Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

Filomat **26:1** (2012), 165–176 DOI: 10.2298/FIL1201165C

Almost Fedosov and metriplectic structures in the geometry of semisprays

Mircea Crasmareanu

Abstract

Given a pair (semispray S, almost symplectic form ω) on a tangent bundle, the family of nonlinear connections N such that ω is recurrent with respect to (S, N) with a fixed recurrent factor is determined by using the Obata tensors. In particular, we obtain a characterization for a pair (N, ω) to be recurrent as well as for the triple $(S, \overset{c}{N}, \omega)$ where $\overset{c}{N}$ is the canonical nonlinear connection of the semispray S. In the particular case of vanishing recurrence factor we get the family of almost Fedosov structures associated to a fixed semispray and almost symplectic structure. For a triple (semispray S, almost symplectic form ω, metric g), a characterization for existence of a corresponding almost metriplectic structure is obtained.

Introduction

In two many cited papers, [9, 10], Yung-chow Wong derived several properties of a recurrent tensor field T on a manifold M endowed with a linear connection ∇. Recall that this means the existence of a 1-form α_T on M such that:

$$\nabla T = \alpha_T \otimes T. \tag{0}$$

For $\alpha_T = 0$ we recover the notion of *parallel* or *covariant constant tensor field*.

The aim of present paper is to extend the notion of recurrence to the geometry of systems of second order differential equations on M or *path geometry* as is usually called. More precisely, given such a system S, it can be considered as a vector field (called *semispray*) on the tangent bundle TM. Then, a type of differential ∇ is naturally associated to S. A main tool in the definition of ∇ is given by a splitting of the iterated tangent bundle $T(TM)$ provided by a distribution N

2010 *Mathematics Subject Classifications.* 58A30; 34A26; 37C10; 53C15.
Key words and Phrases. Semispray, nonlinear connection, recurrent tensor, almost Fedosov structure, almost metriplectic structure.
Received: October 7, 2010; Revision 1, June 16, 2011; Revision 2, June 22, 2011.
Communicated by Vladimir Dragović.
The author would like to thank the referee for valuable remarks and suggestions which have improved the paper both in substance and presentation.

on TM supplementary to the vertical distribution. Such an object N is called *nonlinear connection* and a main result in path geometry is that every S yields such a nonlinear connection, $\overset{c}{N}$, indexed by us with c from canonical.

We treat in detail the recurrence of tensor fields T of type $(0,2)$ in correspondence with the results from [1] (where the metrizability problem is considered) and [5]; therefore, we can say that we search recurrent almost symplectic forms for a given system of second order differential equations, the recurrence factor being fixed. In fact, given a pair (semispray S, almost symplectic structure ω) on TM, the family of nonlinear connections N such that ω is recurrent with respect to (S,N) with a fixed recurrent factor is determined by using the Obata tensors defined by ω. In particular, we obtain a characterization for a pair (N,ω) to be recurrent as well as for the recurrence of the triple $(S,\overset{c}{N},\omega)$. In the particular case of a vanishing recurrence factor we derive the associated almost Fedosov structures. Recall after [6] that a Fedosov structure is a pair (symplectic form ω, symmetric linear connection ∇) with $\nabla\omega = 0$.

In the last section we put together the results from the (almost) symplectic and metric cases into the (almost) metripletic structures. Metriplectic systems were introduced in [8] and these systems combine both conservative and dissipative systems; see also [7]. The underline geometrical setting is provided by a pair (almost symplectic structure, metric) with both objects parallel with respect to a linear connection.

1 Nonlinear connections and semisprays on tangent bundles

Let M be a smooth, n-dimensional manifold for which we denote: $C^\infty(M)$-the algebra of smooth real functions on M, $\mathcal{X}(M)$-the Lie algebra of vector fields on M, $T^r_s(M)$-the $C^\infty(M)$-module of tensor fields of (r,s)-type on M.

A local chart $(U, x = (x^i) = (x^1, ..., x^n))$ on M lifts to a local chart on the tangent bundle TM given by: $(\pi^{-1}(U), (x,y) = (x^i, y^i))$ where $\pi : TM \to M$ is the canonical bundle projection. The kernel of the differential of π is an integrable distribution $V(TM)$ with local basis $(\frac{\partial}{\partial y^i})$. An important element of $V(TM)$ is the *Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$. $V(TM)$ is called *the vertical distribution* and its elements are *vertical vector fields*.

