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PROPERTIES OF I-SUBMAXIMAL IDEAL TOPOLOGICAL SPACES

Erdal Ekici and Takashi Noiri

Abstract

In [2], the notion of I-submaximal ideal topological spaces is introduced and studied. In this paper, several characterizations and further properties of I-submaximal ideal topological spaces are obtained.

1 Introduction

The concept of submaximality of general topological spaces was introduced by Hewitt [12] in 1943. He discovered a general way of constructing maximal topologies. In [3], Alas et al. proved that there can be no dense maximal subspace in a product of first countable spaces, while under Booth's Lemma there exists a dense submaximal subspace in $[0,1]^c$. It is established that under the axiom of constructibility any submaximal Hausdorff space is σ-discrete. Any homogeneous submaximal space is strongly σ-discrete if there are no measurable cardinals. The first systematic study of submaximal spaces was undertaken in the paper of Arhangel'skiĭ and Collins [4]. They gave various necessary and sufficient conditions for a space to be submaximal and showed that every submaximal space is left-separated. This led to the question whether every submaximal space is σ-discrete [4]. The notion of ideal topological spaces was studied by Kuratowski [17] and Vaidyanathaswamy [19]. In 1990, Janković and Hamlett [13] investigated further properties of ideal topological spaces. In [2], properties of I-submaximal ideal topological spaces is studied. In this paper, several characterizations and further properties of I-submaximal ideal topological spaces are obtained. It will be shown that every ideal subspace of an I-submaximal ideal topological space is I-submaximal.

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2 Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively. An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in I$ and $B \subset A$ implies $B \in I$.
(2) $A \in I$ and $B \in I$ implies $A \cup B \in I$.

Given a topological space (X, τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function [17] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X : G \cap A \notin I$ for every $G \in \tau(x)\}$ where $\tau(x) = \{G \in \tau : x \in G\}$. A Kuratowski closure operator $Cl^*(.)$ for a topology $\tau^*(I, \tau)$, called the $\star$-topology, finer than τ, is defined by $Cl^*(A) = A \cup A^*(I, \tau)$ [13]. When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I, \tau)$. For any ideal space (X, τ, I), the collection $\{U \backslash J : U \in \tau$ and $J \in I\}$ is a basis for τ^*. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or simply an ideal space.

Definition 1. *A subset A of an ideal space (X, τ, I) is called*
(1) α-I-open [8] if $A \subset Int(Cl^*(Int(A)))$.
(2) pre-I-open [5] if $A \subset Int(Cl^*(A))$.
(3) semi-I-open [8] if $A \subset Cl^*(Int(A))$.
(4) strongly β-I-open [9] if $A \subset Cl^*(Int(Cl^*(A)))$.
(5) $\star$-dense [6] if $Cl^*(A) = X$.

Lemma 1. *([2]) For a subset A of an ideal space (X, τ, I), the following properties are equivalent:*
(1) A is pre-I-open,
(2) $A = G \cap B$, where G is open and B is $\star$-dense.

Lemma 2. *([1]) Let (X, τ, I) be an ideal space and $A \subset X$. Then A is α-I-open if and only if it is semi-I-open and pre-I-open.*

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Definition 2. *([2]) An ideal space (X, τ, I) is called I-submaximal if every $\star$-dense subset of X is open.*

Theorem 3. *For an ideal space (X, τ, I), the following properties are equivalent:*
(1) X is I-submaximal,
(2) Every pre-I-open set is open,
(3) Every pre-I-open set is semi-I-open and every α-I-open set is open.

Proof. (1) $\Rightarrow$ (2) : It follows from Lemma 4.4 of [2].

(2) $\Rightarrow$ (3) : Suppose that every pre-I-open set is open. Then every pre-I-open set is semi-I-open.

Let $A \subset X$ be an α-I-open set. Since every α-I-open set is pre-I-open, then by (2), A is open.

(3) $\Rightarrow$ (1) : Let A be a $\star$-dense subset of X. Since $Cl^*(A) = X$, then A is pre-I-open. By (3), A is semi-I-open. Since a set is α-I-open if and only if it is semi-I-open and pre-I-open, then A is α-I-open. Thus, by (3), A is open and hence X is I-submaximal. ∎

Lemma 4. *([11]) Let (X, τ, I) be an ideal spaces and A, $B \subset X$. If A is semi-I-open and B is open, then $A \cap B$ is semi-I-open.*

Theorem 5. *For a subset A of an I-submaximal ideal space (X, τ, I), the following are equivalent:*

(1) A is semi-I-open,

(2) A is strongly β-I-open.

