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The upper connected vertex detour number of a graph

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Abstract. For vertices x and y in a connected graph $G = (V, E)$ of order at least two, the detour distance $D(x, y)$ is the length of the longest $x - y$ path in G. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. For any vertex x in G, a set $S \subseteq V$ is an x-detour set of G if each vertex $v \in V$ lies on an $x - y$ detour for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G, denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. A connected x-detour set of G is an x-detour set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-detour set of G is the connected x-detour number of G and is denoted by $cd_x(G)$. A connected x-detour set of cardinality $cd_x(G)$ is called a cd_x-set of G. A connected x-detour set S_x is called a minimal connected x-detour set if no proper subset of S_x is a connected x-detour set. The upper connected x-detour number, denoted by $cd_x^+(G)$, is defined as the maximum cardinality of a minimal connected x-detour set of G. We determine bounds for $cd_x^+(G)$ and find the same for some special classes of graphs. For any three integers a, b and c with $2 \leq a < b \leq c$, there is a connected graph G with $d_x(G) = a$, $cd_x(G) = b$ and $cd_x^+(G) = c$ for some vertex x in G. It is shown that for positive integers R, D and $n \geq 3$ with $R < D \leq 2R$, there exists a connected graph G with detour radius R, detour diameter D and $cd_x^+(G) = n$ for some vertex x in G.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of the shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1, 7] and further studied in [3].

The concept of vertex geodomination number was introduced in [8] and further studied in [9]. For any vertex x in a connected graph G, a set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x - y$ geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a *g_x-set*. The connected vertex geodomination number was introduced and

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studied in [11]. A *connected x-geodominating set* of G is an x-geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-geodominating set of G is the *connected x-geodomination number* of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a cg_x*-set* of G.

Some authors study the analogous concepts based on longest paths (rather than shortest paths) between pairs of vertices. For vertices x and y in a connected graph G, the *detour distance* $D(x, y)$ is the length of the longest $x - y$ path in G. For any vertex u of G, the *detour eccentricity* of u is $e_D(u) = \max \{D(u, v) : v \in V\}$. A vertex v of G such that $D(u, v) = e_D(u)$ is called a *detour eccentric vertex* of u. The *detour radius* R and *detour diameter* D of G are defined by $R = rad_D G = \min \{e_D(v) : v \in V\}$ and $D = diam_D G = \max \{e_D(v) : v \in V\}$ respectively. An $x - y$ path of length $D(x, y)$ is called an $x - y$ *detour*. The *closed interval* $I_D[x, y]$ consists of all vertices lying on some $x - y$ *detour* of G, while for $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set*. The detour number of a graph was introduced in [4] and further studied in [5].

The concept of vertex detour number was introduced in [10]. For any vertex x in a connected graph G, a set S of vertices of G is an *x-detour set* if each vertex v of G lies on an $x - y$ detour in G for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G and is denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x*-set* of G. An elaborate study of results regarding the vertex detour number with several interesting applications is given in [10]. The concept of upper vertex detour number was introduced in [13]. An x-detour set S_x is called a *minimal x-detour set* if no proper subset of S_x is an x-detour set. The *upper x-detour number*, denoted by $d_x^+(G)$, is defined as the maximum cardinality of a minimal x-detour set of G.

The connected x-detour number was introduced and studied in [12,14]. A *connected x-detour set* of G is an x-detour set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-detour set of G is the *connected x-detour number* of G and is denoted by $cd_x(G)$. A connected x-detour set of cardinality $cd_x(G)$ is called a cd_x*-set* of G. For the graph G given in Figure 1.1, the minimum vertex detour sets, the vertex detour numbers, the minimum connected vertex detour sets and the connected vertex detour numbers are given in Table 1.1.

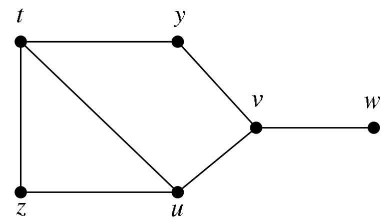

Figure 1.1: A graph G with $d_t(G) = 1$ and $cd_t(G) = 3$

The following theorems will be used in the sequel.

Theorem 1.1. ([6]) *Every nontrivial connected graph has at least two vertices which are not cut vertices.*

Theorem 1.2. ([12]) *If T is any tree of order p, then $cd_x(T) = p$ for any cut vertex x of T.*

Throughout the paper, G denotes a connected graph with at least two vertices.

2. Minimal Connected Vertex Detour Sets

Definition 2.1. Let x be any vertex of a connected graph G. A connected x-detour set S_x is called a *minimal connected x-detour set* if no proper subset of S_x is a connected x-detour set. The *upper connected x-detour number*, denoted by $cd_x^+(G)$, is defined as the maximum cardinality of a minimal connected x-detour set of G.