The tensor field $J \in T^1_1(TM)$ given by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called *the tangent structure*. Two of its properties are: the nilpotence $J^2 = 0$ and $imJ\,(= \ker J) = V(TM)$. A well-known notion in the tangent bundles geometry is:

Definition 1.1([1, p. 336]) A supplementary distribution N to the vertical distribution $V(TM)$:

$$T(TM) = N \oplus V(TM) \tag{1.1}$$

is called *horizontal distribution* or *nonlinear connection*. A vector field belonging to N is called *horizontal*.

A nonlinear connection has a local basis:

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} \tag{1.2}$$

and the functions $\left(N_j^i(x,y)\right)$ are called *the coefficients* of N. A basis of $\mathcal{X}(TM)$ adapted to the decomposition (1.1) is $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ called *Berwald basis*. The dual of the Berwald basis is: $(dx^i, \delta y^i = dy^i + N_j^i dx^j)$.

A second remarkable structure on TM is provided by:

Definition 1.2([1, p. 336]) $S \in \mathcal{X}(TM)$ is called *semispray* if:

$$J(S) = \mathbb{C}. \tag{1.3}$$

In canonical bundle coordinates:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i} \tag{1.4}$$

and the functions $\left(G^i(x,y)\right)$ are *the coefficients* of S. The flow of S is a system of second order differential equations: $\frac{d^2x^i}{dt^2} = 2G^i(x, \frac{dx}{dt})$ and then the geometry of the pair (M,S) is called *path geometry*.

There is an important relationship between semisprays and nonlinear connections. Firstly, a nonlinear connection $N = (N_j^i)$ yields an unique horizontal semispray denoted $S(N)$ with:

$$G^i = \frac{1}{2} N_j^i y^j. \tag{1.5}$$

In other words:

$$S(N) = y^i \frac{\delta}{\delta x^i}. \tag{1.6}$$

Conversely, a semispray S yields a nonlinear connection $\overset{c}{N}$ given by:

$$\overset{c}{N}{}_j^i = \frac{\partial G^i}{\partial y^j}. \tag{1.7}$$

Definition 1.3 A semispray S for which the coefficients $\left(G^i\right)$ are homogeneous of degree 2 with respect to the variables $\left(y^i\right)$ will be called *spray*.

Locally this means, via Euler theorem:

$$2G^i = y^j \frac{\partial G^i}{\partial y^j} \tag{1.8}$$

and then $\overset{c}{N}$ is 1-homogeneous:

$$\overset{c}{N}{}_j^i = y^a \frac{\partial \overset{c}{N}{}_j^i}{\partial y^a} \tag{1.9}$$

which yields that $S = S(N^c)$ and then S is horizontal with respect to $\overset{c}{N}$.

2 Recurrence and almost Fedosov structures in path geometry

2.1 The general problem of recurrent triples

Let us fix a semispray $S = \left(G^i\right)$ and a nonlinear connection $N = \left(N^i_j\right)$. Following [2] let us consider:

Definition 2.1 The *dynamical derivative* associate to the pair (S, N) is the map $\overset{SN}{\nabla} : N \rightarrow N$ given by:

$$\overset{SN}{\nabla} X = \overset{SN}{\nabla}\left(X^i \frac{\delta}{\delta x^i}\right) := \left(S\left(X^i\right) + N^i_j X^j\right) \frac{\delta}{\delta x^i}. \tag{2.1}$$

The dynamical derivative satisfy:

$$\overset{SN}{\nabla}\left(\frac{\delta}{\delta x^i}\right) = N^j_i \frac{\delta}{\delta x^j}, \overset{SN}{\nabla}(X+Y) = \overset{SN}{\nabla} X + \overset{SN}{\nabla} Y, \overset{SN}{\nabla}(fX) = S(f) X + f \overset{SN}{\nabla} X.$$

It is straightforward to extend the action of $\overset{SN}{\nabla}$ to general horizontal tensor fields by requiring to preserve the tensor product and Leibniz rule. Moreover, we will extend $\overset{SN}{\nabla}$ to a special class of tensor fields:

Definition 2.2 A *d-tensor field* (d from distinguished) on TM is a tensor field whose change of components, under a change of canonical coordinates $(x, y) \rightarrow (\widetilde{x}, \widetilde{y})$ on TM, involves only factors of type $\frac{\partial \widetilde{x}}{\partial x}$ and (or) $\frac{\partial x}{\partial \widetilde{x}}$.