Proof. (2) $\Rightarrow$ (1) : Let A be a strongly β-I-open set in X. Put $H = Cl^*(A)$ and $K = A \cup (X \backslash Cl^*(A))$. We have $A = Cl^*(A) \cap K$ and $Cl^*(K) = X$. This implies that $A = H \cap K$, where H is semi-I-open and K is $\star$-dense. Since X is I-submaximal, then K is open. By Lemma 4, $A = H \cap K$ is semi-I-open.

(1) $\Rightarrow$ (2) : It follows from the fact that every semi-I-open set is strongly β-I-open. ∎

Theorem 6. *For an ideal space (X, τ, I), the following properties are equivalent:*

(1) X is I-submaximal,

(2) For all $A \subset X$, if $A \backslash Int(A) \neq \varnothing$, then $A \backslash Int(Cl^*(A)) \neq \varnothing$.

(3) $\tau = \{U \backslash A : U \in \tau \text{ and } Int^*(A) = \varnothing\}$.

Proof. (1) $\Rightarrow$ (2) : Let $A \subset X$ and $A \backslash Int(A) \neq \varnothing$. Suppose that $A \backslash Int(Cl^*(A)) = \varnothing$. Then $A \subset Int(Cl^*(A))$. This implies that A is pre-I-open. Since X is I-submaximal, by Theorem 3, A is open. Thus, $A \backslash Int(A) = A \backslash A = \varnothing$. This is a contradiction.

(2) $\Rightarrow$ (1) : Let A be a pre-I-open set. Then $A \subset Int(Cl^*(A))$.

Suppose that A is not open. Then $A \nsubseteq Int(A)$ and hence $A \backslash Int(A) \neq \varnothing$. By (2), $A \backslash Int(Cl^*(A)) \neq \varnothing$. Thus, $A \nsubseteq Int(Cl^*(A))$. This is a contradiction.

(1) $\Rightarrow$ (3) : Suppose that $\sigma = \{U \backslash A : U \in \tau \text{ and } Int^*(A) = \varnothing\}$.

Let $G \in \tau$. Since $G = G \backslash \varnothing$ and $Int^*(\varnothing) = \varnothing$, then $\tau \subset \sigma$.

Let $G \in \sigma$. Then $G = U \backslash A$, where $U \in \tau$ and $Int^*(A) = \varnothing$. We have $G = U \cap X \backslash A$. Since $Int^*(A) = \varnothing$, then $X \backslash Int^*(A) = Cl^*(X \backslash A) = X$. Since X is I-submaximal, then $X \backslash A$ is open. Thus, G is open. Hence $\sigma \subset \tau$.

(3) $\Rightarrow$ (1) : Let A be a pre-I-open set. By Lemma 1, $A = G \cap B$, where G is open and B is $\star$-dense. We have $Cl^*(B) = X$ and hence $Int^*(X \backslash B) = \varnothing$. This implies that $A = G \backslash (X \backslash B)$ and $Int^*(X \backslash B) = \varnothing$. Thus, by (3), A is open. Hence, by Theorem 3, X is I-submaximal. ∎

Definition 3. *([12]) A topological space (X, τ) is called a submaximal space if each of its dense subset is open.*

Theorem 7. *Let $f : (X, \tau) \to (Y, \sigma, I)$ be an open surjective function. If X is submaximal, then Y is I-submaximal.*

Proof. Let X be submaximal and $A \subset Y$ be a $\star$-dense set. Then A is dense in Y. Since $f^{-1}(A)$ is dense, then $f^{-1}(A)$ is open in X. Since f is an open surjective function, then $A = f(f^{-1}(A))$ is open. Hence, Y is I-submaximal. ∎

Corollary 8. *If $\prod_{i \in I} X_i$ is a submaximal product space of X_i, then X_i is I-submaximal for every $i \in I$.*

Proof. It follows from the fact that for each $i \in I$, the projective function $p_i : \prod_{i \in I} X_i \to X_i$ is an open surjection. ∎

Definition 4. *A subset A of an ideal space (X, τ, I) is called $\star$-codense if $X \backslash A$ is $\star$-dense.*

Theorem 9. *For an ideal space (X, τ, I), the following are equivalent:*
(1) X is I-submaximal,
(2) Every $\star$-codense subset A of X is closed.

