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BANDS OF λ-SIMPLE SEMIGROUPS

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Abstract

Semigroups having a decomposition into a band of semigroups have been studied in many papers. In the present paper we give characterizations of various special types of bands of λ- semigroups and semilattices of matrices of λ- semigroups.

1. Introduction and preliminaries

Semigroups which can be decomposed into a band of left Archimedean semigroups have been studied by many authors. M. S. Putcha [17] proved a general theorem that characterizes such semigroups. Some other characterizations in the general case are given by S. Bogdanović, M. Ćirić and Ž. Popović [7] and P. Protić [14]. Some special decompositions of this type have been also treated in a number of papers. S. Bogdanović [1], [2], [3], P. Protić [13], [14], [15], S. Bogdanović and M. Ćirić [4] and S. Bogdanović, M. Ćirić and B. Novikov [6] studied bands of left Archimedean semigroups whose related band homomorphic images belong to several very important varieties of bands.

In this paper we give some results concerning decompositions into a band of λ-simple semigroups in the general and some special cases (Theorem 2).

Let a semigroup S be a semilattice Y of semigroups S_α, $\alpha \in Y$, and for any $\alpha \in Y$, let S_α be a matrix (left zero band, right zero band) I_α of semigroup S_i, $i \in I_\alpha$. The partition of S whose components are semigroups S_i, $i \in I$, where $I = \cup_{\alpha\in Y} I_\alpha$, will be called a *semilattice-matrix* (*semilattice-left*, *semillatice-right*) *decomposition* of S. All band decompositions are special cases of semilattice-matrix decompositions. The general lattice theoretical properties of semilattice-matrix decompositions of semigroups are investigated by M. Ćirić and S. Bogdanović [11]. A semilattice of matrix of left Archimedean semigroups were studied by S. Bogdanović and M. Ćirić [4].

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It is well known that a band of semigroups from a class $\mathcal{K}$ of a semigroups is a semilattice of matrices of semigroups from $\mathcal{K}$. Semilattices of matrices of λ-simple semigroups are described by Theorem 3. The characterizations of semilattices of hereditary weakly left Archimedean semigroups are given by Theorem 5. At the end semilattice of λ-simple semigroups are described by Theorem 6.

By $\mathbf{Z}^+$ we denote the set of all positive integers. By S^1 we denote a semigroup S with identity 1.

A semigroup in which all its elements are idempotents is a *band*. A commutative band is a *semilattice*. By $\mathcal{B}$ ($\mathcal{S}$) we denote the class of all bands (semilattices).

Let ϱ be an arbitrary binary relation on a semigroup S. The intersection of all transitive relations on S containing ϱ is a transitive relation on S, denoted by ϱ^∞. It is easy to prove that $\varrho^\infty = \cup_{n\in\mathbf{Z}^+}\varrho^n$. The relation ϱ^∞ we call the *transitive closure* of ϱ.

Let ϱ be an arbitrary relation on a semigroup S. Then *radical* $R(\varrho)$ of ϱ is a relation on S defined by:

$$(a,b)\in R(\varrho) \Leftrightarrow (\exists p,q\in\mathbf{Z}^+)\ (a^p,b^q)\in\varrho.$$

The radical $R(\varrho)$ was introduced by L. N. Shevrin in [19].

An equivalence relation ξ is a *left* (*right*) *congruence* if for all $a,b\in S$, $a\,\xi\,b$ implies $ca\,\xi\,cb$ ($ac\,\xi\,bc$). An equivalence ξ is a congruence if it is both left and right congruence. A congruence relation ξ is a *band congruence* on S if S/ξ is a band, i.e. if $a\,\xi\,a^2$, for all $a\in S$.

Let ξ be an equivalence on a semigroup S. By $\xi^\flat$ we define the largest congruence relation on S contained in ξ. It is well-known that

$$\xi^\flat = \{(a,b)\in S\times S\,|\,(\forall x,y\in S^1)\ (xay,xby)\in\xi\}.$$

For an element a of a semigroup S, the left ideal (the ideal) of S generated by a we denote with $L(a)$ ($J(a)$) and it we call *the principal left ideal* (*the principal ideal*) of S *generated by* a. Also, a subsemiogroup $\langle a\rangle$ of a semigroup S generated by one element subset $\{a\}$ of S is a *monogenic* or a *cyclic* subsemigroup of S.

