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Perturbation bounds for the Moore-Penrose inverse of operators

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Abstract. We consider the perturbation bounds for the Moore-Penrose inverse of a given operator on Hilbert space and apply these results to the relative errors of the minimum norm least squares solution of the equation $Ax = b$.

1. The first section

Perturbation bounds for the Moore-Penrose inverse of matrices or operators have been investigated in many recent papers [1, 3, 15, 17–20, 23, 24]. P. A. Wedin [24] presented some perturbation bounds for the Moore-Penrose inverse of matrices under general unitarily invariant norm, the spectral norm and the Frobenius norm. L. Meng and B. Zheng [16] obtained the optimal perturbation bounds for the Moore-Penrose inverse of matrices under the Frobenius norm using singular value decomposition and these results extended the results from [24]. C. Deng and Y. Wei [6] considered the perturbation bound for the Moore-Penrose inverse of operators on Hilbert spaces while the perturbation bounds of linear operators on Banach spaces have been considered in [18, 26]. In this paper, we consider the perturbation bound for the Moore-Penrose inverse of linear operator on Hilbert space using generalized Neumman lemma.

Let H, K be Hilbert spaces and let $L(H, K)$ be the set of all bound linear operators from H to K. The symbols A^*, $r(A)$, $R(A)$ and $N(A)$ stand for the conjugate transpose, the spectral radius, the range and the null space of $A \in L(H, K)$, respectively.

Let $A \in L(H, K)$ has a closed range. Then there is a unique operator $B \in L(K, H)$ such that

$$ABA = A, \tag{1}$$
$$BAB = B, \tag{2}$$
$$(AB)^* = AB, \tag{3}$$
$$(BA)^* = BA.$$

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B is called *the Moore-Penrose inverse of* A and it is denoted by $A^\dagger$.

If B satisfies the equation (1), i.e. $ABA = A$, then B is *the* $\{1\}$*-inverse of* A, where $A\{1\}$ denotes the set of all $\{1\}$-inverses of A. Similarly, we have the notations $A\{2\}$, $A\{1,2\}$, $A\{1,3\}$ and $A\{1,4\}$, etc.

Let $A \in L(H, K)$ has a closed range. Then,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \tag{4}$$

and

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \to \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix}, \tag{5}$$

where A_1 is invertible.

First, we state a definition which is given for Banach spaces but it can be used also for Hilbert spaces:

Definition 1.1. ([11]) Let X, Y and Z be Banach spaces, $T \in L(X, Y)$, $A \in L(X, Z)$ and $D(T) \subset D(A)$. If for some nonnegative constants a and b and every $u \in D(T)$,

$$\|Au\| \le a\|u\| + b\|Tu\|,$$

then A is said to be *T-bounded*.

The next generalized Neumman Lemma [7] is a main tool in this paper. It is proved in [7] in the case when X, Y are Banach spaces but it is also valid when X, Y are Hilbert spaces.

Lemma 1.2. ([7]) *Let* $P \in B(X)$ *be such that for* $\lambda_1 < 1$, $\lambda_2 < 1$ *and every* $x \in X$,

$$\|Px\| \le \lambda_1\|x\| + \lambda_2\|(I + P)x\|.$$

Then $\lambda_1 \in (-1, 1)$, $\lambda_2 \in (-1, 1)$ *and* $I + P$ *is a bijective mapping. Moreover,*

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \le \|(I + P)x\| \le \frac{1+\lambda_1}{1-\lambda_2}\|x\|, \ \textit{for every } x \in X$$

and

$$\frac{1-\lambda_2}{1+\lambda_1}\|y\| \le \|(I + P)^{-1}y\| \le \frac{1+\lambda_2}{1-\lambda_1}\|y\|, \ \textit{for every } y \in Y.$$

Also, we sate a useful lemma which is proved in the matrix case in [5]. The proof is similar but we will give it for the completeness.

