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On an integral-type operator from the Bloch space into the $Q_K(p, q)$ space

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Abstract. Let n be a positive integer, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The boundedness and compactness of the integral operator $(C^n_{\varphi,g} f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi$ from the Bloch and little Bloch space into the spaces $Q_K(p, q)$ and $Q_{K,0}(p, q)$ are characterized.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. Let dA denote the normalized Lebesgue area measure in $\mathbb{D}$ and $g(z, a)$ the Green function with logarithmic singularity at a, i.e. $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathcal{B}$, if

$$\|f\|_b = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

Under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_b$, $\mathcal{B}$ is a Banach space. Let $\mathcal{B}_0$ denote the space of all $f \in \mathcal{B}$ satisfying

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0.$$

This space is called the little Bloch space. Throughout this paper, the closed unit ball in $\mathcal{B}$ and $\mathcal{B}_0$ will be denoted by $\mathbb{B}_{\mathcal{B}}$ and $\mathbb{B}_{\mathcal{B}_0}$ respectively.

Let $p > 0$, $q > -2$, $K : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function. The space $Q_K(p, q)$ consists of those $f \in H(\mathbb{D})$ such that (see [11, 26])

$$\|f\|^p = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z, a))dA(z) < \infty. \tag{1}$$

When $p \geq 1$, $Q_K(p, q)$ is a Banach space with the norm defined by $\|f\|_{Q_K(p,q)} = |f(0)| + \|f\|$. We say that an $f \in H(\mathbb{D})$ belong to the space $Q_{K,0}(p, q)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q K(g(z, a))dA(z) = 0. \tag{2}$$

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When $p = 2, q = 0$, the space $Q_K(p, q)$ equals to Q_K, which was studied, for example, in [3, 4, 10, 23, 25, 27–29]. If $Q_K(p, q)$ consists of just constant functions, we say that it is trivial. $Q_K(p, q)$ is non-trivial if and only if (see [26])

$$\int_0^1 (1 - r^2)^q K(-\log r) r dr < \infty. \tag{3}$$

Throughout this paper, we assume that (3) is satisfied.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$C_\varphi(f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The composition operator has been studied by many researchers on various spaces (see, e.g., [1] and the references therein).

Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. In [6], the author of this paper and Stević defined the generalized composition operator as follows:

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi)) g(\xi) d\xi, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

The boundedness and compactness of the generalized composition operator on Zygmund spaces and Bloch spaces were investigated in [6]. Some related results can be found, for example, in [5, 7, 8, 13, 16, 17, 19, 30, 31]. For related operators in n-dimensional case, see [9, 15, 18, 20–22].

Let n be a nonnegative integer, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Here we study the following integral-type operator

$$(C_{\varphi,g}^n f)(z) = \int_0^z f^{(n)}(\varphi(\xi)) g(\xi) d\xi, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

When $n = 1$, $C_{\varphi,g}^1$ is the generalized composition operator C_φ^g. The purpose of this paper is to study the operator $C_{\varphi,g}^n$. The boundedness and compactness of the operator $C_{\varphi,g}^n$ from the Bloch space $\mathcal{B}$ into $Q_K(p, q)$ and $Q_{K,0}(p, q)$ are completely characterized.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Main result and proof

In order to formulate our main results, we need some auxiliary results which are incorporated in the following lemmas. The following lemma, can be proved in a standard way (see, e.g., Theorem 3.11 in [1]).

Lemma 1. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0, \infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then $C_{\varphi,g}^n : \mathcal{B} \to Q_K(p, q)$ is compact if and only if $C_{\varphi,g}^n : \mathcal{B} \to Q_K(p, q)$ is bounded and for every bounded sequence $\{f_k\}$ in $\mathcal{B}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, $\lim_{k\to\infty} \|C_{\varphi,g}^n f_k\|_{Q_K(p,q)} = 0$.*

Lemma 2 *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0, \infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. If $C_{\varphi,g}^n : \mathcal{B}(\mathcal{B}_0) \to Q_K(p, q)$ is compact, then for any $\varepsilon > 0$ there exists a δ, $0 < \delta < 1$, such that for all f in $\mathbb{B}_{\mathcal{B}}(\mathbb{B}_{\mathcal{B}_0})$,*

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p |g(z)|^p (1 - |z|^2)^q K(g(z, a)) dA(z) < \varepsilon \tag{4}$$

holds whenever $\delta < r < 1$.