Let a and b be elements of a semigroup S. Then:

$$a\,|\,b \Leftrightarrow b\in J(a), \qquad a\,|_l\,b \Leftrightarrow b\in L(a),$$

$$a \longrightarrow b \Leftrightarrow (\exists n\in\mathbf{Z}^+)\ a\,|\,b^n, \qquad a \stackrel{l}{\longrightarrow} b \Leftrightarrow (\exists n\in\mathbf{Z}^+)\ a\,|_l\,b^n,$$
$$\text{and}\quad \longrightarrow\!\!\!\!\!\!\!\!\!\!- \ = \longrightarrow \cap (\longrightarrow)^{-1}.$$

Also, on a semigroup S the relation $\uparrow_l$ is defined by

$$a \uparrow_l b \ \Leftrightarrow\ (\exists n\in\mathbf{Z}^+)\ b^n\in\langle a,b\rangle\, a.$$

Recall that a semigroup S is *left Archimedean* if $a \stackrel{l}{\longrightarrow} b$, for all $a,b\in S$. A semigroup S is *weakly left Archimedean* if $ab \stackrel{l}{\longrightarrow} b$, for all $a,b\in S$. A semigroup S is *hereditary weakly left Archimedean* if

$$(\forall a,b\in S)(\exists i\in\mathbf{Z}^+)\ b^i\in\langle a,b\rangle ab.$$

A semigroup S is *power-joined* if for every $a, b \in S$ there exists $n, m \in \mathbf{Z}^+$ such that $a^n = b^m$.

For an element a of a semigroup S we introduce the following notation

$$\Sigma(a) = \{x \in S \mid a \longrightarrow^{\infty} x\}, \qquad \Lambda(a) = \{x \in S \mid a \stackrel{l}{\longrightarrow} {}^{\infty} x\},$$

$$\Lambda_n(a) = \{x \in S \mid a \stackrel{l}{\longrightarrow} {}^{n} x\}.$$

On a semigroup S we define the following equivalences by

$$a \; \sigma \; b \Leftrightarrow \Sigma(a) = \Sigma(b), \qquad a \; \lambda \; b \Leftrightarrow \Lambda(a) = \Lambda(b),$$

$$a \; \lambda_n \; b \Leftrightarrow \Lambda_n(a) = \Lambda_n(b).$$

In [10] is proved that the relation σ is the greatest semilattice congruence on a semigroup, λ is an equivalence and it is a generalization of the well-known Green's equivalence $\mathcal{L}$.

A semigroup S is λ-*simple* (σ-*simple*, λ_n-*simple*) if $a \, \lambda \, b$ ($a \, \sigma \, b$, $a \, \lambda_n \, b$), for all $a, b \in S$. We denote by Λ the class of all λ-simple semigroups.

2. Special bands of λ-semigroups

For two classes $\mathcal{X}_1$ and $\mathcal{X}_2$ of semigroups, $\mathcal{X}_1 \circ \mathcal{X}_2$ will denote the *Mal'cev product* of $\mathcal{X}_1$ and $\mathcal{X}_2$, i.e. the class of all semigroups S on which there exists a congruence ϱ such that S/ϱ belongs to $\mathcal{X}_2$ and each ϱ-class of S which is a subsemigroup of S belongs to $\mathcal{X}_1$.

By $\mathcal{LZ}$ we denote the variety of left zero bands.

Lemma 1.. *Let S be a semigroup. Then*

$$\Lambda = \Lambda \circ \mathcal{LZ}.$$

Proof. Let S be a left zero band Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$, then $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha, \beta \in Y$, whence $ab \in S_\alpha S_\beta \subseteq S_{\alpha\beta} = S_\alpha$. Hence, $ab, a \in S_\alpha$. So $ab \stackrel{l}{\longrightarrow} {}^{\infty} a$, whence $b \stackrel{l}{\longrightarrow} {}^{\infty} a$. In a similar way it can be prove that $a \stackrel{l}{\longrightarrow} {}^{\infty} b$. Thus $a \stackrel{l}{\longrightarrow} {}^{\infty} \cap (\stackrel{l}{\longrightarrow} {}^{\infty})^{-1} b$ and by Lemma 6 [10] we have that $a \lambda b$. Therefore, S is a λ-simple semigroup.