Lemma 1.3. *Let* $A \in L(H, K)$ *be represented by*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and $R(A)$ *is closed. If* A_{11} *is invertible and* $S_{A_{11}}(A)$ *is a Moore-Penrose invertible, then*

$$A^\dagger = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S_{A_{11}}(A)^\dagger A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S_{A_{11}}(A)^\dagger \\ -S_{A_{11}}(A)^\dagger A_{21}A_{11}^{-1} & S_{A_{11}}(A)^\dagger \end{bmatrix} \tag{6}$$

if and only if

$$N(S_{A_{11}}(A)) \subset N(A_{12}),\ R(A_{21}) \subset R(S_{A_{11}}(A)),\ N(S_{A_{11}}(A)) \subset N(A_{22}), \tag{7}$$

where $S_{A_{11}}(A) = A_{22} - A_{21}A_{11}^{-1}A_{12}$ *is a Schur complement of* A_{11} *in* A.

Proof. $R(A)$ is closed, so $A^\dagger$ exists. Suppose that (7) holds. We will prove that $A^\dagger$ is given by (6).

Denoting by T the right side of (6). From $N(S_{A_{11}}(A)) \subset N(A_{12})$, we get that $A_{12}\Big(I - S_{A_{11}}(A)^\dagger S_{A_{11}}(A)\Big) = 0$ which implies that

$$TA = \begin{bmatrix} I & 0 \\ 0 & S_{A_{11}}(A)^\dagger S_{A_{11}}(A) \end{bmatrix},$$

i.e. $T \in A\{4\}$.

Similarly, applying $R(A_{21}) \subset R(S_{A_{11}}(A))$, we get

$$\Big(I - S_{A_{11}}(A)S_{A_{11}}(A)^\dagger\Big)A_{21} = 0 \tag{8}$$

which induce that

$$AT = \begin{bmatrix} I & 0 \\ 0 & S_{A_{11}}(A)S_{A_{11}}(A)^\dagger \end{bmatrix},$$

i.e. $T \in A\{3\}$.

Analogously, we get $ATA = A$ and $TAT = T$, so $T = A^\dagger$.

Conversely, if $T = A^\dagger$, then from $(TA)^* = TA$ we have that (8) holds which is equivalent with $R(A_{21}) \subset R(S_{A_{11}}(A))$. Similarly, we get that the other two conditions hold. □

2. Perturbation bounds for the Moore-Penrose inverse of an operator

In this section, we will consider the perturbation bounds for the Moore-Penrose inverse of a given operator. Let $A \in L(H,K)$ and let $E \in L(H,K)$ be the perturbation operator of A. Suppose that E is given by

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}. \tag{9}$$

Now, from (4), we have that

$$A + E = \begin{bmatrix} A_1 + E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \to \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}.$$

In the following theorem, we investigate the perturbation bound of $\|(A+E)^\dagger - A^\dagger\|$ in the case when the Moore-Penrose of $A+E$ exists.

Theorem 2.1. *Let $A, E \in L(H,K)$ be such that $A, A+E$ have a closed ranges and let A, E be given by* (4) *and* (9)*, respectively. Suppose that for some $\lambda_1 < 1, \lambda_2 < 1$ and every $x \in H$,*

$$\|EA^\dagger x\| \le \lambda_1\|x\| + \lambda_2\|(I + EA^\dagger)x\| \tag{10}$$

and that $S = E_{22} - E_{21}(A_1 + E_{11})^{-1}E_{12}$ is a Moore-Penrose invertible. Then

$$(A+E)^\dagger = \begin{bmatrix} \Delta^{-1} + \Delta^{-1}E_{12}S^\dagger E_{21}\Delta^{-1} & -\Delta^{-1}E_{12}S^\dagger \\ -S^\dagger E_{21}\Delta^{-1} & S^\dagger \end{bmatrix} \tag{11}$$

if and only if $N(S) \subset N(E_{12})$, $R(E_{21}) \subset R(S)$, $N(S) \subset N(E_{22})$, where $\Delta = A_1 + E_{11}$. In this case,

$$\begin{aligned} \|(A+E)^\dagger - A^\dagger\| \;&\le\; \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|A_1^{-1}E_{11}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^\dagger E_{21}\|\right] \\ &\quad + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^\dagger\| + \|E_{21}S^\dagger\|\right] + \|S^\dagger\| \\ \|(A+E)(A+E)^\dagger - AA^\dagger\| \;&\le\; \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[1 + \|A_1^{-1}E_{11}\| + 2\frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^\dagger E_{21}\|\right] \\ &\quad + 2\frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^\dagger\| + 2\|E_{21}S^\dagger\|\right] + 2\|S^\dagger\|. \end{aligned}$$