Vertex x	d_x-sets	$d_x(G)$	cd_x-sets	$cd_x(G)$
t	$\{y,w\}$, $\{z,w\}$, $\{u,w\}$	2	$\{y,v,w\}$, $\{u,v,w\}$	3
y	$\{w\}$	1	$\{w\}$	1
z	$\{w\}$	1	$\{w\}$	1
u	$\{w\}$	1	$\{w\}$	1
v	$\{y,w\}$, $\{z,w\}$, $\{u,w\}$	2	$\{y,v,w\}$, $\{u,v,w\}$	3
w	$\{y\}$, $\{z\}$, $\{u\}$	1	$\{y\}$, $\{z\}$, $\{u\}$	1

Table 1.1

Example 2.2. For the graph G given in Figure 2.1, the minimum vertex detour sets, the vertex detour numbers, the minimum connected vertex detour sets, the connected vertex detour numbers, the minimal connected vertex detour sets and the upper connected vertex detour numbers are given in Table 2.1.

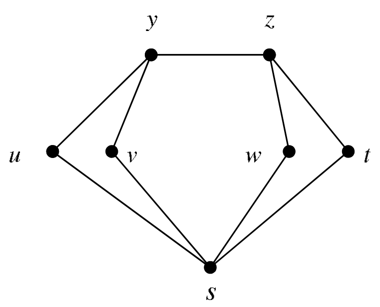

Figure 2.1: The graph G in Example 2.2.

Note 2.3. For any vertex x in a connected graph G, every minimum connected x-detour set is a minimal connected x-detour set.

In the next two theorems we prove certain properties satisfied by every x-detour set of G.

Theorem 2.4. *Let x be any vertex of a connected graph G. If $y \neq x$ is an end vertex of G, then y belongs to every x-detour set of G.*

Proof. Let x be any vertex of G and let $y \neq x$ be an end-vertex of G. Then y is the terminal vertex of an $x - y$ detour and y is not an internal vertex of any detour so that y belongs to every x-detour set of G. □

Theorem 2.5. *Let G be a connected graph with at least one cut-vertex and let S_x be an x-detour set of G for some vertex x. If v is a cut-vertex of G, then every component of $G - v$ contains an element of $S_x \bigcup \{x\}$.*

Proof. Suppose that there is a component B of $G - v$ such that B contains no vertex of $S_x \bigcup \{x\}$. Then clearly, $x \in V - V(B)$. Let $u \in V(B)$. Since S_x is an x-detour set, there exists an element $y \in S_x$ such that u lies in some $x - y$ detour $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in G. Since v is a cut-vertex of G, the $x - u$ subpath of P and the $u - y$ subpath of P both contain v, it follows that P is not a path, contrary to assumption. □

A vertex v in a connected graph G is called a *cut-vertex* if $G - v$ is disconnected. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ in G is called a branch of G at v.

Vertex x	d_x-sets	$d_x(G)$	cd_x-sets	$cd_x(G)$	Minimal connected x-detour sets	$cd_x^+(G)$
s	$\{u,v\},\{u,z\},\{u,w\},\{u,t\}, \{v,z\},\{v,w\},\{v,t\},\{y,z\}, \{y,w\},\{y,t\},\{t,w\}$	2	$\{y, z\}$	2	$\{y,z\}, \{u,v,y\}, \{u,v,s\}, \{u,w, s\},\{u,t,s\}, \{v,w, s\}, \{v,t,s\}, \{w,t,s\}, \{w,t,z\}$	3
y	$\{z\},\{t\},\{w\}$	1	$\{z\},\{t\},\{w\}$	1	$\{z\},\{t\},\{w\},\{u,v,s\}, \{u,v,y\}$	3
z	$\{y\},\{u\},\{v\}$	1	$\{y\},\{u\},\{v\}$	1	$\{y\},\{u\},\{v\},\{w,t,z\}, \{w,t,s\}$	3
u	$\{z\},\{w\},\{v\},\{t\}$	1	$\{z\},\{w\},\{v\},\{t\}$	1	$\{z\},\{w\},\{v\},\{t\}$	1
v	$\{z\},\{w\},\{u\},\{t\}$	1	$\{z\},\{w\},\{u\},\{t\}$	1	$\{z\},\{w\},\{u\},\{t\}$	1
w	$\{y\},\{u\}, \{v\}, \{t\}$	1	$\{y\},\{u\}, \{v\}, \{t\}$	1	$\{y\},\{u\}, \{v\}, \{t\}$	1
t	$\{y\},\{u\}, \{v\}, \{w\}$	1	$\{y\},\{u\}, \{v\},\{w\}$	1	$\{y\},\{u\}, \{v\}, \{w\}$	1

