\#}$ all definitions considered in this paper coincide with the corresponding topological ones.

Example 1. Let $(X, \mathcal{T})$ be a topological space and $u = \mathrm{cl}_\theta$ be the θ-closure operator in $(X, \mathcal{T})$. ($x \in \mathrm{cl}_\theta(A)$ if each closed neighbourhood of x intersects A.) θ-open sets $(X, \mathcal{T})$ form the topology $\mathcal{T}(\tilde{u})$, while the semi-regularization topology of $\mathcal{T}$, whose basis is the family of regular open sets in $(X, \mathcal{T})$, is the topology $\mathcal{T}(\tilde{u})$. (cl_θ is a Kuratowski closure operator if and only if $(X, \mathcal{T})$ is a regular space.) By [8], Corollary 1, Proposition 5 and Theorem 2, connectedness and weak local connectedness are shared by the spaces $(X, \mathcal{T}_s)$, (X, cl_θ) and $(X, \mathcal{T}_\theta)$, while local connectedness of $(X, \mathcal{T}_s)$ implies that of (X, cl_θ).

In particular, let $(X, \mathcal{T})$ be the digital line and (X, u) its θ-closure space $(\mathbb{Z}, \mathrm{cl}_\theta)$. (See [8], Example 4.) A basis for the topology $\mathcal{T}$ is $\mathcal{B} = \{\{2m-1\} | m \in \mathbb{Z}\} \cup \{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$. Since every basis element is regularly open, $\mathcal{T}_s = \mathcal{T}$. The only θ-open sets are the empty set and $\mathbb{Z}$, so the space $(\mathbb{Z}, \mathcal{T}_\theta)$ is indiscrete. Thus the spaces (X, u) and $(X, \tilde{u})$ are not (strongly) compact, but they are locally (strongly) compact, connected and locally connected. The space $(X, \hat{u})$ satisfies all the listed properties.

In general, connectedness is not shared by (X, u) and $(X, \tilde{u})$. It can be seen from the next simple example.

Example 2. Let $X = \{a, b, c\}$, $u(\emptyset) = \emptyset$, $u(\{a\}) = \{a, b\}$, $u(\{b\}) = \{b, c\}$, $u(\{c\}) = \{c, a\}$, $u(\{a, b\}) = u(\{a, c\}) = u(\{b, c\}) = u(X) = X$. Then $\mathrm{int}_u \emptyset = \mathrm{int}_u\{a\} = \mathrm{int}_u\{b\} = \mathrm{int}_u\{c\} = \emptyset$, $\mathrm{int}_u\{a, b\} = \{b\}$, $\mathrm{int}_u\{b, c\} = \{c\}$, $\mathrm{int}_u\{a, c\} = \{a\}$, $\mathrm{int}_u X = X$. The space $(X, \hat{u})$ is indiscrete while $(X, \tilde{u})$ is discrete. The induced topologies $\mathbf{V}$ and $\mathbf{V}^{\#}$ are indiscrete and discrete respectively.
$\mathcal{H}^* = \{\{a, b\}, \{a, c\}, \{b, c\}, X\}$ and its generalized Vietoris topology
$\mathbf{W} = \{\emptyset, \{X\}, \{X, \{a, b\}\}, \{X, \{a, c\}\}, \{X, \{b, c\}\}, \{X, \{a, b\}, \{a, c\}\}, \{X, \{a, b\}, \{b, c\}\}, \{X, \{a, c\}\{b, c\}\}, \mathcal{H}^*\}$. Thus $(\mathcal{H}^*, \mathbf{W})$ is (hyper)connected, while $(\mathbf{A}(X), \mathbf{W})$ is disconnected since $\{a\}^- \cup \{b\}^- = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$

and $\{c\}^+ = \{\{c\}\}$ form a decomposition of $\mathbf{A}(X)$.

All notions not explained here concerning Čech closure spaces can be found in [1] and [7].

3 Some properties of the hyperspaces of (X,u)

In the sequel all considered spaces are T_1.

By $\mathbf{C}(X)$ we denote the family of all nonempty compact subsets in $(X,\hat{u})$, and by $\mathbf{Q}(X)$ the family of all nonempty closed subsets in $(X,\mathcal{T}(\tilde{u}))$.

Lemma 3. *For a space (X,u) the following hold:*

(i) A^+ and A^- are closed sets for every $A\in\mathcal{H}$.

(ii) $\mathrm{cl} < G;G_1,\dots,G_n > \subset < u(G);u(G_1),\dots,u(G_n) >$.