Proof. Since $R(A)$ is closed we can suppose that A and $A^{\dagger}$ are given by (4) and (5), respectively. Also, suppose that E is given by (9). From the Moore-Penrose invertibility of the Schur complement S and by Lemma 1.3 we obtain that $(A+E)^{\dagger}$ is given by (11) if and only if $N(S) \subset N(E_{12})$, $R(E_{21}) \subset R(S)$, and $N(S) \subset N(E_{22})$.

From (10) and Lemma 1.2, we have that $I + EA^{\dagger}$ is invertible and

$$\|A^{\dagger}(I+EA^{\dagger})^{-1}\| = \left\| \begin{bmatrix} A_1^{-1}(I+E_{11}A_1^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \right\| \le \|A^{\dagger}\|\|(I+EA^{\dagger})^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\|. \tag{12}$$

It implies that $A_1 + E_{11}$ is invertible and $\|(A_1+E_{11})^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|$. Now, we will consider the perturbation bound of $\|(A+E)^{\dagger} - A^{\dagger}\|$.

Note that

$$\Delta^{-1} - A_1^{-1} = A_1^{-1}(I+E_{11}A_1^{-1})^{-1} - A_1^{-1} = -A_1^{-1}E_{11}A_1^{-1}(I+E_{11}A_1^{-1})^{-1} = -A_1^{-1}E_{11}\Delta^{-1}. \tag{13}$$

According to (12) and (13), we prove that

$$\|\Delta^{-1} - A_1^{-1}\| \le \|A_1^{-1}E_{11}\|\|A_1^{-1}(I+E_{11}A_1^{-1})^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}E_{11}\|\|A_1^{-1}\|.$$

Now,

$$\|A_1^{-1}E_{11}\Delta^{-1} + \Delta^{-1}E_{12}S^{\dagger}E_{21}\Delta^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|A_1^{-1}E_{11}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right], \tag{14}$$

$$\|-\Delta^{-1}E_{12}S^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|E_{12}S^{\dagger}\|\|A_1^{-1}\|, \tag{15}$$

and

$$\|-\Delta^{-1}E_{21}S^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|E_{21}S^{\dagger}\|\|A_1^{-1}\|. \tag{16}$$

By (5),(11) and (13), we easily obtain

$$\begin{aligned} (A+E)^{\dagger} - A^{\dagger} &= \begin{bmatrix} \Delta^{-1} - A_1^{-1} + \Delta^{-1}E_{12}S^{\dagger}E_{21}\Delta^{-1} & -\Delta^{-1}E_{12}S^{\dagger} \\ -S^{\dagger}E_{21}\Delta^{-1} & S^{\dagger} \end{bmatrix} \\ &= \begin{bmatrix} -A_1^{-1}E_{11}\Delta^{-1} + \Delta^{-1}E_{12}S^{\dagger}E_{21}\Delta^{-1} & -\Delta^{-1}E_{12}S^{\dagger} \\ -S^{\dagger}E_{21}\Delta^{-1} & S^{\dagger} \end{bmatrix}. \end{aligned} \tag{17}$$

From (11), (14)–(17), we have

$$\begin{aligned} \|(A+E)^{\dagger}\| &= \left\| \begin{bmatrix} \Delta^{-1} + \Delta^{-1}E_{12}S^{\dagger}E_{21}\Delta^{-1} & -\Delta^{-1}E_{12}S^{\dagger} \\ -S^{\dagger}E_{21}\Delta^{-1} & S^{\dagger} \end{bmatrix} \right\| \\ &\le \left\|\Delta^{-1} + \Delta^{-1}E_{12}S^{\dagger}E_{21}\Delta^{-1}\right\| + \left\|\Delta^{-1}E_{12}S^{\dagger}\right\| + \left\|S^{\dagger}E_{21}\Delta^{-1}\right\| + \left\|S^{\dagger}\right\| \\ &\le \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[1 + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right] \\ &\quad + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\| \end{aligned} \tag{18}$$

and

$$\begin{aligned} \|(A+E)^{\dagger} - A^{\dagger}\| &= \left\| \begin{bmatrix} -A_1^{-1}E_{11}\Delta^{-1} + \Delta^{-1}E_{12}S^{\dagger}E_{21}\Delta^{-1} & -\Delta^{-1}E_{12}S^{\dagger} \\ -S^{\dagger}E_{21}\Delta^{-1} & S^{\dagger} \end{bmatrix} \right\| \\ &\le \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|A_1^{-1}E_{11}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right] \\ &\quad + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|. \end{aligned} \tag{19}$$