Examples 2.3 i) $\left(\frac{\delta}{\delta x^i}\right)$ and $\left(\frac{\partial}{\partial y^i}\right)$ are components of d-tensor fields of $(1,0)$-type.
ii) $\left(dx^i\right)$ and $\left(\delta y^i\right)$ are components of d-tensor fields of $(0,1)$-type.
iii) $\left(G^i\right)$ are not components of a d-tensor field since a change of coordinates implies:

$$2\widetilde{G}^i = 2\frac{\partial \widetilde{x}^i}{\partial x^j} G^j - \frac{\partial \widetilde{y}^i}{\partial x^j} y^j$$

but it results that given two semisprays $\overset{1}{S}$ and $\overset{2}{S}$ their difference $X = \overset{2}{S} - \overset{1}{S}$ is a vertical (and then d-) vector field.
iv) $\left(N^i_j\right)$ are not components of a d-tensor field since a change of coordinates implies:

$$\frac{\partial \widetilde{x}^j}{\partial x^k} N^k_i = \widetilde{N}^j_k \frac{\partial \widetilde{x}^k}{\partial x^i} + \frac{\partial \widetilde{y}^j}{\partial x^i}.$$

It follows that given two nonlinear connections $\overset{1}{N}$ and $\overset{2}{N}$ their difference $F = \overset{2}{N} - \overset{1}{N} = \left(F^i_j = \overset{2}{N}{}^i_j - \overset{1}{N}{}^i_j\right)$ is a d-tensor field of $(1,1)$-type. Here, we thought of the difference $\overset{2}{N} - \overset{1}{N}$ in terms of associated projectors, namely if v, h_1, h_2 are the projectors given by the decomposition (1.1) then $\overset{2}{N} - \overset{1}{N}$ is corresponding to $h_2 - h_1$ which is a projector together with v.

Definition 2.4 An *almost symplectic structure* ω on TM is a d-tensor field of $(0,2)$-type of the local (diagonal) form: $\omega = \omega_{ij}dx^i \wedge \delta y^j$, which is skew-symmetric and non-degenerate.

It results for the components $\omega_{ij} = \omega_{ij}(x,y) = \omega(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j})$ the following properties:
1) (skew-symmetry) $\omega_{ij} = -\omega_{ji}$,
2) (non-degeneration) $\det(\omega_{ij}) \neq 0$; then there exists the d-tensor field of $(2,0)$-type $\omega^{-1} = (\omega^{ij})$.
An important property of ω is that N and $V(TM)$ are Lagrangian distributions.
Definition 2.5 The *dynamical derivative* of the almost symplectic structure ω with respect to the pair (S,N) is $\overset{SN}{\nabla}\omega : N \times N \to N$ given by:

$$\overset{SN}{\nabla}\omega(X,Y) = S(\omega(X,Y)) - \omega(\overset{SN}{\nabla}X, Y) - \omega(X, \overset{SN}{\nabla}Y). \tag{2.2}$$

The main notion of this section is:
Definition 2.6 i) Let $\alpha \in C^\infty(TM)$. The almost symplectic structure ω is called *α-recurrent* with respect to the pair (S,N) if:

$$\overset{SN}{\nabla}\omega = \alpha\omega. \tag{2.3}$$

Also, the triple (S,N,ω) will be called an *α-recurrent structure*.
ii) An *almost Fedosov structure* on TM is a triple (S,N,ω) which is 0-recurrent i.e. ω is parallel with respect to $\overset{SN}{\nabla}$.