Table 2.1

Corollary 2.6. *Let G be a connected graph with cut-vertices and let S_x be an x-detour set of G. Then every branch of G contains an element of $S_x \bigcup\{x\}$.*

Theorem 2.7. *For any vertex x in a connected graph G, $1 \leq cd_x(G) \leq cd_x^+(G) \leq p$.*

Proof. It is clear from the definition of cd_x-set that $cdx(G) \geq 1$. Since every minimum connected x-detour set is a minimal connected x-detour set, $cd_x(G) \leq cd_x^+(G)$. Also, since $V(G)$ induces a connected x-detour set of G, it is clear that $cd_x^+(G) \leq p$. □

Remark 2.8. For the cycle C_p, $cd_x(C_p) = 1$ for every vertex x in C_p. For any non-trivial tree T with $p \geq 3$, $cd_x^+(T) = p$ for any cut-vertex x in T (See Theorem 2.12(i)). For an end vertex x in the star $G = K_{1,n} (n \geq 3)$, $cd_x(G) = n = cd_x^+(G)$. Also, the inequalities in the theorem can be strict. For the graph G given in Figure 2.1, $cd_s(G) = 2$, $cd_s^+(G) = 3$ and $p = 7$ as in Example 2.2. Thus $1 < cd_s(G) < cd_s^+(G) < p$.

Theorem 2.9. *Let x be any vertex of a connected graph G. If $cd_x^+(G) = p$, then x is a cut-vertex of G.*

Proof. Suppose x is not a cut-vertex of G. Then it follows from the fact that x lies on every $x - y$ detour and so $V - \{x\}$ is a connected x-detour set of G. Thus $cd_x^+(G) \leq p - 1$, which is a contradiction. □

Remark 2.10. The converse of Theorem 2.9 is not true. For the graph G given in Figure 2.2, $cd_x^+(G) = 3 < p$ for the cut vertex x in G.

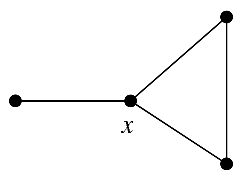

Figure 2.2: A graph G in Remark 2.10 with $cd_x^+(G) = 3 < p$.

Theorem 2.11. *There is no graph G of order p with $cd_x^+(G) = p$ for every vertex x in G.*

Proof. This follows from Theorems 1.1 and 2.9. □

Theorem 2.12. (*i*) *If T is a tree, then* $cd_x^+(T) = p$ *for any cut-vertex x of T.*

(*ii*) *If T is a tree which is not a path, then for an end vertex x,* $cd_x^+(T) = p - D(x, y)$*, where y is the vertex of T with* $deg(y) \geq 3$ *such that* $D(x, y)$ *is minimum.*

(*iii*) *If T is a path, then* $cd_x^+(T) = 1$ *for any end vertex x of T.*

Proof. (i) It is easy to see that in a tree T every vertex x has a unique minimal connected x-detour set. This implies that $cd_x(T) = cd_x^+(T)$ for every vertex x in T. Thus the result follows from Theorem 1.2.

(ii) Let T be a tree which is not a path and x an end vertex of T. Let $S = (V(T) - I_D[x, y]) \bigcup \{y\}$. Clearly, S is a connected x-detour set of T and so $cd_x(T) \leq | S |= p - D(x, y)$. We claim that $cd_x(T) = p - D(x, y)$. Otherwise, there is a connected x-detour set M of T with $| M |< p - D(x, y)$. By Theorem 2.4, every connected x-detour set of T contains all end vertices except possibly x and hence there exists a cut vertex v of T such that $v \in S$ and $v \notin M$. Let $B_1, B_2, \ldots, B_m (m \geq 3)$ be the components of $T - \{y\}$. Assume that x belongs to B_1.

Case 1. $v = y$. Let $z \in B_2$ and $w \in B_3$ be two end vertices of T. Then v lies on the unique $z - w$ detour. Since z and w belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

Case 2. $v \neq y$. Let $v \in B_i$ $(i \neq 1)$. Now, choose an end vertex $u \in B_i$ such that v lies on the $y - u$ detour. Let $a \in B_j$ $(j \neq i, 1)$ be an end vertex of T. Then y lies on the $u - a$ detour. Hence it follows that v lies on the $u - a$ detour. Since u and a belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction. Thus $cd_x(T) = p - D(x, y)$. Since $cd_x^+(T) = cd_x(T)$ for every vertex x in T, $cd_x^+(T) = p - D(x, y)$.

(iii) Let T be a path with end vertices x and y. Then for the vertex x, every vertex of T lies on an $x - y$ detour and so $\{y\}$ is the unique minimal connected x-detour set of T so that $cd_x^+(T) = 1$. □

Corollary 2.13. *For any tree T,* $cd_x^+(T) = p$ *if and only if x is a cut vertex of T.*

The following theorem is an easy consequence of the definition of the upper connected vertex detour number of a graph.