The converse follows immediately. □

Lemma 2.. [6] *Let $\mathcal{X}$ be a class of semigroups and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two classes of bands. Then*

$$\mathcal{X} \circ (\mathcal{B}_1 \circ \mathcal{B}_2) \subseteq (\mathcal{X} \circ \mathcal{B}_1) \circ \mathcal{B}_2.$$

The lattice **LVB** of all varieties of bands was studied by P. A. Birjukov, C. F. Fennemore, J. A. Gerhard, M. Petrich and others. Here we use the characterization of **LVB** given by J. A. Gerhard and M. Petrich in [12]. They defined inductively three systems of words as follows:

$$\begin{array}{lll} G_2 = x_2x_1, & H_2 = x_2, & I_2 = x_2x_1x_2, \\ G_n = x_n\overline{G}_{n-1}, & H_n = x_n\overline{G}_{n-1}x_n\overline{H}_{n-1}, & I_n = x_n\overline{G}_{n-1}x_n\overline{I}_{n-1}, \end{array}$$

(for $n \geq 3$), and they shown that the lattice **LVB** can be represented by the graph given in Figure 1.

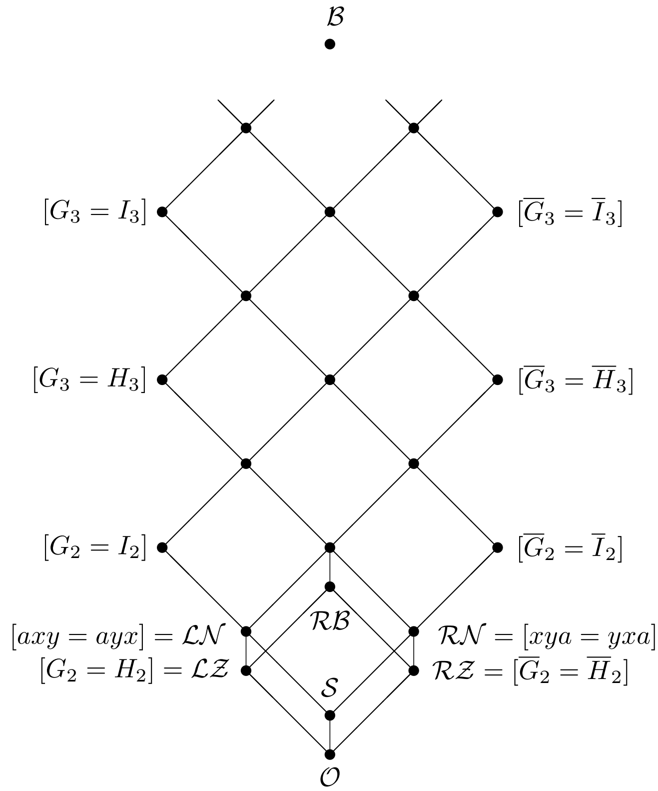

Figure 1.

Theorem 1.. [6] *Let $\mathcal{V}$ be an arbitrary variety of bands. Then*

$$\mathcal{LZ} \circ \mathcal{V} = \begin{cases} \mathcal{LZ}, & \text{if } \mathcal{V} \in [\mathcal{O}, \mathcal{LZ}]; \\ \mathcal{RB}, & \text{if } \mathcal{V} \in [\mathcal{RZ}, \mathcal{RB}]; \\ [G_2 = I_2], & \text{if } \mathcal{V} \in [\mathcal{S}, [G_2 = I_2]]; \\ [G_3 = I_3], & \text{if } \mathcal{V} \in [\mathcal{RN}, [G_3 = H_3]]; \\ [G_{n+1} = I_{n+1}], & \text{if } \mathcal{V} \in \left[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\right], n \geq 2; \\ [G_{n+1} = H_{n+1}], & \text{if } \mathcal{V} \in \left[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\right], n \geq 3. \end{cases}$$

Our next goal is to characterize semigroups from $\Lambda \circ \mathcal{V}$, for an arbitrary variety of bands $\mathcal{V}$.

Theorem 2.. *Let $\mathcal{V}$ be an arbitrary variety of bands. Then*

$$\Lambda \circ \mathcal{V} = \begin{cases} \Lambda, & \text{if } \mathcal{V} \in [\mathcal{O}, \mathcal{LZ}]; \\ \Lambda \circ \mathcal{RZ}, & \text{if } \mathcal{V} \in [\mathcal{RZ}, \mathcal{RB}]; \\ \Lambda \circ \mathcal{S}, & \text{if } \mathcal{V} \in [\mathcal{S}, [G_2 = I_2]]\,; \\ \Lambda \circ \mathcal{RN}, & \text{if } \mathcal{V} \in [\mathcal{RN}, [G_3 = H_3]]\,; \\ \Lambda \circ \left[\overline{G}_n = \overline{I}_n\right], & \text{if } \mathcal{V} \in \left[[\overline{G}_n = \overline{I}_n], [G_{n+1} = I_{n+1}]\right], n \geq 2; \\ \Lambda \circ \left[\overline{G}_n = H_n\right], & \text{if } \mathcal{V} \in \left[[\overline{G}_n = \overline{H}_n], [G_{n+1} = H_{n+1}]\right], n \geq 3. \end{cases}$$