Finally, we will consider the perturbation bound of projection in the following. Obviously, we have the following result

$$(A+E)(A+E)^{\dagger} - AA^{\dagger} = A(A+E)^{\dagger} + E(A+E)^{\dagger} - AA^{\dagger} = A\left[(A+E)^{\dagger} - A^{\dagger}\right] + E(A+E)^{\dagger} \tag{20}$$

According to (18)–(20), we show that

$$\begin{aligned} \|(A+E)(A+E)^{\dagger} - AA^{\dagger}\| &= \left\|A\left[(A+E)^{\dagger} - A^{\dagger}\right] + E(A+E)^{\dagger}\right\| \\ &\leq \|A\|\|(A+E)^{\dagger} - A^{\dagger}\| + \|E\|\|(A+E)^{\dagger}\| \\ &\leq \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[1 + \|A_1^{-1}E_{11}\| + 2\frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right] \\ &\quad +2\frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + 2\|S^{\dagger}\|. \end{aligned}$$

where $k(A_1) = \|A_1^{-1}\|\|A\| = \|A_1^{-1}\|\|A_1\|$.

Therefore, we have finished the proof. $\square$

In the following theorem, we will give the perturbation bound of $\|(A+E)^{\dagger} - A^{\dagger}\|$ under certain condition. At first, we will give Theorem 2.2 and Theorem 2.3 before investigating the perturbation bound of $\|(A+E)^{\dagger} - A^{\dagger}\|$.

Theorem 2.2. *Let $A \in L(H,K)$ has a closed range and let $R(E) \subseteq R(A)$. If E, $A^{\dagger}$ satisfy* (10)*, then*

$$A^{\dagger}(I+EA^{\dagger})^{-1} = (I+A^{\dagger}E)^{-1}A^{\dagger} \in A\{1,2,3\}$$

and

$$\begin{aligned} &\|A^{\dagger}(I+EA^{\dagger})^{-1}\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\| \\ &\|A^{\dagger}(I+EA^{\dagger})^{-1} - A^{\dagger}\| \leq \frac{1+\lambda_2}{1-\lambda_1}\frac{\lambda_1+\lambda_2}{1-\lambda_2}\|A^{\dagger}\| \\ &\|(A+E)A^{\dagger}(I+EA^{\dagger})^{-1} - AA^{\dagger}\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\|\left[\frac{1+\lambda_2}{1-\lambda_1}\|A\| + \|E\|\right]. \end{aligned}$$

where $\lambda_1 < 1, \lambda_2 < 1$.

Proof. Since $E, A^{\dagger}$ satisfy the condition (10) and by Lemma 1.2, we obtain that $(I+EA^{\dagger})^{-1}$ exists and

$$\|(I+EA^{\dagger})^{-1}\| \leq \frac{1+\lambda_2}{1-\lambda_1}. \tag{21}$$

Let $T = A^{\dagger}(I+EA^{\dagger})^{-1}$. By Lemma 2.3 [14], we get that $I + A^{\dagger}E$ is invertible and

$$T = (I+A^{\dagger}E)^{-1}A^{\dagger}. \tag{22}$$

Now, we will prove that $T \in (A+E)^{(1,2,3)}$. So we only need to verify the equations (1), (2), (3) of four Moore-Penrose equations. Note that

$$\begin{aligned} T(A+E)T &= A^{\dagger}(I+EA^{\dagger})^{-1}(A+E)A^{\dagger}(I+EA^{\dagger})^{-1} \\ &= A^{\dagger}(I+EA^{\dagger})^{-1}(AA^{\dagger}+EA^{\dagger})(I+EA^{\dagger})^{-1} \\ &= A^{\dagger}(I+EA^{\dagger})^{-1}(I+EA^{\dagger})AA^{\dagger}(I+EA^{\dagger})^{-1} \\ &= T. \end{aligned}$$

It implies that T is a $\{2\}$-inverse of $A+E$.