Theorem 2.14. (*i*) *For any vertex x in the complete graph* K_p, $cd_x^+(K_p) = 1$.

(*ii*) *For any vertex x in the complete bipartite graph* $K_{m,n}$, $cd_x^+(K_{m,n}) = 1$ *if* $m, n \geq 2$.

(*iii*) *For any vertex x in the wheel* W_p, $cd_x^+(W_p) = 1$.

Theorem 2.15. *For any two integers n and p with* $1 \leq n \leq p$ *and* $p \geq 5$*, there exists a connected graph G with order p and* $cd_x^+(G) = n$ *for some vertex x of G.*

Proof. We prove this theorem by considering four cases.

Case 1. Suppose $n = 1$. Let G be the path of order p. Then by Theorem 2.12(iii), $cd_x^+(G) = 1$ for an end vertex x in G.

Case 2. Suppose $n = 2$. If p is odd, let G be the odd cycle of order p. For any vertex x in G, let y_1 and y_2 be the eccentric vertices of x in G and let x_1 and x_2 be the detour eccentric vertices of x in G. Since $p \geq 5$, x_1, x_2, y_1 and y_2 are distinct. It is clear that $S_1 = \{y_1, y_2\}$, $S_2 = \{x_1\}$ and $S_3 = \{x_2\}$ are the only minimal connected x-detour sets of G and so $cd_x^+(G) = 2$. If p is even, let G be the graph obtained from the odd cycle C_{p-1} by adding a new vertex x and joining x to exactly one vertex of C_{p-1}. Then by a similar argument, it is seen that $cd_x^+(G) = 2$.

Case 3. Suppose $3 \leq n \leq p-1$. Let G be the graph obtained from the path $P_{p-1} : u_1, u_2, \ldots, u_{p-1}$ by adding a new vertex y and joining y to u_{p-n+1}. Then by Theorem 2.12(ii), $cd_x^+(G) = n$ for the end vertex $x = u_1$ in G.

Case 4. Suppose $n = p$. Let G be any tree of order p. Then by Theorem 2.12(i), $cd_x^+(G) = p$ for any cut vertex x in G. □

Remark 2.16. For $2 \leq p \leq 4$, it is straight forward to verify that there is no connected graph G of order p with $n = 2$. Thus Theorem 2.15 is not true for $2 \leq p \leq 4$.

Since any connected x-detour set is also an x-detour set it follows that $d_x(G) \leq cd_x(G)$ and so by Theorem 2.7, we have $d_x(G) \leq cd_x(G) \leq cd_x^+(G)$. Now we have the following realization theorem.

Theorem 2.17. *For any three integers a, b and c with $2 \leq a < b \leq c$, there is a connected graph G with $d_x(G) = a$, $cd_x(G) = b$ and $cd_x^+(G) = c$ for some vertex x in G.*

Proof. Let $F = K_2 \bigcup ((c-b+1)K_1) + \overline{K_2}$, where let $Z = V(K_2) = \{z_1, z_2\}$, $Y = V((c-b+1)K_1) = \{y_1, y_2, \ldots, y_{c-b+1}$ and $U = V(\overline{K_2}) = \{x, y\}$. Let $K_{1,a-2}$ be the star at the vertex w and let $W = \{w_1, w_2, \ldots, w_{a-2}\}$ be the set of end vertices of $K_{1,a-2}$. Let $P_{b-a+1} : u_1, u_2, \ldots, u_{b-a+1}$ be the path of length $b-a$. Let G be the graph obtained from $K_{1,a-2}$, F and P_{b-a+1} by identifying w of $K_{1,a-2}$, x of F and u_{b-a+1} of P_{b-a+1}. The graph G is shown in Figure 2.3.

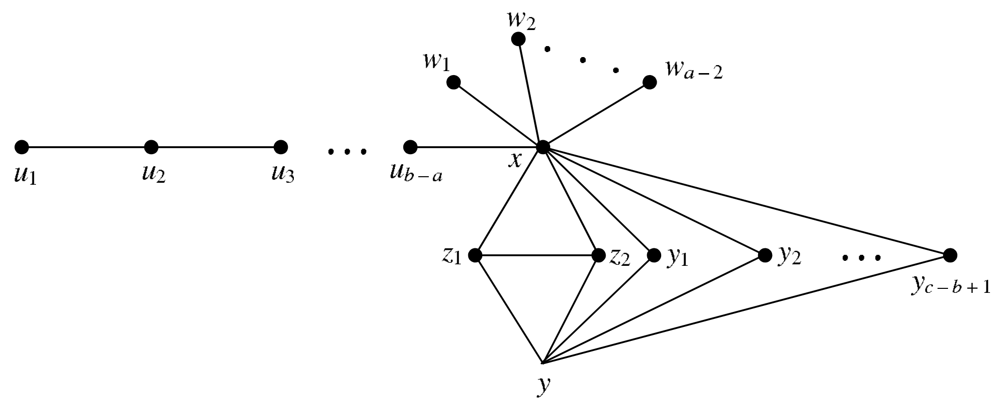

Figure 2.3: A graph G in Theorem 2.17 with $d_x(G) = a$, $cd_x(G) = b$ and $cd_x^+(G) = c$.

