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A regularized method for two dimensional nonlinear heat equation backward in time

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Abstract. In this paper, a non-local boundary value problem method for solving 2-D nonlinear heat equation backward in time is given. Some error estimates between the exact solution and its regularization approximation are provided and numerical examples show that the method works effectively.

1. Introduction

Let T be a positive number. We consider the problem of finding the tempereture $u(x,y,t),\ (x,y,t) \in I \times [0,T]$ such that

$$u_t - \Delta u = f(x,y,t,u(x,y,t)) \qquad (x,y,t) \in I \times (0,T), I = (0,\pi)\times(0,\pi) \tag{1}$$

$$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0 \qquad t \in [0,T] \tag{2}$$

$$u(x,y,T) = \varphi(x,y) \qquad x,y \in I. \tag{3}$$

where $\varphi(x,y), f(x,y,t,z))$ are given. This is a typical example of the inverse and ill-posed problem and for its applications we refer to various excellent literature, e.g. Latt'es-Lions [10] and Tikhonov-Arsenin [16].

As is known, the problem is severely ill-posed, i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It makes difficult to numerical calculations. Hence, a regularization is in order. The linear case was studied extensively in the last four decades by many methods. The literature related to the problem is impressive (see, e.g. [1, 2, 4, 10] and the references therein). In [2], the stochastic methods was used to regularize the problem of finding u

$$\begin{aligned} u'(t) &= -Au(t), \quad t \in [0,T], \\ u(0) &= u^0, \end{aligned}$$

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where A generates an analytic C_0-semigroup. In the pioneering work [10] in 1967, Lattes and Lion present, in a heuristic way, the quasi-reversibility method. They approximates the problem by adding a "corrector" into the main equation. In fact, they considered the problem

$$\begin{aligned} u_t + Au - \epsilon A^* Au &= 0, \quad t \in [0,T], \\ u(T) &= \varphi. \end{aligned}$$

The stability magnitude of the method are of order $e^{c\epsilon^{-1}}$. In [1], the problem was approximated with

$$\begin{aligned} u_t + Au + \epsilon Au_t &= 0, \quad t \in [0,T], \\ u(T) &= \varphi. \end{aligned}$$

The method is useful if we cannot construct clearly the operator A^*. However, the stability order in the case are quite large as in the original quasi-reversibility methods. In [13], using the method, so-called, of stabilized quasi reversibility, K. Miller approximated the problem with

$$\begin{aligned} u_t + f(A)u &= 0, \quad t \in [0,T], \\ u(T) &= \varphi. \end{aligned}$$

He shown that, with appropriate conditions on the "corrector" $f(A)$, the stability magnitude of the method is of order $c\epsilon^{-1}$.

Sixteen years after the pioneering work by Lattes-Lions, in 1983, Showalter [15]presented the quasi-boundary method or non-local boundary value method. He considered the problem

$$\begin{aligned} u_t - Au(t) &= Bu(t), \quad t \in [0,T], \\ u(0) &= \varphi, \end{aligned}$$

and approximated the problem with

$$\begin{aligned} u_t - Au(t) &= Bu(t), \quad t \in [0,T], \\ u(0) + \epsilon u(T) &= \varphi. \end{aligned}$$

where B is a adjoint operator. In his opinion, this method gave a better stability estimate than the other discussed methods. Clark and Oppenheimer, in their paper [4], used the non-local boundary value method to regularize the backward problem with

$$\begin{aligned} u_t + Au(t) &= 0, \quad t \in [0,T], \\ u(T) + \epsilon u(0) &= \varphi. \end{aligned}$$

The authors shown that the stability estimate of the method is of order ϵ^{-1}. Recently, this method has been used effectively in solving the homogeneous parabolic equation backward in time (See [7–9]).

Although there are many papers on the linear homogeneous case of the backward problem, but we only find a few papers on the nonhomogeneous case, and especially, the 2-D nonlinear case of their is very scarce. For the 2-D homogeneous case, we refer the reader to the results in [3, 12]. Very recently, a linear nonhomogeneous case $f(x,y,t,u(x,y,t)) = f(x,y,t)$ of the problem (1)-(3) has been considered by QBV method [17] and truncation method [18].