(iii) If $\{U_\lambda\}$ is a neighbourhood basis of $x\in X$, then $\{(\mathrm{int}_u U_\lambda)^+\}$ is a neighbourhood basis of $\{x\}\in\mathcal{H}^*$.

Proof. (i) The statement follows from the equalities: $(A^+)^c \equiv \mathcal{H}^*\backslash A^+ = (A^c)^-$ and $(A^-)^c \equiv \mathcal{H}^*\backslash A^- = (A^c)^+$.
(ii) Since $u(G),u(G_1),\dots,u(G_n)\in\mathcal{H}^*$, the collections $(u(G))^+,(u(G_1))^+,\dots,(u(G_n))^+$ are closed in $(\mathcal{H}^*,\mathbf{W})$. Thus, $< u(G);u(G_1),\dots,u(G_n) >$ is closed and $\mathrm{cl} < G;G_1,\dots,G_n > \subset < u(G);u(G_1),\dots,u(G_n) >$.
For the converse, let $H \in < u(G);u(G_1),\dots,u(G_n) >$ and $< U;U_1,\dots,U_m >$ be a neighbourhood of H, where $U=\mathrm{int}_u A$ and $U_j=\mathrm{int}_u A_j$, $j=1,\dots,m$. Then $H\subset U$, $H\cap U_j\neq\emptyset$ for $j=1,\dots,m$; $H\subset u(G)$ and $H\cap u(G_i)\neq\emptyset$ for $i=1,\dots,n$. There is $x_i\in H\cap u(G_i)$ for each $i\in\{1,\dots,n\}$, and for the neighbourhood A of x_i, there is a $z_i\in A\cap G_i$. In a similar way, for each $j\in\{1,\dots,m\}$, there is $y_j\in H\cap U_j$ and for the neighbourhood A_j of y_j, there is a $\hat{z}_j\in A_j\cap G$. The set $K=\{z_i \mid i=1,\dots,n\}\cup\{\hat{z}_j \mid j=1,\dots,m\} \in < A;A_1,\dots,A_m > \cap < G;G_1,\dots,G_n > = < A\cap G;A_1\cap G,\dots,A_m\cap G,G_1\cap A,\dots,G_n,\cap A >$.

In fact, $\mathrm{cl} < C;C_1,\dots,C_n > \subset < u(C);u(C_1),\dots,u(C_n) >$ holds for any nonempty sets $C,C_1,\dots,C_n$, $C_i\subset C$.
(iii) Clear. $x\in\mathrm{int}_u U_\lambda \Leftrightarrow \{x\}\in(\mathrm{int}_u U_\lambda)^+$ holds. □

Proposition 1. *For a space (X,u) the following hold:*

(i) $\mathbf{F}(X)$ is dense in $(\mathcal{H}^*,\mathbf{W})$.

(ii) If (X,u) is T_2, then $\mathbf{F}_n(X)$ is closed in $(\mathcal{H}^*,\mathbf{W})$ for all $n\geqslant 1$.

(iii) The natural projection $p:(X,u)^n\to(\mathbf{F}_n(X),\mathbf{V})$ defined by $p(x_1,\dots,x_n)=\{x_1,\dots,x_n\}$, is continuous.

Proof. (i) For each nonempty open set $< G;G_1,\dots,G_n >$, let $x_i\in G_i$, $i=1,\dots,n$. The set $\{x_1,\dots,x_n\}$ is finite and belongs to $< G;G_1,\dots,G_n >$.

(ii) For a fixed $n \in \mathbb{N}$, let $H \in \mathcal{H}^*\backslash\mathbf{F}_n$ and $x_1, \ldots, x_{n+1}$ be distinct points in H. Since the space (X, u) is T_2, let $U_1, \ldots, U_{n+1}$ be pairwise disjoint neighbourhoods of $x_1, \ldots, x_{n+1}$, respectively. Let U, be a neighbourhood of H, and $U_i \subset U$, for $i = 1, \ldots, n{+}1$. (It can be chosen $U = X$.) Then $< \mathrm{int}_u U; \mathrm{int}_u U_1, \ldots, \mathrm{int}_u U_m >$ is a neighbourhood of H, and $< \mathrm{int}_u U; \mathrm{int}_u U_1, \ldots, \mathrm{int}_u U_m > \cap\, \mathbf{F}_n = \emptyset$.