On the other hand, by $R(E) \subset R(A)$ and $R(A) = R(AA^\dagger)$, we easily prove that

$$\begin{aligned}(A+E)T &= (A+E)A^\dagger(I+EA^\dagger)^{-1} \\ &= (AA^\dagger + AA^\dagger EA^\dagger)(I+EA^\dagger)^{-1} \\ &= AA^\dagger(I+EA^\dagger)(I+EA^\dagger)^{-1} \\ &= AA^\dagger. \end{aligned} \tag{23}$$

Thus $[(A+E)T]^* = (A+E)T$. i.e. $T \in (A+E)\{3\}$. According to (23), we also prove that T satisfies the first Moore-Penrose equation as follow

$$\begin{aligned}(A+E)T(A+E) &= (A+E)A^\dagger(I+EA^\dagger)^{-1}(A+E) \\ &= AA^\dagger(A+E) \\ &= AA^\dagger A + AA^\dagger E \\ &= A+E.\end{aligned}$$

Therefore $T \in (A+E)\{1\}$. Thus, we prove that T is a element of the set $(A+E)\{1,2,3\}$. From (22), we have $R(T) = R(A^\dagger)$ and $N(T) = N(A^\dagger)$.

Also,

$$\|T\| = \|(I+A^\dagger E)^{-1}A^\dagger\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^\dagger\|. \tag{24}$$

From (10), we have $\|EA^\dagger x\| \le \lambda_1\|x\| + \lambda_2\|x\| + \lambda_2\|EA^\dagger x\|, \forall x \in H$. Therefore,

$$\|EA^\dagger x\| \le \frac{\lambda_1+\lambda_2}{1-\lambda_2}. \tag{25}$$

According (21) and (25), we can compute that

$$\begin{aligned}\|T - A^\dagger\| &= \|A^\dagger(I+EA^\dagger)^{-1} - A^\dagger\| \\ &= \|A^\dagger(I+EA^\dagger)^{-1}(I-(I+EA^\dagger)\| \\ &\le \|A^\dagger\|\|(I+EA^\dagger)^{-1}\|\|EA^\dagger)\| \\ &\le \frac{1+\lambda_2}{1-\lambda_1}\cdot\frac{\lambda_1+\lambda_2}{1-\lambda_2}\|A^\dagger\|.\end{aligned} \tag{26}$$

In the following, we consider the perturbation bound of $\|(A+E)T - AA^\dagger\|$. It easily follows that

$$(A+E)T - AA^\dagger = AT + ET - AA^\dagger = A\left[T - A^\dagger\right] + ET$$

Therefore, from (24) and (26) we get

$$\|(A+E)T - AA^\dagger\| \le \|A\|\|T - A^\dagger\| + \|E\|\|T\| \le \frac{1+\lambda_2}{1-\lambda_1}\frac{\lambda_1+\lambda_2}{1-\lambda_2}\|A\|\|A^\dagger\| + \frac{1+\lambda_2}{1-\lambda_1}\|E\|\|A^\dagger\|.$$

Thus, we have finished the proof. □

If the condition $R(E) \subseteq R(A)$ in Theorem 2.2 is replaced by $N(A) \subseteq N(E)$, we have the following theorem:

Theorem 2.3. *Let $A \in L(H,K)$ has a closed range and let $N(A) \subseteq N(E)$. If E, $A^\dagger$ satisfy* (10)*, then*

$$A^\dagger(I+EA^\dagger)^{-1} = (I+A^\dagger E)^{-1}A^\dagger \in A\{1,2,4\}$$

and

$$\|A^{\dagger}(I+EA^{\dagger})^{-1}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\| \quad (27)$$
$$\|A^{\dagger}(I+EA^{\dagger})^{-1}-A^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\frac{\lambda_1+\lambda_2}{1-\lambda_2}\|A^{\dagger}\|$$
$$\|(A+E)A^{\dagger}(I+EA^{\dagger})^{-1}-AA^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\|\left[\frac{1+\lambda_2}{1-\lambda_1}\|A\|+\|E\|\right]$$

where $\lambda_1 < 1$, $\lambda_2 < 1$.