The aim of this section is to find all nonlinear connections which together with fixed (S,ω) form an α-recurrent structure. In order to answer at this question, a look at example 2.3 iv) gives necessary a study of two operators, called *Obata* in the following, acting on the space of d-tensor fields of $(1,1)$-type of local (horizontal) forms $X = X^i_j \frac{\delta}{\delta x^i} \otimes dx^j$:

$$O(\omega)^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l - \omega^{ij}\omega_{kl}\right), \quad \overset{*}{O}(\omega)^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \omega^{ij}\omega_{kl}\right). \tag{2.4}$$

The Obata operators are supplementary projectors:

$$O^{ia}_{bj}\overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ia}_{bj}O^{bk}_{la} = 0, \quad O^{ia}_{bj}O^{bk}_{la} = O^{ik}_{lj}, \quad \overset{*}{O}{}^{ia}_{bj}\overset{*}{O}{}^{bk}_{la} = \overset{*}{O}{}^{ik}_{lj} \tag{2.5}$$

(for simplicity we give up to denote ω into these O) and the tensorial equations involving these operators has solutions as follows:
Proposition 2.7 *The system of equations*:

$$\overset{*}{O}(\omega)^{ia}_{bj}\left(X^b_a\right) = A^i_j, \qquad \left(O(\omega)^{ia}_{bj}(X^b_a) = A^i_j\right) \tag{2.6}$$

with X as unknown has a solution if and only if:

$$O(\omega)^{ia}_{bj}\left(A^b_a\right) = 0, \qquad \left(\overset{*}{O}(\omega)^{ia}_{bj}\left(A^b_a\right) = 0\right) \tag{2.7}$$

and then, the general solution is:

$$X_j^i = A_j^i + O(\omega)_{bj}^{ia}\left(Y_a^b\right), \qquad \left(X_j^i = A_j^i + \overset{*}{O}(\omega)_{bj}^{ia}\left(Y_a^b\right)\right) \tag{2.8}$$

with Y *an arbitrary d-tensor field of* $(1,1)$*-type.*

We are ready for one of the main results of paper:

Theorem 2.8 *Set* S, ω *and* α.

i) *The family* $\mathcal{N}(S,\omega,\alpha)$ *of all nonlinear connections* $N = \left(N_j^i\right)$ *such that* (S,N,ω) *is* α*-recurrent is given by:*

$$N_j^i = \frac{1}{2}\overset{c}{N}{}_j^i - \frac{1}{2}\omega^{ia}\omega_{bj}\overset{c}{N}{}_a^b + \frac{1}{2}\omega^{ia}S\left(\omega_{aj}\right) - \frac{\alpha}{2}\delta_j^i + O(\omega)_{bj}^{ia}\left(X_a^b\right). \tag{2.9}$$

ii) *The family* $\mathcal{N}_F(S,\omega)$ *of all nonlinear connections* $N = \left(N_j^i\right)$ *such that* (S,N,ω) *is an almost Fedosov structure on* TM *is given by:*

$$N_j^i = \frac{1}{2}\overset{c}{N}{}_j^i - \frac{1}{2}\omega^{ia}\omega_{bj}\overset{c}{N}{}_a^b + \frac{1}{2}\omega^{ia}S\left(\omega_{aj}\right) + O(\omega)_{bj}^{ia}\left(X_a^b\right). \tag{2.9F}$$

In these relations $X = \left(X_a^b\right)$ *is an arbitrary horizontal d-tensor field of* $(1,1)$*-type. It follows that* $\mathcal{N}(S,\omega,\alpha)$ *is a* $C^\infty(TM)$*-affine module over the* $C^\infty(TM)$*-module of horizontal d-tensor fields of* $(1,1)$*-type.*

Proof. We search $\left(N_j^i\right)$ of the form:

$$N_j^i = \overset{c}{N}{}_j^i + F_j^i \tag{2.10}$$

with $\left(F_j^i\right)$ a d-tensor field of $(1,1)$-type to be determined. The local expression of equation (2.3) is:

$$S\left(\omega_{uv}\right) - \omega_{um}N_v^m - \omega_{mv}N_u^m = \alpha\omega_{uv} \tag{2.11}$$

and inserting (2.10) in (2.11) gives:

$$S\left(\omega_{uv}\right) - \omega_{um}\overset{c}{N}{}_v^m - \omega_{mv}\overset{c}{N}{}_u^m = \omega_{um}F_v^m + \omega_{mv}F_u^m + \alpha\omega_{uv}.$$