In the present paper, we extend the results in [7–9, 17, 18] to the nonlinear case. The 1-D nonlinear case of the problem (1)-(3) is studied in [18]. We shall use the non-local boundary value method to regularize the nonlinear problem. We approximate the problem (1)-(3) by the following problem:

$$u_t^\epsilon - \Delta u^\epsilon = \sum_{n,m=1}^{\infty} \frac{e^{-t(n^2+m^2)}}{\epsilon^{t/T} + e^{-t(n^2+m^2)}} f_{nm}(u^\epsilon)(t) \sin nx \sin my \quad (x,y,t) \in I \times (0,T) \tag{4}$$

$$u^\epsilon(0,y,t) = u^\epsilon(\pi,y,t) = u^\epsilon(x,0,t) = u^\epsilon(x,\pi,t) = 0 \quad (x,y,t) \in I \times [0,T] \tag{5}$$

$$u^\epsilon(x,y,T) + \epsilon u^\epsilon(x,y,0) = \varphi(x,y) - \\ - \sum_{n,m=1}^{\infty}\left(\int_0^T \frac{\epsilon}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)ds\right)\sin nx \sin my \quad (x,y)\in I \tag{6}$$

where $0<\epsilon<1$, $f_{nm}(u)(t) = \frac{4}{\pi^2} < f(x,y,t,u(x,y,t)), \sin nx \sin my >$, $< .,. >$ is inner product in $L^2(I)$. We note the reader that if $f=0$, then the problem (4)-(6) has been considered in [4] under the same form. Moreover, this problem is different as compared to the problem (7)-(9) in [17] (See p.874).

The paper is organized as follows. First we shall show that (4)-(6) is well posed and has a unique solution u^ϵ. Then we also estimate error between a exact solution u of Problem (1)-(3) and approximation solution u^ϵ. Finally, a numerical experiment is given.

2. The well-posedness of regularized problem

Through out this paper, we denote $\|.\|$ be the norm in $L^2(I)$. In the section, we shall study the existence, the uniqueness and the stability of a solution of Problem (4)-(6). In fact, one has

Theorem 2.1. *Let $\varphi \in L^2(I)$ and let $f \in L^\infty([0,\pi]\times[0,\pi]\times[0,T]\times R)$ satisfy*

$$|f(x,y,t,w) - f(x,t,y,u)| \le k|w-u|$$

for a $k>0$ independent of x,y,t,w,u.

Then Problem (4)-(6) has a unique solution $u^\epsilon \in C([0,T];H_0^1(I))\cap C^1((0,T);L^2(I))$ satisfying the following integral equation

$$u^\epsilon(x,y,t) \\ = \sum_{n,m=1}^{\infty}\left(\frac{e^{-t(n^2+m^2)}}{\epsilon+e^{-T(n^2+m^2)}}\varphi_{nm} - \int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)ds\right)\sin nx \sin my \tag{7}$$

where

$$\varphi_{nm} = \frac{4}{\pi^2} < \varphi(x,y), \sin nx \sin my > .$$

The solution also depends continuously on φ.

Proof. The proof is divided into three steps. In Step 1, we shall prove that Problem (4)-(6) is equivalence to problem (7). In Step 2, we prove the existence and the uniqueness of a solution of (7). Finally in Step 3, the stability of the solution is given.

Step 1. Prove that (4)-(6) is equivalence (7)

We divide this Step into two parts.

Part A If u^ϵ satisfies (7) then u^ϵ is the solution of (4)-(6).

We have:

$$u^\epsilon(x,y,t) = \\ = \sum_{n,m=1}^{\infty}\left(\frac{e^{-t(n^2+m^2)}}{\epsilon+e^{-T(n^2+m^2)}}\varphi_{nm} - \int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)ds\right). \\ . \sin nx \sin my \quad 0\le t\le T. \tag{8}$$

This implies that

$$\begin{aligned} u_t^\epsilon(x,y,t) &= \sum_{n,m=1}^{\infty}\left(\frac{-(n^2+m^2)e^{-t(n^2+m^2)}}{\epsilon+e^{-T(n^2+m^2)}}\varphi_{nm} - \int_t^T \frac{-(n^2+m^2)e^{-t(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)ds \right. \\ &\quad \left. + \frac{e^{-t(n^2+m^2)}}{\epsilon^{t/T}+e^{-t(n^2+m^2)}} f_{nm}(u^\epsilon)(t)\right)\sin nx \sin my \\ &= \Delta u^\epsilon + \sum_{n,m=1}^{\infty} \frac{e^{-t(n^2+m^2)}}{\epsilon^{t/T}+e^{-t(n^2+m^2)}} f_{nm}(u^\epsilon)(t)\sin nx\sin my \end{aligned} \tag{9}$$

and

$$\begin{aligned} \epsilon u^\epsilon(x,y,0) &+ u^\epsilon(x,y,T) \\ &= \left[\sum_{n,m=1}^{\infty}\epsilon\left(\frac{1}{\epsilon+e^{-T(n^2+m^2)}}\varphi_{nm} - \int_0^T \frac{1}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)\right)ds \right. \\ &\quad \left. + \sum_{n,m=1}^{\infty}\frac{e^{-T(n^2+m^2)}}{\epsilon+e^{-T(n^2+m^2)}}\varphi_{nm}\right]\sin nx\sin my \\ &= \varphi(x,y) - \sum_{n,m=1}^{\infty}\left(\int_0^T \frac{\epsilon}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)ds\right)\sin nx\sin my. \end{aligned} \tag{10}$$

So u^ϵ is the solution of (4)–(6).

