Filomat 26:2 (2012), 371–377
DOI 10.2298/FIL1202371C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hölder's means and triangles inscribed in a semicircle in Banach spaces

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Abstract. By the Hölder's means, we introduce two classes geometric constants for Banach spaces. We study some geometric properties related to these constants and the stability under norm perturbations of them.

1. Introduction

There are various ways for constructing the means between two positive numbers a and b (see for example [4]). Among them Hölder's means (also called power means) are defined by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \text{ for } p \neq 0,$$
$$M_0(a,b) = \lim_{p\to 0} M_p(a,b) = \sqrt{ab}.$$

In particular, the arithmetic mean $A := M_1$ and the geometric mean $G := M_0$ are well-known. We should note that Hölder's means are positively homogeneous, that is,

$$M_p(ta,tb) = tM_p(a,b)\ (t \geq 0).$$

For two real numbers $p \leq q$,

$$\min(a,b) \leq M_p(a,b) \leq M_q(a,b) \leq \max(a,b),$$

where "=" holds only for the case $a = b$.

Throughout the paper assume that X is a Banach space and denote by S_X and B_X the unit sphere and the unit ball, respectively. Let x, y are two points on the unit sphere S_X of X. Baronti, Casini and Papini [3] defined

$$A_1(X) = \inf_{x\in S_X} \sup_{y\in S_X} M_1\Big(\|x+y\|, \|x-y\|\Big),$$
$$A_2(X) = \sup_{x,y\in S_X} M_1\Big(\|x+y\|, \|x-y\|\Big),$$

2010 *Mathematics Subject Classification*. Primary 46B20
Keywords. Hölder's mean, James constant, modulus of convexity
Received: 10 January 2010; Accepted: 21 June 2010
Communicated by Dragan S. Djordjević
Research supported by Natural Science Foundation of Department of Education, Henan(2011B110023).
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by considering the arithmetic mean of $\|x+y\|$ and $\|x-y\|$. Later, Alonso and Llorens-Fuster introduced

$$t(X) = \inf_{x \in S_X} \sup_{y \in S_X} M_0\Big(\|x+y\|, \|x-y\|\Big),$$

$$T(X) = \sup_{x,y \in S_X} M_0\Big(\|x+y\|, \|x-y\|\Big),$$

by considering the geometric mean between $\|x+y\|$ and $\|x-y\|$.

Based on the idea of the above constants, we will consider Hölder's means of $\|x+y\|$ and $\|x-y\|$, and therefore define two classes new geometric constants, which is more general than the above constants. These constants are also proved to be connected with the well-known modulus of convexity and other geometric properties. The results presented in this paper are more general than the known results about the constants mentioned above.

2. Preliminaries

We begin this section with some definitions and notations. Recall the modulus of convexity of X is a function $\delta_X(\epsilon) : [0,2] \to [0,1]$, defined as

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \ x, y \in S_X, \ \|x-y\| = \epsilon\right\}.$$

The number defined by

$$\epsilon_0(X) = \sup\{\epsilon \in [0,2] : \delta_X(\epsilon) = 0\}$$

is called the characteristic of convexity. A space X is called uniformly convex if $\delta_X(\epsilon) > 0$ for $0 < \epsilon \leq 2$, or equivalently $\epsilon_0(X) = 0$.

The following constants

$$J(X) = \sup_{x,y \in S_X} \min\Big(\|x+y\|, \|x-y\|\Big),$$

$$j(X) = \inf_{x \in S_X} \sup_{y \in S_X} \min\Big(\|x+y\|, \|x-y\|\Big)$$

were defined by Gao in [7] (see also [5]). The constant

$$E(X) = \sup\left\{\|x+y\|^2 + \|x-y\|^2 : \ x, y \in S_X\right\},$$

was also introduced by Gao [6] and recently studied by several authors (see [2, 8, 11]). Both $J(X)$ and $E(X)$ characterize the uniform nonsquareness, that is, X is uniformly non-square if and only if $J(X) < 2$ or $E(X) < 8$ (see[8, 9, 11]). Recall that a Banach space X is called uniformly non-square if for any $x, y \in S_X$ there exists $\delta > 0$, such that either $\|x-y\|/2 \leq 1 - \delta$, or $\|x+y\|/2 \leq 1 - \delta$.