Proof. By Lemma 1 we have that $\Lambda \circ \mathcal{LZ} = \Lambda$. Let $\mathcal{V} \in [\mathcal{V}_1, \mathcal{V}_2]$, whence $[\mathcal{V}_1, \mathcal{V}_2]$ is some of the intervals of the lattice **LVB** from the theorem. By Theorem 1 we have that $\mathcal{V}_2 = \mathcal{LZ} \circ \mathcal{V}_1$, whence

$$\Lambda \circ \mathcal{V}_1 \subseteq \Lambda \circ \mathcal{V} \subseteq \Lambda \circ \mathcal{V}_2 = \Lambda \circ (\mathcal{LZ} \circ \mathcal{V}_1) \subseteq (\Lambda \circ \mathcal{LZ}) \circ \mathcal{V}_1 = \Lambda \circ \mathcal{V}_1 \text{ (by Lemma 1).}$$

Therefore, $\Lambda \circ \mathcal{V}_1 = \Lambda \circ \mathcal{V} = \Lambda \circ \mathcal{V}_2$. □

3. Semilattices of matrices of λ-simple semigroups

By the well-known result of A. H. Clifford, any band of λ-simple semigroups is a semillatice of matrices of λ-simple semigroups. These semigroups will be characterized by the following theorem.

Theorem 3.. *A semigroup S is a semilattice of matrices of λ-simple semigroups if and only if*

$$a \longrightarrow^{\infty} b \implies ab \overset{l}{\longrightarrow}^{\infty} b, \tag{2}$$

for every $a, b \in S$.

Proof. Let S be a semilattice Y of matrices of λ-simple semigroup S_α, $\alpha \in Y$. Assume that $a \longrightarrow^{\infty} b$, for $a \in S_\alpha$, $b \in S_\beta$, $\alpha, \beta \in Y$. Then by Lemma 1.4 [18] or Lemma 9 [10] is $\beta \leq \alpha$, whence $b, ba \in S_\beta$ and by Theorem 1 [4] we have that $ba \cdot b \overset{l}{\longrightarrow}^{\infty} b$, i.e. $ab \overset{l}{\longrightarrow}^{\infty} b$.

Conversely, since every semigroup S is a semilattice Y of semilattice indecomposable semigroups S_α, $\alpha \in Y$, then for $a, b \in S_\alpha$, $\alpha \in Y$ we have that $a\sigma b$ (where σ is corresponding the greatest semilattice congruence on S), whence by Lemma 6 [10] $a \longrightarrow^{\infty} b$. By Lemma 9 [10] we have that $a \longrightarrow^{\infty} b$ in S_α, $\alpha \in Y$. From this it follows by (2) that $ab \overset{l}{\longrightarrow}^{\infty} b$. By Lemma 11 [10] we have that $ab \overset{l}{\longrightarrow}^{\infty} b$ in S_α, $\alpha \in Y$ and by Theorem 1 [4] S_α is a matrix of λ-simple semigroups, for all $\alpha \in Y$. □

The next theorem gives an explanation why the notion "hereditary weakly left Archimedean" is used.

Theorem 4.. *The following conditions on a semigroup S are equivalent:*

(i) *S is hereditary weakly left Archimedean;*
(ii) *any subsemigroup of S is weakly left Archimedean;*
(iii) *$\uparrow_l$ is a symmetric relation on S.*

Proof. (i) $\Longrightarrow$ (ii) Let T be a subsemigroup of S. For $a, b \in T$ we have that $b^i \in \langle a, b\rangle ab \subseteq Tab$, for some $i \in \mathbf{Z}^+$. Hence, T is a weakly left Archimedean semigroup and therefore S is a hereditary weakly left Archimedean semigroup.

(ii) $\Longrightarrow$ (i) Assume $a, b \in S$, then $\langle ba, b\rangle$ is a weakly left Archimedean semigroup, whence

$$b^i \in \langle ba, b\rangle ba \cdot b \subseteq \langle a, b\rangle ab,$$

for some $i \in \mathbf{Z}^+$.