Part B If u^ϵ satisfies (4)–(6) then u^ϵ is the solution of (7).

Infact, taking the inner product the equation (4) with respect to $\sin nx \sin my$ we get in view of (4)

$$\frac{d}{dt}u_{nm}^\epsilon(t) + (n^2+m^2)u_{nm}^\epsilon(t) = \frac{e^{-t(n^2+m^2)}}{\epsilon^{t/T}+e^{-t(n^2+m^2)}} f_{nm}(u^\epsilon)(t) \tag{11}$$

where we recall that

$$u_{nm}^\epsilon(t) = \frac{4}{\pi^2} < u^\epsilon(x,y,t), \sin nx \sin my >,$$

$$f_{nm}(u^\epsilon)(t) = \frac{4}{\pi^2} < f(x,y,t,u^\epsilon(x,y,t)), \sin nx\sin my > .$$

It follows that

$$\begin{aligned} u_{nm}^\epsilon(t) = &\quad e^{-t(n^2+m^2)}u_{nm}^\epsilon(0) \\ &+ \int_0^t e^{-(t-s)(n^2+m^2)} \frac{e^{-s(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2+m^2)}} f_{nm}(u^\epsilon)(s)ds \end{aligned} \tag{12}$$

Hence, we have the Fourier expansion

$$\begin{aligned} u^{\epsilon}(x,y,t) &= \sum_{n,m=1}^{\infty} \Big(e^{-t(n^2+m^2)} u^{\epsilon}_{nm}(0) + \\ & \int_0^t e^{-(t-s)(n^2+m^2)} \frac{e^{-s(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} f_{nm}(u^{\epsilon})(s) ds \Big) \sin nx \sin my \\ &= \sum_{n,m=1}^{\infty} \Big(e^{-t(n^2+m^2)} u^{\epsilon}_{nm}(0) \\ & + \int_0^t \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} f_{nm}(u^{\epsilon})(s) ds \Big) \sin nx \sin my. \end{aligned} \tag{13}$$

Hence

$$\begin{aligned} u^{\epsilon}(x,y,T) &= \sum_{n,m=1}^{\infty} \Big(e^{-T(n^2+m^2)} u^{\epsilon}_{nm}(0) \\ & + \int_0^T \frac{e^{-T(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} f_{nm}(u^{\epsilon})(s) ds \Big) \sin nx \sin my. \end{aligned} \tag{14}$$

This implies that

$$\sum_{n,m=1}^{\infty} ((\epsilon + e^{-T(n^2+m^2)}) u^{\epsilon}_{nm}(0)) \sin nx \sin my = \varphi(x,y) -$$

$$- \sum_{n,m=1}^{\infty} \left(\int_0^T \frac{\epsilon + e^{-T(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} f_{nm}(u^{\epsilon})(s) ds \right) \sin nx \sin my.$$

We obtain

$$u^{\epsilon}_{nm}(0) = \frac{1}{\epsilon + e^{-T(n^2+m^2)}} \varphi_{nm} - \int_0^T \frac{1}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} f_{nm}(u^{\epsilon})(s) ds \qquad \forall n, m. \tag{15}$$

Replacing (15) in (13), we shall receive (7).
This completes the proof of Step 1.

Step 2. The existence and the uniqueness of solution of (7)

Put

$$G(u)(x,y,t) = \Psi(x,y,t) - \sum_{n,m=1}^{\infty} \left(\int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} f_{nm}(u)(s) ds \right) \sin nx \sin my$$

where $\Psi(x,y,t) = \sum_{n,m=1}^{\infty} \frac{e^{-t(n^2+m^2)}}{\epsilon + e^{-T(n^2+m^2)}} \varphi_{nm} \sin nx \sin my$.

We claim that

$$\|G^p(u)(.,.,t) - G^p(v)(.,.,t)\|^2 \le \left(\frac{k}{\epsilon}\right)^{2p} \frac{(T-t)^p C^p}{p!} |||u - v|||^2 \tag{16}$$

for every $p \ge 1$, and $G^p(u) = G(G...G(u))$ for p times, $C = \max\{T, 1\}$ and $|||.|||$ is sup norm in $C([0,T]; L^2(I))$.

We shall prove the latter inequality by induction.