Multiplying the last relation with ω^{ku} we get:

$$\omega^{ku}S\left(\omega_{uv}\right) - \overset{c}{N}{}_v^k - \omega^{ku}\omega_{mv}\overset{c}{N}{}_u^m - \alpha\delta_v^k = F_v^k + \omega^{ku}\omega_{mv}F_u^m = 2\overset{*}{O}{}_{av}^{kb}\left(F_b^a\right). \tag{2.12}$$

The condition (2.7) is satisfied:

$$O(\omega)_{av}^{kb}\left(\omega^{am}S\left(\omega_{mb}\right) - \overset{c}{N}{}_b^a - \omega^{am}\omega_{lb}\overset{c}{N}{}_m^l - \alpha\delta_b^a\right) =$$

$$= \omega^{km}S\left(\omega_{mv}\right) - \overset{c}{N}{}_v^k - \omega^{km}\omega_{vl}\overset{c}{N}{}_m^l - \omega^{km}S\left(\omega_{mv}\right) + \omega^{km}\omega_{vl}\overset{c}{N}{}_m^l + \overset{c}{N}{}_v^k = 0.$$

It follows:

$$F_j^i = \frac{1}{2}\omega^{im}S\left(\omega_{mj}\right) - \frac{1}{2}\overset{c}{N}{}_j^i - \frac{1}{2}\omega^{ia}\omega_{jb}\overset{c}{N}{}_a^b - \frac{\alpha}{2}\delta_j^i + O_{aj}^{ib}\left(X_b^a\right)$$

and returning to (2.10) we have the conclusion. □

In the spray case the equation (2.9) admits a simple form:

Proposition 2.9 *Fix a spray S and an almost symplectic structure ω.*

i) *The family $\mathcal{N}(S,\omega,\alpha)$ is:*

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}\omega^{ia}\omega_{bj}\overset{c}{N}{}^b_a + \frac{1}{2}\omega^{ia}y^m\frac{\delta\omega_{aj}}{\delta x^m} - \frac{\alpha}{2}\delta^i_j + O(\omega)^{ia}_{bj}\left(X^b_a\right). \tag{2.13}$$

ii) *The family $\mathcal{N}_F(S,\omega)$ is:*

$$N^i_j = \frac{1}{2}\overset{c}{N}{}^i_j - \frac{1}{2}\omega^{ia}\omega_{bj}\overset{c}{N}{}^b_a + \frac{1}{2}\omega^{ia}y^m\frac{\delta\omega_{aj}}{\delta x^m} + O(\omega)^{ia}_{bj}\left(X^b_a\right). \tag{2.13F}$$

2.2 Recurrence of a pair (nonlinear connection, almost symplectic form)

Fix a nonlinear connection $N = \left(N^i_j\right)$ and associate to N the semispray $S(N)$.

Definition 2.10 The pair (N,ω) is *α-recurrent* (or *almost Fedosov*) if the triple $(S(N), N, \omega)$ is so.

Since the canonical nonlinear connection for $S(N)$ is:

$$\overset{c}{N}{}^i_j = \frac{1}{2}\left(N^i_j + \frac{\partial N^i_u}{\partial y^j}y^u\right)$$

it follows:

Theorem 2.11 i) *The pair (N,ω) is α-recurrent if and only if:*

$$N^i_j + \omega^{ia}\omega_{bj}N^b_a + \omega^{im}\frac{\partial\omega_{mj}}{\partial y^b}N^b_a y^a = \omega^{im}\frac{\partial\omega_{mj}}{\partial x^a}y^a - \alpha\delta^i_j \tag{2.14}$$

for all $i,j \in \{1,\ldots,n\}$.

ii) *The pair (N,ω) is almost Fedosov if and only if:*

$$N^i_j + \omega^{ia}\omega_{bj}N^b_a + \omega^{im}\frac{\partial\omega_{mj}}{\partial y^b}N^b_a y^a = \omega^{im}\frac{\partial\omega_{mj}}{\partial x^a}y^a \tag{2.14F}$$

for all $i,j \in \{1,\ldots,n\}$.