Proof. We adopt the methods of [24]. We only give the proof for $\mathcal{B}_0$ and the proof for $\mathcal{B}$ is similar. For $f \in \mathbb{B}_{\mathcal{B}_0}$ let $f_s(z) = f(sz)$, $0 < s < 1$. Then $f_s \in \mathbb{B}_{\mathcal{B}_0}$ and $f_s \to f$ uniformly on compact subsets of $\mathbb{D}$ as $s \to 1$.

Since $C^n_{\varphi,g}$ is compact, $\|C^n_{\varphi,g}f_s - C^n_{\varphi,g}f\|_{Q_K(p,q)} \to 0$ as $s \to 1$. That is, for given $\varepsilon > 0$ there exists an $s \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f_s^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))\right|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon. \tag{5}$$

For $r, 0 < r < 1$, using the triangle inequality and (5), we get

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
\le\ & 2^p\varepsilon + 2^p\|f_s^{(n)}\|_\infty^p \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z).
\end{aligned}$$

Now we prove that for given $\varepsilon > 0$ and $\|f_s^{(n)}\|_\infty^p > 0$ there exists a $\delta \in (0,1)$ such that

$$\|f_s^{(n)}\|_\infty^p \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon$$

whenever $\delta < r < 1$.

Set $f_k(z) = z^k \in \mathcal{B}_0$. Since $C^n_{\varphi,g}$ is compact, we get $\lim_{k\to\infty}\|C^n_{\varphi,g}z^k\| \to 0$. Thus, for given $\varepsilon > 0$ and $\|f_s\|_\infty^p > 0$ there exists an $N \in \mathbb{N}$ such that

$$\|f_s\|_\infty^p \cdot \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big(k\cdots(k-n+1)\Big)^p |\varphi^{k-n}(z)|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon$$

whenever $k \ge N > n$. Hence, for $0 < r < 1$,

$$\begin{aligned}
&\Big(N\cdots(N-n+1)\Big)^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi^{N-n}(z)|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
\ge\ & \Big(N\cdots(N-n+1)\Big)^p \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |\varphi^{N-n}(z)|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
\ge\ & \Big(N\cdots(N-n+1)\Big)^p r^{p(N-n)} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z).
\end{aligned} \tag{6}$$

Therefore, for $r \ge [N\cdots(N-n+1)]^{-\frac{1}{N-n}}$, we have

$$\|f_s\|_\infty^p \cdot \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$

Thus we have proved that for any $\varepsilon > 0$ and for each $f \in \mathbb{B}_{\mathcal{B}_0}$ there exists a $\delta = \delta(\varepsilon, f)$ such that

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^p |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon$$

holds whenever $\delta < r < 1$.

The rest of the proof can be completed by using the finite covering property of the set $C^n_{\varphi,g}(\mathbb{B}_{\mathcal{B}_0})$ which is relatively compact in $Q_K(p,q)$ (see, e.g., [24]), and hence we omit it. The proof of this theorem is completed. $\square$

By modifying the proof of Theorem 3.5 of [10], we can prove the following lemma. We omit the details.

Lemma 3. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then $C^n_{\varphi,g} : \mathcal{B} \to Q_{K,0}(p,q)$ is compact if and only if $C^n_{\varphi,g} : \mathcal{B} \to Q_{K,0}(p,q)$ is bounded and*

$$\lim_{|a|\to 1} \sup_{\|f\|_{\mathcal{B}}\le 1} \int_{\mathbb{D}} |(C^n_{\varphi,g} f)'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0. \tag{7}$$

Let $L : X \to Y$ be a linear operator, where X and Y are Banach spaces. Then L is said to be weakly compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, $(L(x_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence, i.e., there is a subsequence $(x_{n_m})_{m\in\mathbb{N}}$ such that for every $\Lambda \in Y^*$, $\Lambda(L(x_{n_m}))_{m\in\mathbb{N}}$ converges (see [2]). Let A^1 denote the space of all $f \in H(\mathbb{D})$ such that $\int_{\mathbb{D}} |f(z)|dA(z) < \infty$. From [32], we know that $(\mathcal{B}_0)^* = A^1$ and $(A^1)^* = \mathcal{B}$. We also know that $A^1 \cong l^1$. Since l^1 has the Schur property, we get the following proposition.