For $p = 1$, we have

$$\begin{aligned}
\|G(u)(.,.,t) - G(v)(.,.,t)\|^2 &= \frac{\pi^2}{4}\sum_{n,m=1}^{\infty}\left[\int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2+m^2)}}(f_{nm}(u)(s)-f_{nm}(v)(s))ds\right]^2\\
&\le \frac{\pi^2}{4}\sum_{n,m=1}^{\infty}\int_t^T\left(\frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2+m^2)}}\right)^2 ds\int_t^T (f_{nm}(u)(s)-f_{nm}(v)(s))^2 ds\\
&\le \frac{\pi^2}{4}\sum_{n,m=1}^{\infty}\frac{1}{\epsilon^2}(T-t)\int_t^T (f_{nm}(u)(s)-f_{nm}(v)(s))^2 ds =\\
&= \frac{1}{\epsilon^2}(T-t)\int_t^T\int_I (f(x,y,s,u(x,y,s))-f(x,y,s,v(x,y,s)))^2 dxdyds\\
&\le \frac{k^2}{\epsilon^2}(T-t)\int_t^T\int_I |u(x,y,s)-v(x,y,s)|^2 dxdyds\\
&\le C\frac{k^2}{\epsilon^2}(T-t)|||u-v|||^2.
\end{aligned}$$

Thus (16) holds. Suppose that (16) holds for $p = j$. We prove that (16) holds for $p = j + 1$. We have

$$\begin{aligned}
\|G^{j+1}(u)(.,.,t) \quad - \quad & G^{j+1}(v)(.,.,t)\|^2 =\\
&= \frac{\pi^2}{4}\sum_{n,m=1}^{\infty}\left[\int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T}+e^{-s(n^2}+m^2)}\left(f_{nm}(G^j(u))(s)-f_{nm}(G^j(v))(s)\right)ds\right]^2\\
&\le \frac{\pi^2}{4}\frac{1}{\epsilon^2}\sum_{n,m=1}^{\infty}\left[\int_t^T |f_{nm}(G^j(u))(s)-f_{nm}(G^j(v))(s)|ds\right]^2\\
&\le \frac{\pi^2}{4}\frac{1}{\epsilon^2}(T-t)\int_t^T\sum_{n,m=1}^{\infty}|f_{nm}(G^j(u))(s)-f_{nm}(G^j(v))(s)|^2 ds\\
&\le \frac{1}{\epsilon^2}(T-t)k^2\int_t^T \|G^j(u)(.,.,s)-G^j(v)(.,.,s)\|^2 ds\\
&\le \frac{1}{\epsilon^2}(T-t)k^2\left(\frac{k}{\epsilon}\right)^{2j}\int_t^T\frac{(T-s)^j}{j!}dsC^j|||u-v|||^2\\
&\le \left(\frac{k}{\epsilon}\right)^{2(j+1)}\frac{(T-t)^{j+1}}{(j+1)!}C^{j+1}|||u-v|||^2.
\end{aligned}$$

Therefore

$$|||G^p(u)-G^p(v)||| \le \left(\frac{k}{\epsilon}\right)^p\frac{T^{p/2}}{\sqrt{p!}}C^p|||u-v|||$$

for all $u, v \in C([0,T]; L^2(I))$.

We consider $G : C([0,T]; L^2(I)) \to C([0,T]; L^2(I))$.

Since $\lim_{p\to\infty}\left(\frac{k}{\epsilon}\right)^p \frac{T^{p/2}C^p}{\sqrt{p!}} = 0$, there exists a positive integer number p_0, such that G^{p_0} is a contraction.
It follows that the equation $G^{p_0}(u) = u$ has a unique solution $u^\epsilon \in C([0,T];L^2(I))$.
We claim that $G(u^\epsilon) = u^\epsilon$. In fact, one has

$$G(G^{P_0}(u^\epsilon)) = G(u^\epsilon).$$

Hence

$$G^{P_0}(G(u^\epsilon)) = G(u^\epsilon).$$

By the uniqueness of the fixed point of G^{P_0}, one has $G(u^\epsilon) = u^\epsilon$, i.e., the equation $G(u) = u$ has a unique solution $u^\epsilon \in C([0,T];L^2(I))$.
From Part A, we complete the proof of Step 2.
Step 3. The solution of the problem (4) – (6) depends continuously on φ in $L^2(I)$.
Let u and v be two solutions of (4) – (6) corresponding to the values φ and ω.
From (7) one has

$$\begin{aligned}
&\|u(.,.,t) - v(.,.,t)\|^2 \\
= &\ \frac{\pi^2}{4}\sum_{n,m=1}^{\infty}\left|\frac{e^{-t(n^2+m^2)}}{\epsilon + e^{-T(n^2+m^2)}}(\varphi_{nm} - \omega_{nm}) - \int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}}(f_{nm}(u)(s) - f_{nm}(v)(s)ds)\right|^2 \\
\le &\ \frac{\pi^2}{2}\sum_{n,m=1}^{\infty}\left(\frac{e^{-t(n^2+m^2)}}{\epsilon + e^{-T(n^2+m^2)}}|\varphi_{nm} - \omega_{nm}|\right)^2 \\
&+\frac{\pi^2}{2}\sum_{n,m=1}^{\infty}\left(\int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}}|f_{nm}(u)(s) - f_{nm}(v)(s)|ds\right)^2
\end{aligned} \tag{17}$$