(i) $\Longrightarrow$ (iii) Let $a, b \in S$ such that $a \uparrow_l b$, i.e. $b^n \in \langle a, b\rangle a$, for some $n \in \mathbf{Z}^+$. Then $b^n = xa$, for some $x \in \langle a, b\rangle$. For x and a there exists $m \in \mathbf{Z}^+$, $y \in \langle x, a\rangle \subseteq \langle a, b\rangle$ such that $a^m = yax = yb^n$, i.e. $b \uparrow_l a$.

(iii) $\Longrightarrow$ (i) Let $a, b \in S$, then $b \uparrow_l ab$, whence $ab \uparrow_l b$, i.e. $b^i \in \langle ab, b\rangle ab \subseteq \langle a, b\rangle ab$, for some $i \in \mathbf{Z}^+$. □

T. Tamura [20] proved that in the general case a semilattices of Archimedean semigroups are not subsemigroup closed. Here, we prove that a semilattices of hereditary weakly Archimedean semigroups are subsemigroup closed. By the following theorem we generalize some results obtained in [5].

Theorem 5.. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of hereditary weakly left Archimedean semigroups;*
(ii) $(\forall a, b \in S)\ a \longrightarrow b \implies (\exists i \in \mathbf{Z}^+)\ b^i \in \langle a, b\rangle ab;$
(iii) *every subsemigroup of S is a semilattice of hereditary weakly left Archimedean semigroups.*

Proof. (i) $\Longrightarrow$ (ii) Let S be a semilattice Y of hereditary weakly left Archimedean semigroups S_α, $\alpha \in Y$. Assume $a, b \in S$ such that $a \longrightarrow b$. If $a \in S_\alpha$, $b \in S_\beta$ for some $\alpha, \beta \in Y$, then $\beta \leq \alpha$, whence $b, ba \in S_\beta$. Now

$$b^n \in \langle ba, b\rangle bab \subseteq \langle a, b\rangle ab,$$

for some $n \in \mathbf{Z}^+$. Hence, (ii) holds.

(ii) $\Longrightarrow$ (i) Assume $a, b \in S$. Since $a \longrightarrow ab$, then by the hypothesis $a \cdot ab \uparrow_l ab$, i.e. $(ab)^n \in \langle a, ab\rangle a^2b$, for some $n \in \mathbf{Z}^+$. Now by Theorem 1 [9] S is a semilattice Y of Archimedean semigroups S_α, $\alpha \in Y$. Further, assume $\alpha \in Y$, $a, b \in S_\alpha$. Then $a \longrightarrow b$, so by the hypothesis $b^n \in \langle a, b\rangle ab$, for some $n \in \mathbf{Z}^+$. Therefore, S_α, $\alpha \in Y$ is an hereditary weakly left Archimedean semigroup.

(ii) $\Longrightarrow$ (iii) Let T be a subsemigroup of S and $a, b \in T$ such that $a \longrightarrow b$ in T, then $a \longrightarrow b$ in S and by (ii), $b^n \in \langle a, b\rangle ab \subseteq Tab$, for some $n \in \mathbf{Z}^+$. Thus, T is a semilattice of hereditary weakly left Archimedean semigroups.

(iii) $\Longrightarrow$ (i) This implication follows immediately. □

A semilattices of λ-simple semigroups were described in [6] and [9]. Here, by the following theorem we give some new interesting characterizations of these semigroups.

Theorem 6.. *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of λ-simple semigroups;*

(ii) $(\forall a,b \in S)\ a \longrightarrow^{\infty} b \implies a \stackrel{l}{\longrightarrow}{}^{\infty} b$;

(iii) $(\forall a,b \in S)\ a {\longrightarrow\!\!\!\!\!-}^{\infty} b \implies a \stackrel{l}{\longrightarrow}{}^{\infty} b$

Proof. (i) $\implies$ (ii) Let S be a semilattice Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a,b \in S$ such that $a \longrightarrow {}^{\infty} b$. Then by Lemma 1.4 (2) [18] (or Lemma 9 (b) [10]) $a \in S_\alpha$, $b \in S_\beta$, for some $\alpha,\beta \in Y$ and $\beta \leq \alpha$, whence $ba, b \in S_\beta$. So $ba \stackrel{l}{\longrightarrow}{}^{\infty} b$. Since $a \stackrel{l}{\longrightarrow} ba \stackrel{l}{\longrightarrow}{}^{\infty} b$, we then have that $a \stackrel{l}{\longrightarrow}{}^{\infty} b$.