Proposition 1. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)(Q_{K,0}(p,q))$ is weakly compact if and only if $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)(Q_{K,0}(p,q))$ is compact.*

Proposition 2. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_{K,0}(p,q)$ is compact if and only if $C^n_{\varphi,g} : \mathcal{B} \to Q_{K,0}(p,q)$ is bounded.*

Proof. From Gantmacher's theorem (see [2]), we know that an operator $L : X \to Y$ is weakly compact if and only if $L^{**}(X^{**}) \subset Y$, where L^{**} and X^{**} is the second adjoint of L and X respectively. From Proposition 1, we see that $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_{K,0}(p,q)$ is compact if and only if $C^n_{\varphi,g}((\mathcal{B}_0)^{**}) \subset Q_{K,0}(p,q)$. Since $(\mathcal{B}_0)^{**} \cong \mathcal{B}$, the result follows. □

Theorem 1. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then the following statements are equivalent.*

(i) $C^n_{\varphi,g} : \mathcal{B} \to Q_K(p,q)$ is bounded;

(ii) $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)$ is bounded;

(iii)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a))dA(z) < \infty. \tag{8}$$

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Let $f \in \mathcal{B}$. Set $f_s(z) = f(sz)$ for $0 < s < 1$, then we get $f_s \in \mathcal{B}_0$ and $\|f_s\|_b \le \|f\|_b$. Thus, by the assumption for all $f \in \mathcal{B}$ we have

$$\|C^n_{\varphi,g} f_s\|_{Q_K(p,q)} \le \|C^n_{\varphi,g}\|\|f_s\|_b \le \|C^n_{\varphi,g}\|\|f\|_b. \tag{9}$$

By [14], there exist two Bloch functions f_1 and f_2 satisfying

$$\frac{1}{1-|z|^2} \le |f_1'(z)| + |f_2'(z)|, \quad z \in \mathbb{D}.$$

We choose $g_1(z) = f_1(z) - zf_1'(0)$, $g_2(z) = f_2(z) - zf_2'(0)$. By the well-known result (see [33])

$$(1-|z|^2)^2|f''(z)| + |f'(0)| \asymp (1-|z|^2)|f'(z)|,$$

we see that $g_1, g_2 \in \mathcal{B}$ and

$$\frac{1}{(1-|z|^2)^2} \le |g_1''(z)| + |g_2''(z)|, \quad z \in \mathbb{D}.$$

Following this rule, we see that there exist $h_1, h_2 \in \mathcal{B}$ and

$$\frac{1}{(1-|z|^2)^n} \leq |h_1^{(n)}(z)| + |h_2^{(n)}(z)|, \ \ z \in \mathbb{D}.$$

Replacing f in (9) by h_1 and h_2 respectively and using the following elementary inequality

$$\left(a_1 + a_2\right)^p \leq \begin{cases} a_1^p + a_2^p & , \quad p \in (0,1] \\ 2^{p-1}(a_1^p + a_2^p) & , \quad p \geq 1 \end{cases}, \ \ a_i \geq 0, \ \ i = 1, 2,$$

we obtain that

$$\begin{aligned} & \int_{\mathbb{D}} \frac{|s^n g(z)|^p}{(1-|s\varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a)) dA(z) \\ \leq \ & C \int_{\mathbb{D}} \left(|h_1^{(n)}(s\varphi(z))|^p + |h_2^{(n)}(s\varphi(z))|^p \right) |s^n g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\ = \ & C \int_{\mathbb{D}} \left(|(C^n_{\varphi,g} h_{1s})'(z)|^p + |(C^n_{\varphi,g} h_{2s})'(z)|^p \right) (1-|z|^2)^q K(g(z,a)) dA(z) \\ = \ & C\|C^n_{\varphi,g} h_{1s}\|^p_{Q_K(p,q)} + C\|C^n_{\varphi,g} h_{2s}\|^p_{Q_K(p,q)} \\ \leq \ & C\|C^n_{\varphi,g}\|^p (\|h_1\|^p_{\mathcal{B}} + \|h_2\|^p_{\mathcal{B}}) < \infty \end{aligned} \tag{10}$$

hold for all $a \in \mathbb{D}$ and $s \in (0,1)$. This estimate and Fatou's Lemma give (8).