Proof. The proof is similar as in the Theorem 2.2. □

From Theorem 2.2 and Theorem 2.3, we obtain the perturbation bound of $\|(A+E)^{\dagger}-A^{\dagger}\|$:

Theorem 2.4. *Let* $A \in L(H,K)$ *has a closed range and let* $R(E) \subseteq R(A)$ *and* $N(A) \subseteq N(E)$. *If* E, $A^{\dagger}$ *satisfy* (10), *then*

$$(A+E)^{\dagger} = A^{\dagger}(I+EA^{\dagger})^{-1} = (I+A^{\dagger}E)^{-1}A^{\dagger}$$

and

$$\|(A+E)^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\| \quad (28)$$
$$\|(A+E)^{\dagger}-A^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\frac{\lambda_1+\lambda_2}{1-\lambda_2}\|A^{\dagger}\| \quad (29)$$
$$\|(A+E)(A+E)^{\dagger}-AA^{\dagger}\| \le \frac{1+\lambda_2}{1-\lambda_1}\|A^{\dagger}\|\left[\frac{1+\lambda_2}{1-\lambda_1}\|A\|+\|E\|\right],$$

where $\lambda_1 < 1$, $\lambda_2 < 1$.

3. Applications

In this section, we present the perturbation bound of the least squares solution of minimal norm for the linear operator equation (see [22, Chapter 9])

$$Ax = b. \quad (30)$$

Let $A \in L(H,K)$ has a close range and $b \in K$. The minimum norm least squares problem is presented by

$$\min_{x\in H}\|x\| \text{ such that } \|b-Ax\| = \min_{z\in H}\|b-Az\| \quad (31)$$

where $\|\cdot\|$ is the norm of H or K induced by its inner product $(\cdot,\cdot)$. It is well-known that $x = A^{\dagger}b$ is the least squares solution of minimal norm of (30). Let E and f be perturbed operator of A and b, respectively. Then the equation (31) reduces to the following equation

$$(A+E)x = b+f \quad (32)$$

and in this case equivalent problem is presented by

$$\min_{x\in H}\|x\| \text{ such that } \|b+f-(A+E)z\| = \min_{z\in H}\|b+f-(A+E)z\|. \quad (33)$$

Evidently, if $R(A+E)$ is closed a unique solution of (33) is given by $\bar{x} = (A+E)^{\dagger}(b+f)$.

Theorem 3.1. *Let* $A, E \in L(H,K)$ *be such that* $A, A+E$ *have a close ranges. Suppose that for some* $\lambda_1 < 1, \lambda_2 < 1$ *and every* $x \in H$,

$$\|EA^{\dagger}x\| \le \lambda_1\|x\| + \lambda_2\|(I+EA^{\dagger})x\|$$

and that $S = E_{22} - E_{21}(A_1 + E_{11})^{-1}E_{12}$ is a Moore-Penrose invertible. Then the least square solutions of minimal norm for equations (30) *and* (32) *exist and*

$$\begin{aligned}\frac{\|\bar{x}-x\|}{\|x\|} \leq\ & \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[\|A_1^{-1}E_{11}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right] \\ & + \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|\|A_1\| \\ & + \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[1 + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right]\frac{\|f\|}{\|b\|} \\ & + \left\{\frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|\|A_1\|\right\}\frac{\|f\|}{\|b\|}.\end{aligned}$$

where $k(A_1) = \|A_1^{-1}\|\|A\| = \|A_1^{-1}\|\|A_1\|$ and

$$\begin{aligned}& A_1 = P_AAP_A, E_{11} = P_AEP_A, E_{12} = P_AEP_A^{\perp}, E_{21} = P_A^{\perp}EP_A, \\ & E_{22} = P_A^{\perp}EP_A^{\perp}, S = E_{22} - E_{21}(A_1 + E_{11})^{-1}E_{12}.\end{aligned} \tag{34}$$