Proof. (1) $\Rightarrow$ (2) : Let A be a $\star$-codense subset of X. Since $X \backslash A$ is $\star$-dense, then $X \backslash A$ is open. Thus, A is closed.
(2) $\Rightarrow$ (1) : It is similar to that of (1) $\Rightarrow$ (2). ∎

Definition 5. *A subset A of an ideal space (X, τ, I) is called*
(1) a t-I-set [8] if $Int(A) = Int(Cl^*(A))$.
(2) semi-I-regular [16] if A is a t-I-set and semi-I-open.
(3) an AB_I-set [16] if $A = U \cap V$, where $U \in \tau$ and V is a semi-I-regular set.

Theorem 10. *For an ideal space (X, τ, I), the following are equivalent:*
(1) X is I-submaximal,
(2) Every pre-I-open set is an AB_I-set,
(3) Every $\star$-dense set is an AB_I-set.

Proof. (1) $\Rightarrow$ (2) : Let $A \subset X$ be a pre-I-open set. Since X is I-submaximal, by Theorem 3, A is open. It follows from Proposition 2 of [16] that A is an AB_I-set.
(2) $\Rightarrow$ (3) : Let $A \subset X$ be a $\star$-dense set. Since every $\star$-dense set is pre-I-open, then by (2), A is an AB_I-set.
(3) $\Rightarrow$ (1) : Let $A \subset X$ be a $\star$-dense set. By (3), A is an AB_I-set. Since every $\star$-dense set is pre-I-open, then A is pre-I-open. Since A is pre-I-open and an AB_I-set, by Proposition 4 of [16], A is open. Hence, X is I-submaximal. ∎

4 Subspaces

Recall that if (X, τ, I) is an ideal topological space and A is a subset of X, then (A, τ_A, I_A), where τ_A is the relative topology on A and $I_A = \{A \cap J : J \in I\}$ is an ideal topological space.

Lemma 11. *([14]) Let (X, τ, I) be an ideal topological space and $B \subset A \subset X$. Then $B^*(\tau_A, I_A) = B^*(\tau, I) \cap A$.*

Lemma 12. *([10]) Let (X, τ, I) be an ideal topological space and $B \subset A \subset X$. Then $Cl^*_A(B) = Cl^*(B) \cap A$.*

Theorem 13. *If (X, τ, I) is an I-submaximal ideal space and $A \subset X$, then (A, τ_A, I_A) is I-submaximal.*

Proof. Let B be a $\star$-dense set in (A, τ_A, I_A). Let $U = B \cup (X \backslash A)$. By Lemma 12, we have

$$Cl^*(U) = Cl^*(B \cup (X\backslash A)) \supset$$
$$Cl^*(B) \cup Cl^*(X\backslash A) \supset Cl^*_A(B) \cup Cl^*(X\backslash A)$$
$$= A \cup Cl^*(X\backslash A) = X.$$

Therefore, U is a $\star$-dense set in (X, τ, I). Since X is I-submaximal, then U is open in X. Thus, $B = U \cap A$ and B is open in (A, τ_A, I_A). Hence, (A, τ_A, I_A) is I-submaximal. ∎

Definition 6. *([8]) A subset A of an ideal space (X, τ, I) is called a B_I-set if $A = U \cap V$, where $U \in \tau$ and V is a t-I-set.*

Theorem 14. *For an ideal space (X, τ, I), the following are equivalent:*
(1) X is I-submaximal,
(2) Every subset of X is a B_I-set,
(3) Every strongly β-I-open set is a B_I-set,
(4) Every $\star$-dense subset of X is a B_I-set.