Proof From (2.9) it results that (N,ω) is α-recurrent if and only if:

$$\overset{*}{O}(\omega)^{iv}_{uj}\left(N^u_v + \omega^{ua}\omega_{vb}N^b_a + \omega^{um}\frac{\partial\omega_{mv}}{\partial y^b}N^b_a y^a + \alpha\delta^u_v\right) = \overset{*}{O}(\omega)^{iv}_{uj}\left(\omega^{um}\frac{\partial\omega_{mv}}{\partial x^a}y^a\right)$$

and a straightforward computation yields the conclusion. □

Example 2.12 (Basic almost symplectic structures) Let us consider $\omega_M = \omega_M(x) = \omega_{ij}(x)dx^i \wedge dx^j$ an almost symplectic structure on M. Then we associate an almost symplectic structure in our framework as in Definition 2.4 and then the relation

(2.14) becomes:

$$N^i_j + \omega^{ia}\omega_{jb}N^b_a = \omega^{im}\frac{\partial \omega_{mj}}{\partial x^a}y^a - \alpha\delta^i_j. \tag{2.15}$$

Recall that a symmetric linear connection on M with coefficients $\left(\Gamma^i_{jk}(x)\right)$ yields the nonlinear connection with the coefficients:

$$N^i_j = \Gamma^i_{ja}y^a. \tag{2.16}$$

Then the associated semispray $S(N)$ is a spray:

$$G^i = \frac{1}{2}\Gamma^i_{jk}y^j y^k. \tag{2.17}$$

Inserting (2.16) in (2.15) we get:

$$\left(\omega^{im}\frac{\partial \omega_{mj}}{\partial x^a} - \Gamma^i_{ja} - \omega^{iu}\omega_{jv}\Gamma^v_{ua}\right)y^a = \alpha\delta^i_j. \tag{2.18}$$

But multiplying the last equation with ω_{ik} we arrive at:

$$\left(\frac{\partial \omega_{jk}}{\partial x^a} - \omega_{ki}\Gamma^i_{ja} - \omega_{ji}\Gamma^i_{ka}\right)y^a = \alpha\omega_{jk} \tag{2.19}$$

which is the usual Christoffel process for the almost symplectic case replaced in the recurrent framework of TM. So, we verified the condition (2.14) in the basic almost symplectic setting.

Let us point out the rôle of homogeneity of the spray (2.17). We remark from (2.19) that α must be a 1-homogeneous on y, i.e. $\alpha(x, y) = \alpha_a(x)y^a$, and then we have:

$$\frac{\partial \omega_{jk}}{\partial x^a} - \omega_{ki}\Gamma^i_{ja} - \omega_{ji}\Gamma^i_{ka} = \alpha_a\omega_{jk} \tag{2.20}$$

for all $a, j, k \in \{1, ..., n\}$. By considering the 1-form $\alpha_\omega = \alpha_a(x)dx^a$ we recover the starting formula (0) from Introduction for $T = \omega_M$.

A very important remark is that α_ω must satisfy a necessary condition in order to be a recurrence form for an almost symplectic structure. Namely, cycling (2.20) and then summing up, we get:

$$\frac{\partial \omega_{jk}}{\partial x^a} + \frac{\partial \omega_{ka}}{\partial x^j} + \frac{\partial \omega_{aj}}{\partial x^k} = \alpha_a\omega_{jk} + \alpha_j\omega_{ka} + \alpha_k\omega_{aj}. \tag{2.20C}$$

If ω_M is a symplectic structure, namely $d\omega_M = 0$, then the left-hand side of above relation vanishes and then α must be zero which yield a Fedosov structure.

2.3 Recurrence of a pair (semispray, almost symplectic form)

Let us fix a semispray $S = \left(G^i\right)$ and the almost symplectic form ω.
Definition 2.13 The pair (S,ω) is called *α-recurrent* (or *almost Fedosov*) if the triple $\left(S, \overset{c}{N}, \omega\right)$ is so.