First, we show that $d_x(G) = a$ for the vertex x in G. Let S_x be any x-detour set of G. By Theorem 2.4, $W \bigcup \{u_1\} \subseteq S_x$. It is clear that no $y_j (1 \leq j \leq c-b+1)$ lies on any $x-z$ detour with $z \in W \bigcup \{u_1\}$. Thus $W \bigcup \{u_1\}$ is not an x-detour set of G. Now, since $W \bigcup \{u_1, z_1\}$ is an x-detour set of G, it follows that $d_x(G) = a$.

Now, we show that $cd_x(G) = b$. Let S'_x be any connected x-detour set of G. Since any connected x-detour set of G is also an x-detour set of G, it follows that S'_x contains $W \bigcup \{u_1\}$. Let $M = \{u_2, u_3, \ldots, u_{b-a}, x\}$. Since the induced subgraph $G[S'_x]$ is connected, $M \subseteq S'_x$. Thus $M \bigcup W \bigcup \{u_1\} \subseteq S'_x$. It is clear that no y_j $(1 \leq j \leq c-b+1)$ lies on any $x-z$ detour with $z \in M \bigcup W \bigcup \{u_1\}$. Thus $M \bigcup W \bigcup \{u_1\}$ is not an x-detour set of G. Now, since $M \bigcup W \bigcup \{u_1, z_1\}$ is a connected x-detour set of G, it follows that $cd_x(G) = b$.

Next, we prove that $cd_x^+(G) = c$. Let $N = M \bigcup W \bigcup Y \bigcup \{u_1\}$. Then, it is clear that N is a connected x-detour set of G. We claim that N is a minimal connected x-detour set of G. Assume, suppose such is not, so that N is not a minimal connected x-detour set of G. Then there exists a proper subset T of N such that T is a connected x-detour set of G. Let $s \in N$ and $s \notin T$. Since every connected x-detour set of G contains $M \bigcup W \bigcup \{u_1\}$, it follows that $s \in Y$. Without loss of generality, we may assume that $s = y_1$. Now, y_1 does not lie on any $x - y_j$ detour for $j = 2, 3, \ldots, c-b+1$. Also, y_1 does not lie on any $x-z$ detour with $z \in M \bigcup W \bigcup \{u_1\}$. Hence it follows that T is not an x-detour set of G, which is a contradiction. Thus N is a minimal connected x-detour set of G and so $cd_x^+(G) \geq\mid N \mid= c$. Next, we prove that $cd_x^+(G) = c$. Suppose that $cd_x^+(G) > c$. Let N' be a minimal connected x-detour set of G with $\mid N' \mid> c$. Then there exists at least one vertex, say, $v \in N'$ such that $v \notin N$. Thus $v \in \{y, z_1, z_2\}$.

Case 1. $v \in \{z_1, z_2\}$, say $v = z_1$. Since $M \bigcup W \bigcup \{u_1, z_1\}$ is a connected x-detour set of G and also it is a proper subset of N', it follows that N' is not a minimal connected x-detour set of G, which is a contradiction.

Case 2. $v = y$. Since $\mid N' \mid> c$ and $v \notin \{z_1, z_2\}$, N is a proper subset of N', it follows that N' is not a minimal connected x-detour set of G, which is a contradiction.

Thus there is no minimal connected x-detour set N' of G with $\mid N' \mid> c$. Hence $cd_x^+(G) = c$. □

Remark 2.18. The graph G of Figure 2.3 contains exactly three minimal connected x-detour sets, namely $M \bigcup W \bigcup \{u_1, z_1\}$, $M \bigcup W \bigcup \{u_1, z_2\}$ and $M \bigcup W \bigcup Y \bigcup \{u_1\}$. This example shows that there is no "Intermediate Value Theorem" for minimal connected x-detour sets, that is, if n is an integer such that $cd_x(G) < n < cd_x^+(G)$, then there does not necessarily exist a minimal connected x-detour set of cardinality n in G.