One has for $s > t$ and $\alpha > 0$

$$\begin{aligned}
\frac{e^{-t(n^2+m^2)}}{\alpha + e^{-s(n^2+m^2)}} &= \frac{e^{-t(n^2+m^2)}}{(\alpha + e^{-s(n^2+m^2)})^{t/s}(\alpha + e^{-s(n^2+m^2)})^{1-t/s}} \\
&= \frac{1}{(\alpha e^{s(n^2+m^2)} + 1)^{t/s}(\alpha + e^{-s(n^2+m^2)})^{1-t/s}} \\
&\le \alpha^{t/s-1}.
\end{aligned}$$

Letting $\alpha = \epsilon, s = T$, we get

$$\frac{e^{-t(n^2+m^2)}}{\epsilon + e^{-T(n^2+m^2)}} \le \epsilon^{t/T-1}. \tag{18}$$

Letting $\alpha = \epsilon^{s/T}$, we get

$$\frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} \le \epsilon^{t/T-s/T}. \tag{19}$$

Hence, from (19) it follows that

$$\begin{aligned}
\|u(.,.,t) - v(.,.,t)\|^2 \;\le\;& 2\epsilon^{2(t/T-1)}\|\varphi - \omega\|^2 \\
+\;& 2k^2(T-t)\epsilon^{2t/T}\int_t^T \epsilon^{-2s/T}\|u(.,.,s) - v(.,.,s)\|^2 ds.
\end{aligned}$$

Hence

$$\begin{aligned}\epsilon^{-2t/T}\|u(.,.,t)-v(.,.,t)\|^2 &\le \epsilon^{-2}\|\varphi-\omega\|^2\\ &+ 2k^2(T-t)\int_t^T \epsilon^{-2s/T}\|u(.,.,s)-v(.,.,s)\|^2 ds.\end{aligned}$$

Using Gronwall's inequality we have

$$\|u(.,.,t)-v(.,.,t)\| \le 2\epsilon^{t/T-1}\exp(k^2(T-t)^2)\|\varphi-\omega\|.$$

This completes the proof of Step 3 and the proof of our theorem. □

3. Regularization and error estimate.

We first have a uniqueness result

Theorem 3.1. *Let φ, f be as in Theorem 2.1. If $\frac{\partial f}{\partial z}(x,y,t,z)$ is bounded on $I\times(0,T)\times R$ then Problem (1) − (3) has at most one solution $u \in C([0,T];H_0^1(I)) \cap C^1((0,T);L^2(I))$.*

Proof. Let $M > 0$ be such that

$$\left|\frac{\partial f}{\partial z}(x,y,t,z)\right| \le M$$

for all $(x,y,t,z) \in I\times(0,T)\times R$.

Let $u_1(x,y,t)$ and $u_2(x,y,t)$ be two solutions of Problem (1) − (3) such that $u_1, u_2 \in C([0,T];H_0^1(I)) \cap C^1((0,T);L^2(I))$.

Put $w(x,y,t) = u_1(x,y,t) - u_2(x,y,t)$.

Then w satisfies the equation

$$w_t(x,y,t) - \Delta w(x,y,t) = f(x,y,t,u_1(x,y,t)) - f(x,y,t,u_2(x,y,t)).$$

Thus

$$w_t(x,y,t) - \Delta w(x,y,t) = \frac{\partial f}{\partial z}(x,y,t,\overline{u}(x,y,t))w(x,y,t),$$

for some $\overline{u}(x,y,t)$.

It follows that

$$(w_t - \Delta w)^2 \le M^2 w^2.$$

Now $w(0,y,t) = w(\pi,y,t) = w(x,0,t) = w(x,\pi,t) = 0$ and $w(x,y,T) = 0$. Hence by the Lees-Protter theorem [5], p. 373,

$$w = 0$$

which gives $u_1(x,y,t) = u_2(x,y,t)$ for all $t \in [0,T]$. The proof is completed. □

Despite the uniqueness, Problem (1)-(3) is still ill-posed.

Hence, a regularization has to resort. We have the following result

Theorem 3.2. *Let φ, f be as in Theorem 2.1. Suppose Problem (1)-(3) has a unique solution $u(x,y,t)$ in $C([0,T];H_0^1(I)) \cap C^1((0,T);L^2(I))$ which satisfies $\int_0^T \sum_{n,m=1}^{\infty} e^{2s(n^2+m^2)} f_{nm}^2(u)(s)ds < \infty$. Then*

$$\|u(.,.,t) - u^\epsilon(.,.,t)\| \le \sqrt{M}exp\left(\frac{3k^2T(T-t)}{2}\right)\epsilon^{t/T}$$

for every $t \in [0,T]$, *where* $M = 3\|u(0)\|^2 + 3\pi^2 T \int_0^T \sum_{n,m=1}^{\infty} e^{2s(n^2+m^2)} f_{nm}^2(u)(s)ds$ *and* u^ϵ *is the unique solution of Problem (4)-(6).*

Remark 3.3. 1. If $f(x,y,t,u(x,y,t)) = 0$ then the error in this Theorem is similar to the results obtained in [4].
2. If the final value φ satisfies the condition $\sum_{n,m=1}^{\infty} e^{2T(n^2+m^2)}\varphi_{nm}^2 < \infty$, then by direct transform, the condition on f in Theorem 3.2 is accepted.