Inserting $\overset{c}{N}$ in the left-hand-side of (2.9) we get:
Theorem 2.14 i) *The pair (S,ω) is α-recurrent if and only if*:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia}\omega_{jb}\frac{\partial G^b}{\partial y^a} - \omega^{ia}S(\omega_{aj}) = -\alpha\delta^i_j \tag{2.21}$$

for all $i,j \in \{1,\dots,n\}$.
ii) *The pair (S,ω) is almost Fedosov if and only if*:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia}\omega_{jb}\frac{\partial G^b}{\partial y^a} - \omega^{ia}S(\omega_{aj}) = 0 \tag{2.21F}$$

for all $i,j \in \{1,\dots,n\}$.
iii) *The spray S makes α-recurrent the almost symplectic form ω if and only if*:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia}\omega_{jb}\frac{\partial G^b}{\partial y^a} - \omega^{ia}y^m\frac{\delta\omega_{aj}}{\delta x^m} = -\alpha\delta^i_j \tag{2.22}$$

for all $i,j \in \{1,\dots,n\}$.
iv) *The spray S makes almost Fedosov the almost symplectic form ω if and only if*:

$$\frac{\partial G^i}{\partial y^j} + \omega^{ia}\omega_{jb}\frac{\partial G^b}{\partial y^a} - \omega^{ia}y^m\frac{\delta\omega_{aj}}{\delta x^m} = 0 \tag{2.22F}$$

for all $i,j \in \{1,\dots,n\}$.
Proof. The left-hand-side of (2.9) becomes:

$$\overset{*}{O}(\omega)^{iv}_{uj}\left(\overset{c}{N}{}^u_v + \omega^{ua}\omega_{vb}\overset{c}{N}{}^b_a - \omega^{um}S(\omega_{mv})\right) = -\alpha\delta^i_j \tag{2.23}$$

and the computations give (2.21). □

3 Almost metriplectic structures in path geometry

Suppose that, in addition to the pair (semispray S, nonlinear connection N), we are given on TM a pair (almost symplectic structure ω, metric g) where we use the following:
Definition 3.1 A *metric* g on TM is a d-tensor field of $(0,2)$-type of local Sasaki form: $g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$, which is symmetric and non-degenerate.

It results for the components $g_{ij} = g(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})$ the following properties:
1) (symmetry) $g_{ij} = g_{ji}$,

2) (non-degeneration) $\det(g_{ij}) \neq 0$; then there exists the d-tensor field of $(2,0)$-type $g^{-1} = (g^{ij})$.
The name is justified from the fact that g is a Riemannian metric on TM for which N and $V(TM)$ are mutually orthogonal distributions.

Let the corresponding Obata operators:

$$O(g)^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k \delta^j_l - g^{ij} g_{kl}\right), \quad \overset{*}{O}(g)^{ij}_{kl} = \frac{1}{2}\left(\delta^i_k \delta^j_l + g^{ij} g_{kl}\right). \tag{3.1}$$

Definition 3.2 The data (S, N, ω, g) is an *almost metriplectic structure* if:

$$\overset{SN}{\nabla} \omega = \overset{SN}{\nabla} g = 0 \tag{3.2}$$

where $\overset{SN}{\nabla} g$ is exactly as in formula (2.2) with ω replaced by g.

Let us fix (S, ω, g). The set $\mathcal{N}(S, g)$ of nonlinear connections making parallel the metric g is given with a formula similar to $(2.9F)$, [1, p. 339]:

$$N^i_j = \frac{1}{2} \overset{c}{N}{}^i_j - \frac{1}{2} g^{ia} g_{bj} \overset{c}{N}{}^b_a + \frac{1}{2} g^{ia} S(g_{aj}) + O(g)^{ia}_{bj}\left(Y^b_a\right) \tag{3.3}$$

through a similar proof; see also [4] or [5]. We derive then:
Proposition 3.3 *Let S, ω and g be given. There exists a nonlinear connection N such that (S, N, ω, g) is an almost metriplectic structure on TM if and only if there are two horizontal d-tensor fields of $(1,1)$-type, X and Y, such that:*

$$\omega^{ia} S(\omega_{aj}) - \omega^{ia}\omega_{jb} \overset{c}{N}{}^b_a + O(\omega)^{ia}_{bj}\left(2X^b_a\right) = g^{ia} S(g_{aj}) - g^{ia} g_{jb} \overset{c}{N}{}^b_a + O(g)^{ia}_{bj}\left(2Y^b_a\right). \tag{3.4}$$