We now define two classes constants by considering Hölder's means.

Definition 2.1. Let p be a real number and let

$$H_p(X) = \sup_{x,y \in S_X} M_p\Big(\|x+y\|, \|x-y\|\Big),$$

$$h_p(X) = \inf_{x \in S_X} \sup_{y \in S_X} M_p\Big(\|x+y\|, \|x-y\|\Big).$$

Remark 2.2. (1) $\sqrt{2} \leq J(X) \leq H_p(X) \leq 2$ for $p \in \mathbb{R}$.
(2) Obviously $H_p(X) \leq H_q(X)$ and $h_p(X) \leq h_q(X)$ if $p \leq q$.

3. Some properties

First let us state an identity between the modulus of convexity and $H_p(X)$. We know (see [1, 3]) that

$$A_2(X) = \sup_{\epsilon\in[0,2]} M_1\Big(\epsilon, 2(1-\delta_X(\epsilon))\Big),$$
$$T(X) = \sup_{\epsilon\in[0,2]} M_0\Big(\epsilon, 2(1-\delta_X(\epsilon))\Big).$$

We will show this fact is also true for the general cases.

Theorem 3.1. *For any Banach space X,*

$$H_p(X) = \sup_{\epsilon\in[0,2]} M_p\Big(\epsilon, 2(1-\delta_X(\epsilon))\Big).$$

Proof. From the definition of $\delta_X(\epsilon)$,

$$\sup_{x,y\in S_X, \|x-y\|=\epsilon} \|x+y\| = 2(1-\delta_X(\epsilon))$$

for any $\epsilon \in [0,2]$, which yields

$$\sup_{x,y\in S_X, \|x-y\|=\epsilon} M_p\Big(\epsilon, \|x+y\|\Big) = M_p\Big(\epsilon, 2(1-\delta_X(\epsilon))\Big).$$

Therefore we have

$$\begin{aligned} H_p(X) &= \sup\{M_p(\|x-y\|, \|x+y\|) : x, y \in S_X\} \\ &= \sup_{\epsilon\in[0,2]} \{M_p\Big(\epsilon, \|x+y\|\Big) : x, y \in S_X, \|x-y\| = \epsilon\} \\ &= \sup_{\epsilon\in[0,2]} M_p\Big(\epsilon, 2(1-\delta_X(\epsilon))\Big). \end{aligned}$$

This completes the proof. □

From Theorem 3.1, one can readily get the following.

Corollary 3.2. *For any Banach space X,*

$$\max\Big(J(X), M_p(\epsilon_0, 2)\Big) \le H_p(X).$$

It was shown that a Banach space X is uniformly non-square if and only if $A_2(X)$ or $T(X) < 2$. So it is readily seen that this fact is true for $H_p(X)$ whenever $p \le 1$. Further we will show $H_p(X)$ can also characterize uniform nonsquareness for all $p \in \mathbb{R}$.

Theorem 3.3. *X is uniformly non-square if and only if $H_p(X) < 2$.*

Proof. ⇒) Assume that X is uniformly nonsquare. Then for any $x, y \in S_X$, there exists $\delta > 0$, such that either $\|x-y\| \le 2(1-\delta)$ or $\|x+y\| \le 2(1-\delta)$. Hence

$$M_p(\|x+y\|, \|x-y\|) \le M_p(2, 2(1-\delta)).$$

Since x, y are arbitrary,

$$H_p(X) \le M_p(2, 2(1-\delta)) < \max(2, 2(1-\delta)) = 2,$$

where the strict inequality follows from $2 \neq 2(1-\delta)$.