Theorem 2.19. *For any three positive integers b, c and n with $b \geq 3$ and $b \leq n \leq c$, there exists a connected graph G with $cd_x(G) = b$, $cd_x^+(G) = c$ and a minimal connected x-detour set of cardinality n for some vertex x in G.*

Proof. Let $l = n - b + 1$ and $m = c - n + 1$. Let $F_1 = (K_2 \bigcup lK_1) + \overline{K_2}$, where let $Z_1 = V(K_2) = \{z_1, z_2\}$, $Y_1 = V(lK_1) = \{y_1, y_2, \ldots, y_l\}$ and $U_1 = V(\overline{K_2}) = \{u_1, u_2\}$. Similarly let $F_2 = (K_2 \bigcup mK_1) + \overline{K_2}$, where let $Z_2 = V(K_2) = \{z_3, z_4\}$, $Y_2 = V(mK_1) = \{x_1, x_2, \ldots, x_m\}$ and $U_2 = V(\overline{K_2}) = \{u_3, u_4\}$. Let $K_{1,b-3}$ be the star at the vertex x and let $W = \{w_1, w_2, \ldots, w_{b-3}\}$ be the set of end vertices of $K_{1,b-3}$. Let G be the graph obtained from $K_{1,b-3}$, F_1 and F_2 by identifying x of $K_{1,b-3}$, u_1 of U_1 and u_3 of U_2. The graph G is shown in Figure 2.4. It follows from Theorem 2.4 that for the vertex x, the vertices of W belong to every minimal connected x-detour set of G.

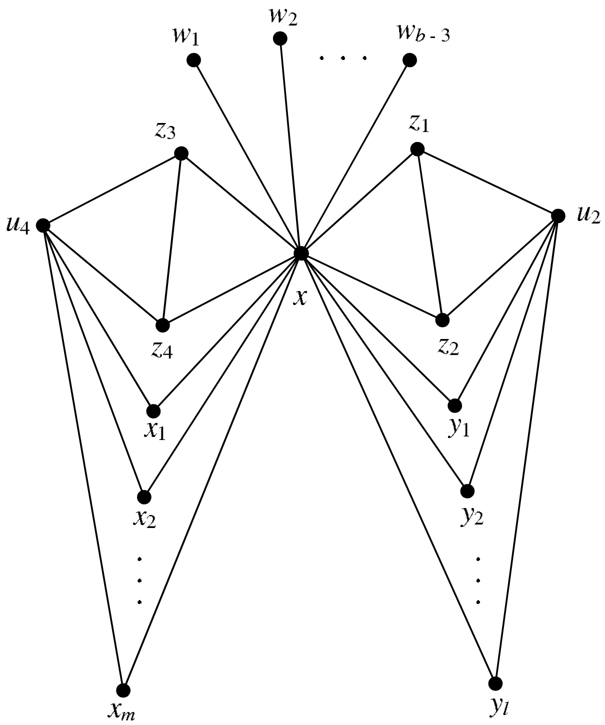

Figure 2.4: A graph G in Theorem 2.19 with $cd_x(G) = b$, $cd_x^+(G) = c$.

First, we show that $cd_x(G) = b$ for the vertex x in G. Let S_x be any connected x-detour set of G. Since the induced subgraph $G[S_x]$ is connected, x must belongs to S_x. Since x is a cut vertex of G, it is clear that $W \bigcup \{x, y\}$ where $y \in V(G) - (W \bigcup \{x\})$ is not an x-detour set of G. Now, each vertex of F_1 lies on an $x - z_i$ detour $(i = 1, 2)$ and each vertex of F_2 lies on an $x - z_j$ detour (j = 3, 4), it follows that $S = W \bigcup \{x, z_i, z_j\}$ $(i = 1, 2$ and $j = 3, 4)$ is an x-detour set of G. Also, the induced subgraph $G[S]$ is connected and so $cd_x(G) = b$.

Next, we show that $cd_x^+(G) = c$. Let $M = W \bigcup Y_1 \bigcup Y_2 \bigcup \{x\}$. It is clear that M is a connected x-detour set of G. We claim that M is a minimal connected x-detour set of G. Assume, to the contrary, that M is not a minimal connected x-detour set. Then there is a proper subset T of M such that T is a connected x-detour set of G. Let $s \in M$ and $s \notin T$. Since every connected x-detour set of G contains $W \bigcup \{x\}$, $s \in Y_1 \bigcup Y_2$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x - y_j$ detour, where $j = 2, 3, \ldots, l$ and y_1 does not lie on any $x - x_j$ detour, where $j = 1, 2, \ldots, m$, it follows that T is not an x-detour set of G, which is a contradiction. Thus M is a minimal connected x-detour set of G and so $cd_x^+(G) \geq \mid M \mid = c$.