$(iii) \Rightarrow (i)$. By the following well-known result (see [33])

$$|f^{(n)}(z)| \leq \frac{C\|f\|_{\mathcal{B}}}{(1-|z|^2)^n}, \ \ f \in \mathcal{B}, \tag{11}$$

we see that (iii) implies (i). This completes the proof of Theorem 1. □

Theorem 2. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then the following statements are equivalent:*

(i) $C^n_{\varphi,g} : \mathcal{B} \to Q_K(p,q)$ is compact;

(ii) $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)$ is compact;

(iii) $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)$ is weakly compact;

(iv)

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) < \infty \tag{12}$$

and

$$\lim_{r \to 1} \sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a)) dA(z) = 0. \tag{13}$$

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Leftrightarrow (iii)$. It follows from Proposition 1.

$(ii) \Rightarrow (iv)$. Assume that $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)$ is compact. By taking $f = \frac{1}{n!} z^n \in \mathcal{B}_0$ we get (12). Now we choose the function $f(z) = \frac{1}{4} \sum_{k=m}^{\infty} z^{2^k}$, where $m = [\frac{lnn}{ln2}] + 1$. Then by [24], we see that $f \in \mathbb{B}_{\mathcal{B}}$. Choose a sequence $\{\lambda_j\}$ in $\mathbb{D}$ which converges to 1 as $j \to \infty$, and let $f_j(z) = f(\lambda_j z)$ for $j \in \mathbb{N}$. Then, $f_j \in \mathbb{B}_{\mathcal{B}_0}$ for all $j \in \mathbb{N}$ and $\|f_j\|_{\mathcal{B}} \leq C$. Let $f_{j,\theta}(z) = f_j(e^{i\theta} z)$. Then $f_{j,\theta} \in \mathbb{B}_{\mathcal{B}_0}$. Replace f by $f_{j,\theta}$ in (2) and then integrate both sides with respect to θ. By Fubini's Theorem, Parseval's identity and the inequality $2^k \cdots (2^k - n + 1) \geq (2^k - n)^n$, we obtain

$$\begin{aligned}
\varepsilon &\geq \frac{1}{2\pi} \int\limits_{|\varphi(z)|>r} \left(\int_0^{2\pi} |f_j^{(n)}(e^{i\theta}\varphi(z))|^p d\theta \right) |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&= \frac{1}{4^p 2\pi} \int\limits_{|\varphi(z)|>r} \int_0^{2\pi} \left| \sum_{k=[\log_2 n]+1}^{\infty} 2^k \cdots (2^k - n + 1)(\lambda_j \varphi(z))^{2^k - n} e^{i\theta(2^k - n)} \right|^p d\theta |\lambda_j|^{np} |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&= \frac{1}{4^p} \int\limits_{|\varphi(z)|>r} \left(\sum_{k=[\log_2 n]+1}^{\infty} [2^k \cdots (2^k - n + 1)]^p |\lambda_j \varphi(z)|^{p(2^k - n)} \right) |\lambda_j|^{np} |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&\geq \frac{1}{4^p} \int\limits_{|\varphi(z)|>r} \left(\sum_{k=[\log_2 n]+1}^{\infty} (2^k - n)^{np} |\lambda_j \varphi(z)|^{p(2^k - n)} \right) |\lambda_j|^{np} |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z). \qquad (14)
\end{aligned}$$

Let

$$G(r) = \sum_{k=[\log_2 n]+1}^{\infty} (2^k - n)^{np} r^{p(2^k - n)}.$$

Since $\log r \geq 2(r-1)$ in the interval $[\frac{1}{2}, 1)$, we get

$$r^{p(2^k - n)} \geq \exp\{2p(2^k - n)(r-1)\}, \qquad r \in [\frac{1}{2}, 1).$$

Hence

$$\begin{aligned}
G(r) &\geq \sum_{k=[\log_2 n]+1}^{\infty} (2^k - n)^{np} \exp\{2p(2^k - n)(r-1)\} \\
&= (1-r)^{-np} \sum_{k=[\log_2 n]+1}^{\infty} [(2^k - n)(1-r)]^{np} \exp\{-2p(2^k - n)(1-r)\}. \qquad (15)
\end{aligned}$$

After some calculations, we see that there exists a positive constant c_0 such that

$$G(r) \geq c_0 (1-r)^{-np}, \qquad r \in [\frac{3}{4}, 1).$$

Therefore, for $\delta < r < 1$ and for sufficiently large j, (14) gives

$$\sup_{a\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} \frac{|\lambda_j|^{np} |g(z)|^p}{(1-|\lambda_j \varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a)) dA(z) < C\varepsilon.$$

By Fatou's Lemma we get (13).