Proof. (1) $\Rightarrow$ (2) : It follows from Theorem 3.2 of [18].
(2) $\Rightarrow$ (3) : Obvious.
(3) $\Rightarrow$ (4) : It follows from the fact that every $\star$-dense subset of X is a strongly β-I-open set.
(4) $\Rightarrow$ (1) : It follows from Theorem 3.2 of [18]. ∎

5 Further Properties

Definition 7. *([7]) An ideal space (X, τ, I) is said to be $\star$-extremally disconnected if $\star$-closure of every open subset A of X is open.*

Lemma 15. *([7]) For an ideal space (X, τ, I), the following properties are equivalent:*
(1) X is $\star$-extremally disconnected,
(2) Every semi-I-open set is pre-I-open,
(3) The $\star$-closure of every strongly β-I-open subset of X is open,
(4) Every strongly β-I-open set is pre-I-open.

Theorem 16. *For an ideal space (X, τ, I), the following properties are equivalent:*
(1) X is I-submaximal and $\star$-extremally disconnected,
(2) Any subset of X is strongly β-I-open if and only if it is open.

Proof. (1) $\Rightarrow$ (2) : Let X be I-submaximal and $\star$-extremally disconnected. By Lemma 15, every strongly β-I-open set is pre-I-open. By Theorem 3, every pre-I-open set is open. Thus, every strongly β-I-open set is open. The converse follows from the fact that every open set is strongly β-I-open.

(2) $\Rightarrow$ (1) : Suppose that any subset of X is strongly β-I-open if and only if it is open. Since every strongly β-I-open set is open and so pre-I-open, by Lemma 15, X is $\star$-extremally disconnected. Since every pre-I-open set is open, by Theorem 3, X is I-submaximal. ∎

Corollary 17. *For an ideal space (X, τ, I), if X is I-submaximal and $\star$-extremally disconnected, the following are equivalent for a subset $A \subset X$:*
(1) A is strongly β-I-open,
(2) A is semi-I-open,
(3) A is pre-I-open,
(4) A is α-I-open,
(5) A is open.

Proof. It follows from Theorem 16. ∎

Lemma 18. *([16]) Every AB_I-set is semi-I-open in an ideal topological space (X, τ, I).*

Theorem 19. *For an ideal space (X, τ, I), if X is I-submaximal and $\star$-extremally disconnected, the following properties are equivalent for a subset $A \subset X$:*
(1) A is semi-I-open,
(2) A is an AB_I-set.

Proof. (1) $\Rightarrow$ (2) : Let A is semi-I-open. Since X is $\star$-extremally disconnected, by Lemma 15, every semi-I-open set is pre-I-open. Since X is I-submaximal, by Theorem 10, every pre-I-open set is an AB_I-set.

(2) $\Rightarrow$ (1) : It follows from Lemma 18. ∎

Definition 8. *([15]) A subset A of an ideal space (X, τ, I) is called weakly I-local closed if $A = U \cap V$, where $U \in \tau$ and V is a $\star$-closed set.*

Theorem 20. *For an ideal space (X, τ, I), the following properties are equivalent:*
(1) X is I-submaximal,
(2) Every subset of X is weakly I-local closed,
(3) Every subset of X is a union of a $\star$-open subset and a closed subset of X,
(4) Every $\star$-dense subset of X is an intersection of a $\star$-closed subset and an open subset of X.

Proof. (1) $\Rightarrow$ (2) : It follows from Theorem 3.2 of [18].

(2) $\Leftrightarrow$ (3) : Let $A \subset X$. By (2), we have $X \backslash A = U \cap K$, where U is open and K is $\star$-closed in X. This implies that $A = (X \backslash U) \cup (X \backslash K)$, where $X \backslash U$ is closed and $X \backslash K$ is $\star$-open in X. The converse is similar.

(2) $\Rightarrow$ (4) : Obvious.

(4) $\Rightarrow$ (1) : Let $A \subset X$ be a $\star$-dense set. Then $A = U \cap B$, where U is open and B is $\star$-closed. Since $A \subset B$ and so B is $\star$-dense, then $Int(B) = Int(Cl^*(B)) = Int(X) = X$. Hence $B = X$ and $A = U$ is open. Thus, X is I-submaximal. ∎

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Addresses:

Erdal Ekici

Department of Mathematics, Canakkale Onsekiz Mart University, Terzioglu Campus, 17020 Canakkale, Turkey

E-mail: eekici@comu.edu.tr

Takashi Noiri

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumomoto-ken, 869-5142, Japan

E-mail: t.noiri@nifty.com