Now we prove that $cd_x^+(G) = c$. Suppose that $cd_x^+(G) > c$. Let N be a minimal connected x-detour set of G with $\mid N \mid > c$. Then there exists at least one vertex, say $v \in N$ such that $v \notin M$. Thus $v \in \{u_2, u_4, z_1, z_2, z_3, z_4\}$.

Case 1. Suppose $v \in \{z_1, z_2\}$, say $v = z_1$. Clearly, every vertex of F_1 lies on an $x - z_1$ detour and the induced subgraph $G[(N - V(F_1)) \bigcup \{v, x\}]$ is connected and so $(N - V(F_1)) \bigcup \{v, x\}$ is a connected x-detour set of G and it is a proper subset of N, which is a contradiction to N a minimal connected x-detour set of G.

Case 2. Suppose $v \in \{z_3, z_4\}$. The proof is similar to Case 1.

Case 3. Suppose $v = u_2$. It is clear that no vertex of $Y_1 \bigcup Y_2$ is an internal vertex of any $x - z$ detour for any $z \in V(G) - \{z_1, z_2, z_3, z_4\}$. Since $v \notin \{z_1, z_2, z_3, z_4\}$, $N \subseteq V(G) - \{z_1, z_2, z_3, z_4\}$ and so $Y_1 \bigcup Y_2 \subset N$. It follows that $M = W \bigcup Y_1 \bigcup Y_2 \bigcup \{x\} \subset N$, which is a contradiction to N a minimal connected x-detour set of G.

Case 4. Suppose $v = u_4$. The proof is similar to Case 3.

Thus there is no minimal connected x-detour set N of G with $|N| > c$. Hence $cd_x^+(G) = c$.

Finally, we show that there is a minimal connected x-detour set of cardinality n. Let $S = W \bigcup Y_1 \bigcup \{z_3, x\}$. It is clear that S is a connected x-detour set of G. We claim that S is a minimal connected x-detour set of G. Assume, to the contrary, that S is not a minimal connected x-detour set. Then there is a proper subset T of S such that T is a connected x-detour set of G. Let $s \in S$ and $s \notin T$. Since x is a cut vertex of G and T is a connected x-detour set of G, it is clear that $s = y_i$ for some $i = 1, 2, \ldots, l$. Without loss of generality, we may assume that $s = y_1$. Since y_1 does not lie on any $x - z$ detour with $z \in T$, it follows that T is not an x-detour set of G, which is a contradiction. Thus S is a minimal connected x-detour set of G with cardinality $|S| = n$. Hence the theorem. □

For every connected graph G, $rad_D G \le diam_D G \le 2rad_D G$. Chartrand, Escuadro and Zhang[2] showed that every two positive integers a and b with $a \le b \le 2a$ are realizable as the detour radius and detour diameter, respectively, of some connected graph. This theorem can also be extended so that the upper connected vertex detour number can be prescribed when $a < b \le 2a$.

Theorem 2.20. *For positive integers R, D and $n \ge 3$ with $R < D \le 2R$, there exists a connected graph G with $rad_D G = R$, $diam_D G = D$ and $cd_x^+(G) = n$ for some vertex x in G.*

Proof. If $R = 1$, then $D = 2$. Take $G = K_{1,n}$. Then by Theorem 2.12(ii), $cd_x^+(G) = n$ for an end vertex x in G. Now, let $R \ge 2$. We construct a graph G with the desired properties as follows.

Let $C_{R+1} : v_1, v_2, \ldots, v_{R+1}, v_1$ be a cycle of order $R + 1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let H be a graph obtained from C_{R+1} and P_{D-R+1} by identifying v_1 in C_{R+1} and u_0 in P_{D-R+1}. Now, add $n - 2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H by joining each vertex $w_i (1 \le i \le n - 2)$ to the vertex u_{D-R-1} and obtain the graph G of Figure 2.5. Now $rad_D G = R$, $diam_D G = D$ and G has $n - 1$ end vertices. Let $S = \{w_1, w_2, \ldots, w_{n-2}, u_{D-R}\}$ be the set of all end vertices of G. Let $x = v_2$. Then by Theorem 2.4, every minimal connected x-detour set of G contains S. Also, it follows that every minimal connected x-detour set of G contains the cut vertex u_{D-R-1}. But it is clear that $S \bigcup \{u_{D-R-1}\}$ is a connected x-detour set of G and so $S \bigcup \{u_{D-R-1}\}$ is the unique minimal connected x-detour set of G so that $cd_x^+(G) = n$. □