Proof. Since $R(A)$ and $R(A+E)$ are closed, it follows that $x = A^{\dagger}b$ and $\bar{x} = (A+E)^{\dagger}(b+f)$.
Note that

$$\bar{x} - x = (A+E)^{\dagger}(b+f) - A^{\dagger}b = \left[(A+E)^{\dagger} - A^{\dagger}\right]b - (A+E)^{\dagger}f. \tag{35}$$

Since $Ax = b$ and according to (35), we have

$$\|Ax\| = \|b\| \leq \|A\|\|x\|, \frac{\|b\|}{\|A\|} \leq \|x\| \tag{36}$$

and

$$\|\bar{x} - x\| \leq \left\|\left[(A+E)^{\dagger} - A^{\dagger}\right]\right\|\|b\| + \|(A+E)^{\dagger}\|\|f\|. \tag{37}$$

From (18), (19) and (37), we obtain

$$\begin{aligned}\|\bar{x}-x\| \leq\ & \left\|\left[(A+E)^{\dagger} - A^{\dagger}\right]\right\|\|b\| + \|(A+E)^{\dagger}\|\|f\| \\ \leq\ & \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|A_1^{-1}E_{11}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right]\|b\| \\ & + \left\{\frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|\right\}\|b\| \\ & + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[1 + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right]\|f\| \\ & + \left\{\frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|\right\}\|f\|.\end{aligned}$$

where $A_1, E_{11}, E_{12}, E_{21}, E_{22}, S$ are given by (34).
Applying (36), we get

$$\begin{aligned}\frac{\|\bar{x}-x\|}{\|x\|} \leq\ & \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[\|A_1^{-1}E_{11}\| + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right] \\ & + \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|\|A_1\| \\ & + \frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[1 + \frac{1+\lambda_2}{1-\lambda_1}\|A_1^{-1}\|\|E_{12}S^{\dagger}E_{21}\|\right]\frac{\|f\|}{\|b\|} \\ & + \left\{\frac{1+\lambda_2}{1-\lambda_1}k(A_1)\left[\|E_{12}S^{\dagger}\| + \|E_{21}S^{\dagger}\|\right] + \|S^{\dagger}\|\|A_1\|\right\}\frac{\|f\|}{\|b\|}.\end{aligned}$$

where $k(A_1) = \|A_1^{-1}\|\|A\| = \|A_1^{-1}\|\|A_1\|$.

Therefore, we have finished the proof. $\square$

Theorem 3.2. *Let $A, E \in L(H, K)$ be such that $R(A)$ and $R(A+E)$ are closed and let $R(E) \subseteq R(A)$ and $N(A) \subseteq N(E)$. If* (10) *holds, then the least squares solutions of minimal norm for equations* (30) *and* (32) *exist and*

$$\frac{\|\bar{x}-x\|}{\|x\|} \leq k(A)\frac{1+\lambda_2}{1-\lambda_1}\left\{\frac{\lambda_1+\lambda_2}{1-\lambda_2}+\frac{\|f\|}{\|b\|}\right\}$$

where $k(A) = \|A^\dagger\|\|A\|$ is the condition number.

Proof. Similarly as in Theorem 3.1, we have $x = A^\dagger b$, $\bar{x} = (A+E)^\dagger(b+f)$ and

$$\bar{x} - x = \left[(A+E)^\dagger - A^\dagger\right] b - (A+E)^\dagger f. \tag{38}$$

According to the inequalities in (28), (29) and from (38), (36), we obtain

$$\begin{aligned}\frac{\|\bar{x}-x\|}{\|x\|} &\leq \left\|\left[(A+E)^\dagger - A^\dagger\right]\right\|\|b\| + \|(A+E)^\dagger\|\|f\| \\ &\leq \frac{1+\lambda_2}{1-\lambda_1}\cdot\frac{\lambda_1+\lambda_2}{1-\lambda_2}\|A^\dagger\|\|b\| + \frac{1+\lambda_2}{1-\lambda_1}\|A^\dagger\|\|f\| \\ &\leq k(A)\cdot\frac{1+\lambda_2}{1-\lambda_1}\left\{\frac{\lambda_1+\lambda_2}{1-\lambda_2}+\frac{\|f\|}{\|b\|}\right\},\end{aligned}$$