$(iv) \Rightarrow (i)$. Assume that (12) and (13) hold. Let $\{f_j\}$ be a sequence in $\mathbb{B}_{\mathcal{B}}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. We need to show that $\{C^n_{\varphi,g} f_j\}$ converges to 0 in $Q_K(p,q)$ norm. By (13) for given $\varepsilon > 0$ there is an r, such that

$$\sup_{a\in\mathbb{D}} \int\limits_{|\varphi(z)|>r} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a)) dA(z) < \varepsilon$$

when $0 < r < 1$. Therefore, by (10) we have

$$\begin{aligned} &\int_{\mathbb{D}} |(C^n_{\varphi,g} f_j)'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ = &\Big\{ \int_{|\varphi(z)|\le r} + \int_{|\varphi(z)|>r} \Big\} |f_j^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ \le &\ \|f_j\|^p_{\mathcal{B}} \varepsilon + \sup_{|w|\le r} |f_j^{(n)}(w)|^p \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z). \end{aligned} \tag{16}$$

¿From the assumption, we see that $\{f_j^{(n)}\}$ also converges to 0 uniformly on compact subsets of $\mathbb{D}$ by Cauchy's estimates. It follows that $\|C^n_{\varphi,g} f_j\|_{Q_K(p,q)} \to 0$ since $\sup_{|w|\le r} |f_j^{(n)}(w)|^p \to 0$ as $j \to \infty$. Thus

$$\|C^n_{\varphi,g} f_j\|^p_{Q_K(p,q)} = \|C^n_{\varphi,g} f_j\|^p \to 0, \text{ as } j \to \infty.$$

By Lemma 1, $C^n_{\varphi,g} : \mathcal{B} \to Q_K(p,q)$ is compact. □

Theorem 3. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then the following statements are equivalent:*

(i) $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_{K,0}(p,q)$ is bounded;

(ii)

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0 \tag{17}$$

and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a))dA(z) < \infty. \tag{18}$$

Proof. $(i) \Rightarrow (ii)$. Assume that $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_{K,0}(p,q)$ is bounded. Then it is obvious that $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)$ is bounded. By Theorem 1, (18) holds. Taking $f(z) = \frac{1}{n!} z^n$ and using the boundness of $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_{K,0}(p,q)$, we get (17).

$(ii) \Rightarrow (i)$. Suppose that (17) and (18) hold. From Theorem 1, we see that $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_K(p,q)$ is bounded. To prove that $C^n_{\varphi,g} : \mathcal{B}_0 \to Q_{K,0}(p,q)$ is bounded, it suffices to prove that $C^n_{\varphi,g} f \in Q_{K,0}(p,q)$ for any $f \in \mathcal{B}_0$. Let $f \in \mathcal{B}_0$. For every $\varepsilon > 0$, we can choose $\rho \in (0,1)$ such that $|f^{(n)}(w)|(1-|w|^2)^n < \varepsilon$ for all $w \in \mathbb{D} \setminus \rho\overline{\mathbb{D}}$. Then by (11) we have

$$\begin{aligned} &\int_{\mathbb{D}} |(C^n_{\varphi,g} f)'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ = &\Big(\int_{|\varphi(z)|>\rho} + \int_{|\varphi(z)|\le\rho} \Big) |f^{(n)}(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\ \le &\ \varepsilon \int_{|\varphi(z)|>\rho} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}} (1-|z|^2)^q K(g(z,a))dA(z) + \frac{\|f\|^p_{\mathcal{B}}}{(1-\rho^2)^{np}} \int_{|\varphi(z)|\le\rho} |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z), \end{aligned}$$

which together with the assumed conditions imply the desired result. □

Theorem 4. *Let $p > 0$, $q > -2$ and K be a nonnegative nondecreasing function on $[0,\infty)$. Assume that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$ and n is a positive integer. Then the following statements are equivalent:*