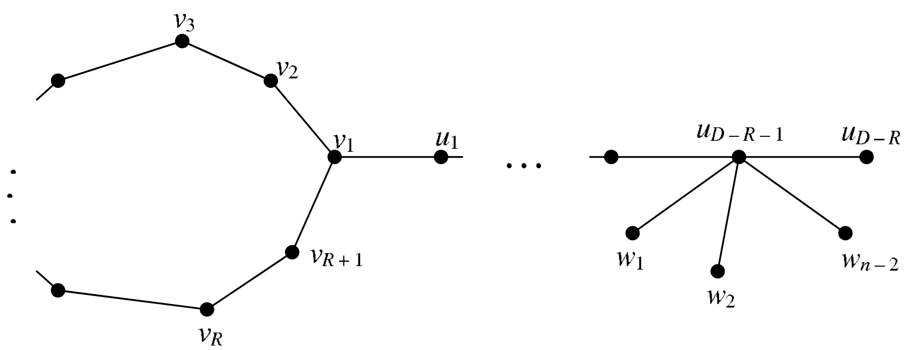

Figure 2.5: A graph G in Theorem 2.20 with $rad_D G = R$, $diam_D G = D$ and $cd_x^+(G) = n$.

The graph G of Figure 2.5 is the smallest graph with the properties described in Theorem 2.20. We leave the following problem as an open question.

Problem 2.21. *For positive integers R, D and $n \geq 3$ with $R \leq D$, does there exist a connected graph G with $rad_D G = R$, $diam_D G = D$ and $cd_x^+(G) = n$ for some vertex x of G?*

In the following, we construct a graph of prescribed order, detour diameter and upper connected vertex detour number under suitable conditions.

Theorem 2.22. *For each triple D, n and p of integers with $4 \leq D \leq p-1$ and $3 \leq n \leq p$, there is a connected graph G of order p, detour diameter D and $cd_x^+(G) = n$ for some vertex x of G.*

Proof. We prove this theorem by considering three cases.

Case 1. Suppose $3 \leq n \leq p-D+1$. Let G be a graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by (i) adding $n-1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining each vertex $v_i (1 \leq i \leq n-1)$ to u_1 and (ii) adding $p-D-n+1$ new vertices $w_1, w_2, \ldots, w_{p-D-n+1}$ and joining each vertex $w_i (1 \leq i \leq p-D-n+1)$ to both u_1 and u_3. The graph G has order p and detour diameter D and is shown in Figure 2.6(i). Let $S = \{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all end vertices of G. Let $x = u_D$. Then by an argument similar to the proof of Theorem 2.20, $S \bigcup \{u_1\}$ is the unique minimal connected x-detour set of G so that $cd_x^+(G) = n$.

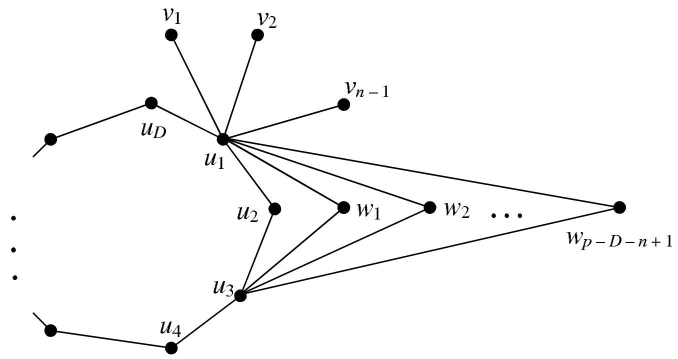

Figure 2.6(i) : A graph G in Case 1 of Theorem 2.22.

Case 2. Suppose $p-D+2 \leq n \leq p-1$. Let $P_{D+1} : u_0, u_1, u_2, \ldots, u_D$ be a path of length D. Add $p-D-1$ new vertices $v_1, v_2, \ldots, v_{p-D-1}$ to P_{D+1} and join each $v_i (1 \leq i \leq p-D-1)$ to u_{p-n}, there by producing the graph G of Figure 2.6(ii). The graph G has order p and detour diameter D. Since G is a tree, by Theorem 2.12(ii), $cd_x^+(G) = p - (p-n) = n$ for the vertex $x = u_0$.

Case 3. Suppose $n = p$. Let G be any tree of order p and detour diameter D. Then by Theorem 2.12(i), $cd_x^+(G) = p$ for any cut vertex x in G. □

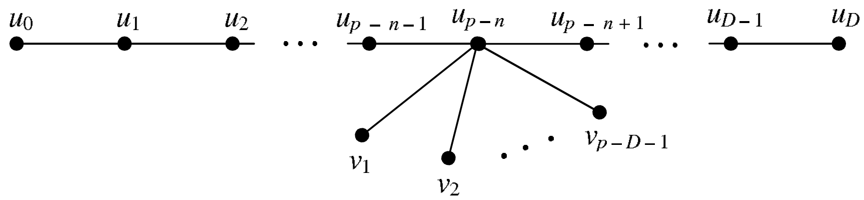

Figure 2.6(ii) : A graph G in Case 2 of Theorem 2.22.

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