Then N is given by (2.9) *or* (3.3).
Example 3.4 (Almost Hermitian structures) Let (g_M, J) be an almost Hermitian structure on M i.e. J is an almost complex structure, $J^2 = -1_{TM}$, compatible with $g_M = (g_{ij}(x))$, [3, p. 90]:

$$g(J\cdot, J\cdot) = g(\cdot, \cdot). \tag{3.5}$$

Then:

$$\omega_M(\cdot, \cdot) = g_M(J\cdot, \cdot) \tag{3.6}$$

is an almost symplectic structure on M. Let (J^i_j) be the components of J, this means:

$$J^i_a J^a_j = -\delta^i_j \tag{3.7}$$

and then $\omega_M = (\omega_{ij}(x))$ with:

$$\omega_{ij} = g\left(J\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = g\left(J^a_i \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^j}\right) = g_{aj} J^a_i \tag{3.8}$$

with inverse:

$$\omega^{ij} = -g^{ia} J^j_a. \tag{3.9}$$

The condition (3.4) becomes:

$$g^{ir}g_{sb}J^a_rJ^s_j \overset{c}{N}{}^b_a - g^{ib}g_{cj}J^a_bS(J^c_a) + O(\omega)^{ia}_{bj}\left(2X^b_a\right) = 2g^{ia}S\left(g_{aj}\right) - g^{ia}g_{jb}\overset{c}{N}{}^b_a + O(g)^{ia}_{bj}\left(2Y^b_a\right) \tag{3.10}$$

which can be written:

$$g^{ir}g_{sb}(\delta^a_r\delta^s_j + J^a_rJ^s_j)\overset{c}{N}{}^b_a + O(\omega)^{ia}_{bj}\left(2X^b_a\right) = 2g^{ia}S\left(g_{aj}\right) + g^{ib}g_{cj}J^a_bS(J^c_a) + O(g)^{ia}_{bj}\left(2Y^b_a\right). \tag{3.11}$$

Another form of this relation is:

$$g^{ir}g_{sb}\overset{*}{O}(J)^{as}_{rj}\left(\overset{c}{N}{}^b_a\right) + O(\omega)^{ia}_{bj}\left(2X^b_a\right) = 2g^{ia}S\left(g_{aj}\right) + g^{ib}g_{cj}J^a_bS(J^c_a) + O(g)^{ia}_{bj}\left(2Y^b_a\right). \tag{3.12}$$

References

[1] I. Bucataru, Metric nonlinear connections, Differential Geom. Appl. 25 (2007) 335-343. MR2330462 (2008e:53141)

[2] I. Bucataru, O. Constantinescu, M. F. Dahl, A geometric setting for systems of ordinary differential equations, Int. J. Geom. Methods Mod. Phys. 8 (2011) DOI No: 10.1142/S0219887811005701.

[3] A. Cannas da Silva, Lectures on symplectic geometry, Lecture Notes in Mathematics, 1764, Springer-Verlag, Berlin, 2001. MR1853077 (2002i:53105)

[4] M. Crasmareanu, Metrizable systems of autonomous second order differential equations, Carpathian J. Math. 25 (2009) 163-176. MR2731192

[5] M. Crasmareanu, Recurrent metrics in the geometry of second order differential equations, Bulletin of the Iranian Mathematical Society, in press.

[6] I. Gelfand, V. Retakh, M. Shubin, Fedosov manifolds, Adv. Math. 136 (1998) 104-140, MR1623673 (99d:53023).

[7] P. Guha, Metriplectic structure, Leibniz dynamics and dissipative systems, J. Math. Anal. Appl. 326 (2007) 121-136. MR2277771 (2007k:37117)

[8] Ph. J. Morrison, A paradigm for joined Hamiltonian and dissipative systems, Phys. D 18 (1986) 410-419. MR0838354 (87i:58065)

[9] Y.-c. Wong, Recurrent tensors on a linearly connected differentiable manifold, Trans. Amer. Math. Soc. 99 (1961) 325-341. MR0121751 (22 #12485)

[10] Y.-c. Wong, Existence of linear connections with respect to which given tensor fields are parallel or recurrent, Nagoya Math. J. 24 (1964) 67-108. MR 0174015 (30 #4222)

Faculty of Mathematics, University "Al. I. Cuza", Iaşi, 700506, România
E-mail: mcrasm@uaic.ro

http://www.math.uaic.ro/∼mcrasm