(i) $C^n_{\varphi,g}:\mathcal{B}\to Q_{K,0}(p,q)$ *is bounded;*
(ii) $C^n_{\varphi,g}:\mathcal{B}\to Q_{K,0}(p,q)$ *is compact;*
(iii) $C^n_{\varphi,g}:\mathcal{B}_0\to Q_{K,0}(p,q)$ *is compact;*
(iv) $C^n_{\varphi,g}:\mathcal{B}_0\to Q_{K,0}(p,q)$ *is weakly compact;*
(v)

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}}(1-|z|^2)^qK(g(z,a))dA(z)=0. \tag{19}$$

Proof. By Proposition 2 we see that (*i*) ⇔ (*iii*). By Proposition 1 we see that (*iii*) ⇔ (*iv*). (*ii*) ⇒ (*i*) is obvious. Now we prove that (*i*) ⇒ (*v*) ⇒ (*ii*).

First assume that $C^n_{\varphi,g}:\mathcal{B}\to Q_{K,0}(p,q)$ is bounded. From the proof of Theorem 1, we choose functions $f_1,f_2\in\mathcal{B}$ such that

$$\frac{1}{(1-|z|^2)^n}\le|f_1^{(n)}(z)|+|f_2^{(n)}(z)|,\ \ z\in\mathbb{D}. \tag{20}$$

¿From the assumption we get $C^n_{\varphi,g}f_1, C^n_{\varphi,g}f_2\in Q_{K,0}(p,q)$. Therefore, by (**??**) and (20) we have

$$\begin{aligned}
&\lim_{|a|\to 1}\int_{\mathbb{D}}\frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}}(1-|z|^2)^qK(g(z,a))dA(z)\\
\le\ &\lim_{|a|\to 1}\int_{\mathbb{D}}\Big(|f_1^{(n)}(\varphi(z))|+|f_2^{(n)}(\varphi(z))|\Big)^p|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
\le\ &C\lim_{|a|\to 1}\int_{\mathbb{D}}\Big(|f_1^{(n)}(\varphi(z))|^p+|f_2^{(n)}(\varphi(z))|^p\Big)|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
=\ &C\lim_{|a|\to 1}\int_{\mathbb{D}}(|(C^n_{\varphi,g}f_1)'(z)|^p+|(C^n_{\varphi,g}f_2)'(z)|^p)(1-|z|^2)^qK(g(z,a))dA(z)\\
=\ &0,
\end{aligned}$$

which implies the desired result.

Assume that (19) holds. Let

$$h_{p,q,\varphi,K}(a)=\int_{\mathbb{D}}\frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}}(1-|z|^2)^qK(g(z,a))dA(z).$$

By the assumption, we have that for every $\varepsilon>0$, there is a $t\in(0,1)$ such that for $|a|>t$, $h_{p,q,\varphi,K}(a)<\varepsilon$. Similarly to the proof of Lemma 2.3 of [12], we see that $h_{p,q,\varphi,K}$ is continuous on $|a|\le t$, hence is bounded on $|a|\le t$. Therefore $h_{p,q,\varphi,K}$ is bounded on $\mathbb{D}$. From Theorem 1, $C^n_{\varphi,g}:\mathcal{B}\to Q_K(p,q)$ is bounded. We first prove that $C^n_{\varphi,g}:\mathcal{B}\to Q_{K,0}(p,q)$ is bounded. For any $f\in\mathcal{B}$, by (10) we have

$$\int_{\mathbb{D}}|(C^n_{\varphi,g}f)'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\le\|f\|^p_{\mathcal{B}}\int_{\mathbb{D}}\frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{np}}(1-|z|^2)^qK(g(z,a))dA(z), \tag{21}$$

which together with (19) imply that $C^n_{\varphi,g}:\mathcal{B}\to Q_{K,0}(p,q)$ is bounded. Fix $f\in\mathbb{B}_{\mathcal{B}}$. The righthand side of (21) tends to 0, as $|a|\to 1$ by (19). From Lemma 3, we see that $C^n_{\varphi,g}:\mathcal{B}\to Q_{K,0}(p,q)$ is compact. The proof of the theorem is completed. □

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