(iii) $(\mathbf{F}_n(X), \mathbf{V})$ is a topological space so the mapping $p : (X^n, v) \to (\mathbf{F}_n(X), \mathbf{V})$ is continuous if and only if the mapping $p : (X^n, \hat{v}) \to (\mathbf{F}_n(X), \mathbf{V})$ is continuous, where $\hat{v}$ is the topological modification of v. Since the topological modification of the product space is the product of topological modifications, p is continuous if and only if it is continuous as the mapping from the nth product of the topological space $(X, \mathcal{T}(\hat{u}))$ into $(\mathbf{F}_n(X), \mathbf{V})$, which follows from [6], Proposition 2.4.3. □

Proposition 2. *(i) If A is dense in (X, u), then $\mathbf{F}(A)$ and $\mathbf{2}^A$ are dense in $(\mathbf{A}(X), \mathbf{V})$.*

(ii) For a T_1 space (X, u), the spaces $(\mathbf{2}^X, \mathbf{V})$, $(\mathcal{H}^*, \mathbf{W})$ and $(\mathbf{Q}(X), \mathbf{V}^{\#})$ are T_1.

(iii) If (X, u) is regular, then $(\mathbf{2}^X, \mathbf{W})$ is T_2.

Proof. (i) If A is dense in (X, u), then A is dense in $(X, \hat{u})$ and the statement follows from definitions and Corollary 5.2.4 of [6].

(ii) Let $H_1, H_2 \in \mathcal{H}^*$, $H_1 \neq H_2$ and $x_1 \in H_1\backslash H_2$. Then $H_1 \in (H_2^c)^- = \mathcal{W}_1$ holds and $H_2 \notin \mathcal{W}_1$. If there is $x_2 \in H_2$, then $H_2 \in (H_1^c)^- = \mathcal{W}_2$ and $H_1 \notin \mathcal{W}_2$ holds. If $H_2 \subset H_1$, then $H_2 \in (\{x_1\}^c)^+ = \mathcal{W}_2$ and $H_1 \notin \mathcal{W}_2$.

(iii) Let $A_1, A_2 \in \mathbf{2}^X$, $A_1 \neq A_2$, $A_1 = u(A_1)$, $A_2 = u(A_2)$ and $x_1 \in A_1\backslash A_2$. By regularity of (X, u) there are disjoint neighbourhoods U of x and V of A_2. Hence $x \in \mathrm{int}_u(U)$ and $A_2 \subset \mathrm{int}_u(V)$ imply $A_1 \in (\mathrm{int}_u(U))^- = \mathcal{W}_1$, $A_2 \in (\mathrm{int}_u(V))^+ = \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$. □

Proposition 3. *A space (X, u) is connected if and only if its nth product (X^n, v), $(n = 1, 2, \ldots)$, is connected.*

Proof. The spaces (X, u) and its product (X^n, v) are connected if and only if their topological modification $(X, \hat{u})$ and $(X^n, \hat{v})$ are connected. Since $(X^n, \hat{v})$ is the nth product of $(X, \hat{u})$, the statement follows. □

Proposition 4. *If σ is a connected collection in $(\mathbf{A}(X), \mathbf{W})$, and one of whose elements is connected, then $E = \cup\{S \in \sigma\}$ is connected.*

Proof. Let $S_0 \in \sigma$ be a connected subset. If $E = \cup\{S \in \sigma\}$ is not connected, then $E = A \cup B$ where A and B are disjoint, nonempty and open-and-closed. Since S_0 is connected, $S_0 \subset A$ or $S_0 \subset B$ holds. Suppose $S_0 \subset A$; then $S_0 \in A^+$. It follows that A^+ and B^- form a decomposition of σ. □

Theorem 1. *Let $\mathbf{F}(X) \subset \sigma \subset \mathbf{A}(X)$.*

(i) If one of the spaces (X,u), $(\mathbf{F}(X)_n,\mathbf{V})$, $n\in\mathbb{N}$, or $(\sigma,\mathbf{V})$ is connected, then all of them are connected.
(ii) Connectedness of one of the spaces $(\mathbf{F}(X)_n,\mathbf{W})$, $n\in\mathbb{N}$, or $(\sigma,\mathbf{W})$ implies connectedness of (X,u).

Proof. (i) Follows by Lemma 1 and Theorem 4.10 of [6].
(ii) In a similar way, by Lemma 1, Proposition 4 and the fact that $\mathbf{V}\subset\mathbf{W}$. □

Note that connectedness of all $(\mathbf{F}(X)_n,\mathbf{W})$, $n\in\mathbb{N}$, implies connectedness of $(\mathbf{F}(X),\mathbf{W})$ and hence of $(\sigma,\mathbf{W})$.