(ii) $\implies$ (i) Let (ii) hold. By Theorem 1 [10] every semigroup S is a semilattice Y of σ-simple semigroups S_α, $\alpha \in Y$. Then for $a,b \in S_\alpha$, $\alpha \in Y$, by Theorem 1.1 [18] we have that $a {-\!\!-}^{\infty} b$, and by Lemma 1.4 (3) [18] $a {-\!\!-}^{\infty} b$ in S_α, $\alpha \in Y$, whence $a \longrightarrow {}^{\infty} b$ in S_α, $\alpha \in Y$. So by hypothesis $a \stackrel{l}{\longrightarrow}{}^{\infty} b$ and by Lemma 11 (a) [10] $a \stackrel{l}{\longrightarrow}{}^{\infty} b$ in S_α, $\alpha \in Y$, since $a,b \in S_\alpha$. Thus $a \stackrel{l}{\longrightarrow}{}^{\infty} b$ in S_α, $\alpha \in Y$, for all $a,b \in S_\alpha$ and by Lemma 6 [10] S_α, $\alpha \in Y$ is a λ-simple semigroup. Therefore, S is a semilattice of λ-simple semigroups.

(i) $\implies$ (iii) Let S be a semilattice Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a,b \in S$ such that $a {-\!\!-}^{\infty} b$. Then by Lemma 1.4 (3) [18] $a,b \in S_\alpha$ and $a {-\!\!-}^{\infty} b$ in S_α, for some $\alpha \in Y$, whence $a\lambda b$ and by Lemma 6 (iv) [10] $a \stackrel{l}{\longrightarrow}{}^{\infty}\ b$.

(iii) $\implies$ (i) Let (iii) hold. Since every semigroup S is a semilattice Y of σ-simple semigroups S_α, $\alpha \in Y$, then for $a,b \in S_\alpha$, $\alpha \in Y$, by Theorem 1.1 [18] we have that $a {-\!\!-}^{\infty} b$, whence $a \stackrel{l}{\longrightarrow}{}^{\infty} b$ and $a(\stackrel{l}{\longrightarrow}{}^{\infty})^{-1} b$ in S_α. Thus $a \stackrel{l}{\longrightarrow}{}^{\infty} \cap (\stackrel{l}{\longrightarrow}{}^{\infty})^{-1} b$ and by Lemma 6 (iv) [18] S_α is a λ-simple semigroup.

$\square$

Problem 1. *By $\mathcal{M}$ we denote the class of all matrices (rectangular bands). Let*

$$\Lambda \circ \mathcal{M}^{k+1} = \left(\Lambda \circ \mathcal{M}^{k}\right) \circ \mathcal{M}, \quad k \in \mathbf{Z}^{+}.$$

Describe the structure of semigroups from the following classes

$$\Lambda \circ \mathcal{M}^{k+1}, \quad \left(\Lambda \circ \mathcal{M}^{k+1}\right) \circ \mathcal{B}, \quad \left(\Lambda \circ \mathcal{M}^{k+1}\right) \circ \mathcal{S}.$$

The previous problem can be formulated in the same way if instead the class Λ we take the class of all power-joined semigroups or the class of all λ_n-simple semigroups.

4. Some remarks on λ-equivalence

In this section we give some characterization of λ congruence.

Lemma 3.. *The following conditions on a semigroup S are equivalent:*

(i) λ *is a congruence;*

(ii) $\lambda = \lambda^\flat$*;*

(iii) λ *is a band congruence.*

Proof. This assertion follows by Lemma 2.2 [8]. □

Lemma 4.. *The following conditions on a semigroup S are equivalent:*

(i) $\lambda^\flat$ *is a band congruence;*

(ii) $\lambda^\flat = R(\lambda^\flat)$*;*

(iii) $(\forall a \in S)(\forall x, y \in S^1)\ (xay, xa^2y) \in \lambda$.

Proof. (i)⇔(ii) This follows by Lemma 2.1 [8] and Lemma 2.3 [8].
(i)⇔(iii) This follows by Lemma 2.4 [8]. □

Corollary 1.. *If $S \in \Lambda \circ \mathcal{B}$, then*

$$(\forall a \in S)(\forall x, y \in S^1)\ (xay, xa^2y) \in \lambda.$$

Proof. Let S be a band Y of λ-simple semigroups S_α, $\alpha \in Y$. Assume $a \in S$ and $x, y \in S$, then $xay, xa^2y \in S_\alpha$, for some $\alpha \in y$, whence $(xay, xa^2y) \in \lambda$. □

Problem 2. *Is the converse of the Corollary 1 holds?*

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