where $k(A) = \|A^\dagger\|\|A\|$ is condition number. $\square$

References

[1] A. Ben-Israel, On error bounds for generalized inverses, SIAM J. Numer. Anal. 3 (1966) 585–592.
[2] A. Ben-Israel, T. N. E. Greville, Generalized Inverse: Theory and Applications, second ed. SpringerVerlag, New York, 2002.
[3] L. Cai, W. Xu, W. Li, Additive and multiplicative perturbation bounds for the Moore-Penrose inverse, Linear Algebra Appl. 434 (2011) 480–489.
[4] G. Chen, M. Wei, Y. Xue, Perturbation Analysis of the Least Squares Solution in Hilbert Spaces, Linear Algebra Appl. 244 (1996) 69–80.
[5] D. S. Cvetković-Ilić, Expression of the Drazin and MP-inverse of partitioned matrix and quotient identity of generalized Schur complement, Appl. Math. Comput. 213(1) (2009) 18–24.
[6] C. Deng, Y. Wei, Perturbation analysis for the Moore-Penrose inverse for a class of bound operators in Hilbert spaces, J. Korean Math. Soc. 47 (2010) 831–843.
[7] J. Ding, New perturbation results on pseudo-inverses of linear operators in Banach spaces, Linear Algebra Appl. 362 (2003) 229–235.
[8] D. S. Djordjević , V. Rakočević, Lectures on Generalized Inverse, Faculty of Sciences and Mathematics, University of Niš, Niš, 2008.
[9] R. G. Douglas, On majorization, factorizations, Recent applications of generalized inverses, pp. 233–249, Res. Notes in Math., 66, Pitman, Boston, Mass-London, 1982.
[10] M. R. Hestencs, Relative hermitian matrices, Pacific J. Math. 11 (1961) 225–245.
[11] T. Kato, Perturbation Theory for Linear Operators, Springer-verlag, Berlin, 1984.
[12] C. F. King, A note on Drazin inverse, I, Pacific J. Math. 70 (1977) 383–390.
[13] D. C. Lay, Spectral analysis using ascent, descent, nullity and defect, Math. Ann. 184 (1970) 197–214.
[14] J. J. Koliha, V. Rakocević, Invertibility of the difference of idempotents, Linear and Multilinear Algebra 51 (2003) 97–110.
[15] W. Li, Multiplicative perturbation bounds for spectral and singular value decompositions, J. Comput. Appl. Math. 217 (2008) 243–251.
[16] L. Meng, B. Zheng, The optimal perturbation bounds for the Moore-Penrose inverse under the Frobenius norm, Linear Algebra Appl. 432 (2010) 956–963.
[17] M. Z. Nashed, Generalized Inverse and Applications, Academic Press, New York, 1976.
[18] Q. Huang, W. Zhai, Perturbations and expressions for generalized inverses in Banach spaces and Moore-Penrose inverses in Hilbert spaces of closed linear operators, Linear Algebra Appl. 435 (2011) 117–127.
[19] G. W. Stewart, On the continuity of the generalized inverses, SIAM J. Appl. Math. 17 (1969) 33–45.
[20] G. W. Stewart, On the perturbation of the pseudo-inverse, projections, and linear squares problems, SIAM J. Rev. 19 (1977) 634–662.
[21] G. W. Stewart, J. G. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.

[22] G. Wang, Y. Wei, S. Qiao, Generalized Inverses: Theory and Computations, Science Press, Beijing, 2004.
[23] P. Å. Wedin, On pseudoinverses of perturbed matrices, Lund Un. Comp. Sc. Tech. Rep., 1969.
[24] P. Å. Wedin, Perturbation theory for pseudo-inverses, BIT. 13 (1973) 217–232.
[25] Y. Wei, S. Qiao, The representation and approximation of the Drazin inverse of a linear operator in Hilbert space, Appl. Math. Comput. 138 (2003) 77–89.
[26] X. Yang, Y. Wang, Some new perturbation theorems for generalized inverses of linear operators in Banach spaces, Linear Algebra Appl. 433 (2010) 1939–1949.