Proof. Suppose the Problem (1)-(3) has an exact solution $u \in C([0,T]; H_0^1(I)) \cap C^1((0,T); L^2(I))$, we get the following formula

$$u(x,y,t) = \sum_{n,m=1}^{\infty} \Big(e^{-(t-T)(n^2+m^2)}\varphi_{nm} - \int_t^T e^{-(t-s)(n^2+m^2)} f_{nm}(u)(s)ds\Big) \sin nx \sin my \tag{20}$$

and

$$\begin{aligned} u(x,y,T) &= \sum_{n,m=1}^{\infty} \Big(e^{-T(n^2+m^2)} u_{nm}(0) + \int_0^T e^{-(T-s)(n^2+m^2)} f_{nm}(u)(s)ds\Big) \sin nx \sin my \\ &= \sum_{n,m=1}^{\infty} \varphi_{nm} \sin nx \sin my \end{aligned}$$

where $u_{nm}(0) = \frac{4}{\pi^2} < u(x,y,0), \sin nx \sin my >$ (see [2]).
Hence $e^{-T(n^2+m^2)}u_{nm}(0) + \int_0^T e^{-(T-s)(n^2+m^2)} f_{nm}(u)(s)ds = \varphi_{nm}$.
From (7), (21) and (20), we get

$$\begin{aligned} &|u_{nm}(t) - u_{nm}^\epsilon(t)| \\ = &\Big|\frac{\epsilon e^{-t(n^2+m^2)}}{e^{-T(n^2+m^2)}(\epsilon + e^{-T(n^2+m^2)})}\varphi_{nm} - \int_t^T \frac{\epsilon^{s/T} e^{-t(n^2+m^2)}}{e^{-s(n^2+m^2)}(\epsilon^{s/T} + e^{-s(n^2+m^2)})} f_{nm}(u)(s)ds \\ &- \int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} (f_{nm}(u)(s) - f_{nm}(u^\epsilon)(s))ds\Big| \\ \le &\Big|\frac{\epsilon e^{-t(n^2+m^2)}}{\epsilon + e^{-T(n^2+m^2)}} u_{nm}(0) + \int_0^T \frac{\epsilon e^{-t(n^2+m^2)}}{e^{-s(n^2+m^2)}(\epsilon + e^{-T(n^2+m^2)})} f_{nm}(u)(s)ds \\ &- \int_t^T \frac{\epsilon^{s/T} e^{-t(n^2+m^2)}}{e^{-s(n^2+m^2)}(\epsilon^{s/T} + e^{-s(n^2+m^2)})} f_{nm}(u)(s)ds\Big| \\ &+ \int_t^T \frac{e^{-t(n^2+m^2)}}{\epsilon^{s/T} + e^{-s(n^2+m^2)}} |f_{nm}(u)(s) - f_{nm}(u^\epsilon)(s)|ds. \end{aligned} \tag{21}$$

From (20)-(21) and (22), we have

$$\begin{aligned}
|u_{nm}(t)-u^{\epsilon}_{nm}(t)| &\leq \epsilon.\epsilon^{t/T-1}|u_{nm}(0)|+\int_0^T \epsilon.\epsilon^{t/T-1}|\frac{f_{nm}(u)(s)}{e^{-s(n^2+m^2)}}|ds\\
&\quad +\int_t^T \epsilon^{s/T}.\epsilon^{t/T-s/T}|\frac{f_{nm}(u)(s)}{e^{-s(n^2+m^2)}}|ds+\int_t^T \epsilon^{t/T-s/T}|f_{nm}(u)(s)-f_{nm}(u^{\epsilon})(s)|ds\\
&\leq \epsilon^{t/T}|u_{nm}(0)|+2\epsilon^{t/T}\int_0^T |\frac{f_{nm}(u)(s)}{e^{-s(n^2+m^2)}}|ds\\
&\quad +\epsilon^{t/T}\int_t^T \epsilon^{-s/T}|f_{nm}(u)(s)-f_{nm}(u^{\epsilon})(s)|ds.
\end{aligned}$$