Proposition 5. *(Feeble) local connectedness of (X,u) implies (feeble) local connectedness of $(\sigma,\mathbf{V})$, where $\mathbf{F}(X)\subset\sigma\subset\mathbf{C}(X)$.*

Proof. By Lemma 2, (feeble) local connectedness of (X,u) implies (feeble) local connectedness of its topological modification $(X,\hat{u})$. Local connectedness of $(\sigma,\mathbf{V})$ follows by applying Theorem 4.12 in [6]. We modify the same proof for feeble local connectedness. □

Proposition 6. *(i) If $(\mathcal{H}^*,\mathbf{W})$ is (countably) compact, then (X,u) is strongly (countably) compact.*

(ii) The space $(\mathbf{2}^X,\mathbf{V})$ is compact if and only if $(X,\hat{u})$ is compact.

(iii) The space $(\mathbf{Q}(X),\mathbf{V}^{\#})$ is compact if and only if $(X,\tilde{u})$ is compact, that is, (X,u) is strongly compact.

Proof. (i) Let $\{G_\alpha\}$ be a (countable) interior cover of X. Then $\cup\{\operatorname{int}_u G_\alpha\}=X$. The collection $\{(\operatorname{int}_u G_\alpha)^-\}$ is a (countable) open cover of $\mathcal{H}^*$ and there is a finite subcover $\{(\operatorname{int}_u G_{\alpha_i})^-\,|i=1,\dots m\}$. Hence $\{G_{\alpha_i},|\,i=1,\dots m\}$ is a finite interior cover of X, a subcover of $\{G_\alpha\}$.

(ii) and (iii) follow by [6], Theorem 4.2. □

The next two statements are a weaker form of [6], Teorem 2.5.

Proposition 7. *If $\mathcal{C}$ is a compact collection in $(\mathcal{H}^*,\mathbf{W})$ consisting of strongly compact elements, then $K=\cup\{C\in\mathcal{C}\}$ is compact.*

Proof. Let $\{G_\alpha\}$ be an interior cover of K. For each $C\in\mathcal{C}$ there is a finite subcollection $\{G_{C,1},\dots,G_{C,n(C)}\}$ whose interiors cover C. Put $G_C=\cup\{G_{C,1},\dots,G_{C,n(C)}\}$. Then $\{<\operatorname{int}_u G_C;\operatorname{int}_u G_{C,1},\dots,\operatorname{int}_u G_{C,n(C)}>\}$, where $C\in\mathcal{C}$, is an open cover of $\mathcal{C}$, so there is a finite subcover $\{<\operatorname{int}_u G_{C_i};\operatorname{int}_u G_{C_i,1},\dots,\operatorname{int}_u G_{C_i,n(C_i)}>\,|\,i=1,\dots,m\}$. The collection $\{G_{C_i},G_{C_i,1},\dots,G_{C_i,n(C_i)}\,|\,i=1,\dots,m\}$ is a finite cover of K, a subcollection of $\{G_\alpha\}$. □

Proposition 8. *If (X,u) is a regular space and $\mathcal{C}$ is a compact collection in $(\mathcal{H}^*,\mathbf{W})$ consisting of closed elements, then $K=\cup\{C\in\mathcal{C}\}$ is closed.*

Proof. Let $\mathcal{C} = \{A_\lambda = u(A_\lambda) | \lambda \in \Lambda\}$. Let $x \in u(K)$. By regularity, for each neighbourhood U of x there is a neighbourhood V of x such that $x \in V \subset u(V) \subset U$ holds. Then $x \in u(K)$ implies $V \cap K \neq \emptyset$, hence there is a $\lambda \in \Lambda$ such that $u(V) \cap A_\lambda \neq \emptyset$, i. e. $A_\lambda \in (u(V))^-$. By Lemma 3(i) the collection $\mathcal{C} \cap (u(V))^-$ is closed in $\mathcal{C}$, and $\{\mathcal{C} \cap (u(V))^- \,|\, V \in \mathcal{N}(x)\}$ has the finite intersection property since $V_1, \ldots, V_n \in \mathcal{N}(x)$ imply $V_1 \cap \ldots \cap V_n \in \mathcal{N}(x)$. Since $(\mathcal{C}, \mathbf{W})$ is compact, there is a nonempty subcollection $\mathcal{E} \subset \mathcal{C}$. For each $E \in \mathcal{E}$, $E \in \cap\{(u(V))^- | V \in \mathcal{N}(x)\}$ implies $E \cap U \neq \emptyset$ for each $U \in \mathcal{N}(x)\}$, hence $x \in u(E) = E$. Since $E \subset K$, $x \in K$. Hence $u(K) = K$. □

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