$\Leftarrow$) Conversely, assume $H_p(X) < 2$. Let $\delta = 1 - (H_p(X)/2)$. Then $\delta > 0$. Thus we can deduce that either

$$\|x-y\| \leq 2(1-\delta) \text{ or } \|x+y\| \leq 2(1-\delta)$$

for any $x, y \in S_X$. In fact if $\|x-y\| \leq 2(1-\delta)$, then we are done. If not, then form the definition of $H_p(X)$,

$$\begin{aligned} \|x+y\| &\leq \left(2H_p^p(X) - \|x-y\|^p\right)^{1/p} \\ &\leq \left(2H_p^p(X) - (2(1-\delta))^p\right)^{1/p} \\ &= H_p(X) = 2(1-\delta). \end{aligned}$$

Therefore X is uniformly non-square. □

The following is a necessary condition for uniform nonsquareness in terms of the constant $h_p(X)$.

Proposition 3.4. *Let $p \in \mathbb{R}$. Then the following are equivalent.*

1. $j(X) = 2$;
2. $h_p(X) = 2$.

Moreover, each of above implies that X is a not-uniformly non-square infinite dimensional space. Thus $h_p(X) < 2$ whenever X is finite dimensional.

Proof. From the inequality

$$j(X) \leq h_p(X) \leq 2,$$

the above conditions are equivalent. Since $j(X) = 2$ implies that X is a not-uniformly non-square infinite dimensional space (cf.[1, Proposition 9]), thus the rest assertion follows. □

Finally we end this section by computing the value of $H_p(X)$ and $h_p(X)$ for the ℓ_r space. It has been shown that in such space

$$J(\ell_r) = \sqrt{\frac{E(\ell_r)}{2}} = \max\left(2^r, 2^{1-1/r}\right)$$

(see [7, 11]).

Theorem 3.5. *(1) Let $p \leq 2$. Then $H_p(\ell_r) = \max(2^r, 2^{1-1/r})$ for any $r \geq 1$.*

(2) Let $2 \leq p \leq r$. Then $H_p(\ell_r) = 2^{1-1/r}$.

Proof. (1) From

$$J(\ell_r) \leq H_p(\ell_r) \leq \sqrt{\frac{E(\ell_r)}{2}},$$

we know (1) holds obviously.

(2) Recall the Clarkson's inequality

$$(\|x+y\|^r + \|x-y\|^r)^{1/r} \leq 2^{1/r}(\|x\|^r + \|y\|^r)^{1-1/r},$$

which implies that

$$M_p(\|x+y\|, \|x-y\|) \leq M_r(\|x+y\|, \|x-y\|) \leq 2^{1-1/r},$$

for any $x, y \in S_{\ell_r}$. Thus $H_p(\ell_r) \leq 2^{1-1/r}$. Since

$$H_p(\ell_r) \geq J(\ell_r) = 2^{1-1/r},$$

and then (2) holds. □

Theorem 3.6. *(1) For any Banach space, $h_p(X) \geq M_p(1, 3/2)$.*
(2) Let $p \leq 1 \leq r \leq 2$. Then $h_p(\ell_r) = 2^{1/r}$.

Proof. (1) By Proposition 10 in [1], we know that for any $x \in S_X$,

$$\sup_{y \in S_X} M_p(\|x+y\|, \|x-y\|) \geq M_p(1, 3/2),$$

and so (1) holds obviously.

(2) This identity follows from

$$j(\ell_r) \leq h_p(\ell_r) \leq A_1(\ell_r)$$

and $j(\ell_r) = A_1(\ell_r) = 2^{1/r}$ (see [3, 7]). □

4. Stability under norm perturbations

We first show that "S_X" can be replaced by "B_X" in the definition of $H_p(X)$.