We have in view of the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$

$$\begin{aligned}
&\|u(.,.,t)-u^{\epsilon}(.,.,t)\|^2\\
=\ & \frac{\pi^2}{4}\sum_{n,m=1}^{\infty}|u_{nm}(t)-u^{\epsilon}_{nm}(t)|^2\\
\leq\ & \frac{3\pi^2}{4}\sum_{n,m=1}^{\infty}\epsilon^{2t/T}|u_{nm}(0)|^2+3\pi^2\sum_{n,m=1}^{\infty}\epsilon^{2t/T}(\int_0^T|\frac{1}{e^{-s(n^2+m^2)}}f_{nm}(u)(s)|ds)^2+\\
& \frac{3\pi^2}{4}\sum_{n,m=1}^{\infty}\epsilon^{2t/T}(\int_t^T\epsilon^{-s/T}|f_{nm}(u)(s)-f_{nm}(u^{\epsilon})(s)|ds)^2\\
\leq\ & 3\epsilon^{2t/T}\|u(0)\|^2+3\pi^2T\epsilon^{2t/T}\int_0^T\sum_{n,m=1}^{\infty}(\frac{1}{e^{-s(n^2+m^2)}}f_{nm}(u)(s))^2ds+\\
& \frac{3\pi^2}{4}(T-t)\epsilon^{2t/T}\int_t^T\epsilon^{-2s/T}\sum_{n,m=1}^{\infty}(f_{nm}(u)(s)-f_{nm}(u^{\epsilon})(s))^2ds\\
\leq\ & 3\epsilon^{2t/T}\|u(0)\|^2+3\pi^2T\epsilon^{2t/T}\int_0^T\sum_{n,m=1}^{\infty}e^{2s(n^2+m^2)}f^2_{nm}(u)(s)ds+\\
& 3(T-t)\epsilon^{2t/T}\int_t^T\epsilon^{-2s/T}\|f(.,.,s,u(.,.,s))-f(.,.,s,u^{\epsilon}(.,.,s))\|^2ds\\
\leq\ & \epsilon^{2t/T}(3\|u(0)\|^2+3\pi^2T\int_0^T\sum_{n,m=1}^{\infty}e^{2s(n^2+m^2)}f^2_{nm}(u)(s)ds+\\
& 3k^2T\int_t^T\epsilon^{-2s/T}\|u(.,.,s)-u^{\epsilon}(.,.,s)\|^2ds.
\end{aligned}$$

Hence

$$\epsilon^{-2t/T}\|u(.,.,t)-u^{\epsilon}(.,.,t)\|^2 \leq M+3k^2T\int_t^T \epsilon^{-2s/T}\|u(.,.,s)-u^{\epsilon}(.,.,s)\|^2 ds$$

where $M=3\|u(0)\|^2+3\pi^2T\int\limits_0^T\sum\limits_{n,m=1}^{\infty}e^{2s(n^2+m^2)}f_{nm}^2(u)(s)ds$.
By using Gronwall's inequality, we get:

$$\epsilon^{-2t/T}\|u(.,.,t)-u^{\epsilon}(.,.,t)\|^2 \leq Me^{3k^2T(T-t)}.$$

Finally

$$\|u(.,.,t)-u^{\epsilon}(.,.,t)\|^2 \leq Me^{3k^2T(T-t)}\epsilon^{2t/T}.$$

This completes the proof of Theorem 3.2. □

One has

Theorem 3.4. *Let φ, f be as in Theorem 2.1 and let $u\in C([0,T];H_0^1(I))\cap C^1((0,T);L^2(I))$ be a solution of Problem (1)-(3) such that $\frac{\partial u}{\partial t}\in L^2((0,T);L^2(I))$ and $\int\limits_0^T\sum\limits_{n,m=1}^{\infty}e^{2s(n^2+m^2)}f_{nm}^2(u)(s)ds<\infty$. Then for all $\epsilon>0$ there exists a t_ϵ such that*

$$\|u(.,.,0)-u^{\epsilon}(.,.,t_\epsilon)\|\leq \sqrt[4]{8}C\sqrt[4]{T}\left(ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}$$

where

$$N=\sqrt{\int\limits_0^T\left\|\frac{\partial u}{\partial t}(.,.,s)\right\|^2ds},$$

$$C=\max\left\{\sqrt{3\|u_0(.,.,0)\|^2+3\pi^2T\int\limits_0^T\sum_{n,m=1}^{\infty}e^{2s(n^2+m^2)}f_{nm}^2(u)(s)ds}\,\exp\left(\frac{3k^2T^2}{2}\right),N\right\}$$

and u^ϵ is the unique solution of Problem (4)-(6).