Theorem 4.1. *For any Banach space X,*

$$H_p(X) = \sup_{x,y \in B_X} M_p\Big(\|x+y\|, \|x-y\|\Big),$$
$$h_p(X) = \inf_{x \in S_X} \sup_{y \in B_X} M_p\Big(\|x+y\|, \|x-y\|\Big).$$

Proof. Let $u \in S_X, v \in B_X$. It follows from [10, p.60] that there exist $x, y \in S_X$, such that

$$\|u - v\| = \|x - y\|, \ \|u + v\| \leq \|x + y\|.$$

Thus we have

$$M_p(\|u-v\|, \|u+v\|) \leq M_p(\|x+y\|, \|x-y\|) \leq \sup_{x,y \in S_X} M_p\Big(\|x+y\|, \|x-y\|\Big) = H_p(X),$$

which implies that

$$H_p(X) \geq \sup_{x \in S_X, y \in B_X} M_p\Big(\|x+y\|, \|x-y\|\Big).$$

On the other hand, let $u, v \in B_X$ and assume without loss of generality that $\|u\| \geq \|v\| > 0$. Then

$$M_p(\|u-v\|, \|u+v\|) = \|u\| M_p\left(\left\|\frac{u}{\|u\|} - \frac{v}{\|u\|}\right\|, \left\|\frac{u}{\|u\|} + \frac{v}{\|u\|}\right\|\right) \leq \sup_{x \in S_X, y \in B_X} M_p\Big(\|x+y\|, \|x-y\|\Big).$$

This implies that

$$\sup_{x \in S_X, y \in B_X} M_p\Big(\|x+y\|, \|x-y\|\Big) \geq \sup_{x,y \in B_X} M_p\Big(\|x+y\|, \|x-y\|\Big)$$

and so the first identity follows. Similarly, we get the second identity. □

Recall the Banach-Mazur distance between isomorphic Banach spaces X and Y is defined as

$$d(X, Y) = \inf\{\|T\| \cdot \|T^{-1}\|\},$$

where the infimum is taken over all bicontinuous linear operators T from X onto Y. In [9] the authors studied the relation between $J(X)$ and $J(Y)$ for isomorphic Banach spaces X and Y. We now study the relation between $H_p(X)$ and $H_p(Y)$ and the idea of the following proof is taken form [9, Theorem 5].

Theorem 4.2. *Let X and Y be isomorphic Banach spaces. Then*

$$\frac{H_p(X)}{d(X, Y)} \leq H_p(Y) \leq H_p(X)d(X, Y). \tag{1}$$

In particular, $H_p(X) = H_p(Y)$ if X and Y are isometric.

Proof. Let $x_1, x_2 \in S_X$. It follows form the definition of Banach-Mazur distance that for any $\epsilon > 0$, there exists an operator T from X onto Y such that

$$\|T\| \cdot \|T^{-1}\| \leq d(X, Y)(1 + \epsilon).$$

Set

$$y_1 = \frac{Tx_1}{\|T\|},\ y_2 = \frac{Tx_2}{\|T\|}.$$

Thus $y_1, y_2 \in B_Y$ and

$$\begin{aligned} M_p(\|x_1 + x_2\|, \|x_1 - x_2\|) &= \|T\| M_p(\|T^{-1}(y_1 + y_2)\|, \|T^{-1}(y_1 - y_2)\|) \\ &\leq d(X, Y)(1 + \epsilon) M_p(\|y_1 + y_2\|, \|y_1 - y_2\|) \\ &\leq d(X, Y)(1 + \epsilon) H_p(Y), \end{aligned}$$

which gives

$$H_p(X) \leq d(X, Y)(1 + \epsilon)H_p(Y).$$

Since ϵ is arbitrary, the left side of (1) follows. Similarly, we get the right side. □

Applying the above, one can easily get the following.

Corollary 4.3. *Let $X_1 = (X, \|\cdot\|_1)$ and $X_2 = (X, \|\cdot\|_2)$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms in X such that*

$$\alpha\|\cdot\|_1 \leq \|\cdot\|_2 \leq \beta\|\cdot\|_1 \ (0 < \alpha \leq \beta).$$

Then

$$\frac{\alpha}{\beta} H_p(X_1) \leq H_p(X_2) \leq \frac{\beta}{\alpha} H_p(X_1).$$

In particular, $H_p(X^{**}) = H_p(X)$.

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