Proof. First, using the Galerkin method (see, e.g.,[10]), we can show that the assumption on u_t holds if $u(.,.,0)\in H_0^1(I)$.
We have

$$u(x,y,t)-u(x,y,0)=\int\limits_0^t\frac{\partial u}{\partial s}(x,y,s)ds$$

It follows that

$$\|u(.,.,0)-u(.,.,t)\|^2\leq t\int\limits_0^t\left\|\frac{\partial u}{\partial t}(.,.,s)\right\|^2ds\leq N^2t.$$

Using Theorem 3.2, we have

$$\begin{aligned}\|u(.,.,0)-u^{\epsilon}(.,.,t)\| &\leq \|u(.,.,0)-u(.,.,t)\|+\|u(.,.,t)-u^{\epsilon}(.,.,t)\|\\ &\leq C(\sqrt{t}+\epsilon^{t/T}).\end{aligned}$$

For every ϵ, there exists t_ϵ such that $\sqrt{t_\epsilon} = \epsilon^{t_\epsilon/T}$, i.e. $\dfrac{lnt_\epsilon}{t_\epsilon} = \dfrac{2ln\epsilon}{T}$.
Using inequality $\ln t > -\frac{1}{t}$ for every $t > 0$, we get

$$\|u(.,.,0) - u^\epsilon(.,.,t_\epsilon)\| \le \sqrt[4]{8}C\sqrt[4]{T}\left(ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}.$$

This completed the proof of Theorem 3.4. □

Theorem 3.5. *Let φ, f be as in Theorem 2.1. Assume that the exact solution u of (1)–(3) corresponding to φ satisfies*

$$u \in C([0,T];L^2(I)) \cap L^2(0,T;H_0^1(I)) \cap C^1((0,T);L^2(I)),$$

$$\frac{\partial u}{\partial t} \in L^2((0,T);L^2(I))$$

and

$$\int_0^T \sum_{n=1}^\infty e^{2s(n^2+m^2)} f_{nm}^2(u)(s)ds < \infty.$$

Let $\varphi_\epsilon \in L^2(I)$ be a measured data such that

$$\|\varphi_\epsilon - \varphi\| \le \epsilon.$$

Then there exists a function U^ϵsatisfying

$$\|U^\epsilon(.,.,t) - u(.,.,t)\| \le (2+\sqrt{M})\exp(\frac{3k^2T(T-t)}{2})\epsilon^{t/T}, \quad for\ \ every\ \ \ t \in (0,T)$$

and

$$\|U^\epsilon(.,.,0) - u(.,.,0)\| \le \sqrt[4]{8}\sqrt[4]{T}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}\left(\exp(k^2T^2) + C\right)$$

where $M = 3\|u(.,.,0)\|^2 + 3\pi^2 T \int\limits_0^T \sum\limits_{n,m=1}^\infty e^{2s(n^2+m^2)} f_{nm}^2(u)(s)ds$ and C is defined in Theorem 3.4.

Proof. Let u^ϵ be the solution of problem (4)-(6) corresponding to φ and let w^ϵ be the solution of problem (4)-(6) corresponding to φ_ϵ where $\varphi, \varphi_\epsilon$ are in right hand side of (6).
Using Theorem 3.4, there exists a t_ϵ such that

$$\sqrt{t_\epsilon} = \epsilon^{t_\epsilon/T} \tag{22}$$

and

$$\|u^\epsilon(.,.,t_\epsilon) - u(.,.,0)\| \le \sqrt[4]{8}C\sqrt[4]{T}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}. \tag{23}$$

Put

$$U^\epsilon(.,.,t) = \begin{cases} w^\epsilon(.,.,t), & 0 < t < T, \\ w^\epsilon(.,.,t_\epsilon), & t = 0 \end{cases}.$$

Using Theorem 3.2 and Step 3 in Theorem 2.1, we get:

$$\begin{aligned} \|u^\epsilon(.,.,t) - u(.,.,t)\| &\le \|w^\epsilon(.,.,t) - u^\epsilon(.,.,t)\| + \|u^\epsilon(.,.,t) - u(.,.,t)\| \\ &\le (2+\sqrt{M})\exp\left(\frac{3k^2T(T-t)}{2}\right)\epsilon^{t/T}, \end{aligned}$$

for every $t \in (0, T)$.
From Step 3 in Theorem 2.1, we have

$$\begin{aligned} \|U^\epsilon(.,.,0) - u(.,.,0)\| &\leq \|w^\epsilon(.,.,t_\epsilon) - u^\epsilon(.,.,t_\epsilon)\| + \|u^\epsilon(.,.,t_\epsilon) - u(.,.,0)\| \\ &\leq 2\epsilon^{t_\epsilon/T}\exp(k^2T^2) + \sqrt[4]{8}C\sqrt[4]{T}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4} \\ &\leq \sqrt[4]{8}\sqrt[4]{T}\left(\ln\left(\frac{1}{\epsilon}\right)\right)^{-1/4}\left(\exp(k^2T^2) + C\right). \end{aligned}$$

This completed the proof of Theorem. □

4. A numerical experiment

In this section, a example is devised for verifying the validity of the proposed method. Our main purpose in this section is to give a simple analytical result. So, we don't present some numerical experiments with random perturbation. We consider

$$-\Delta u + u_t = f(u) + g(x, y, t)$$

where

$$f(u) = \begin{cases} u^4 & u \in [-e^{10}, e^{10}] \\ -\frac{e^{30}}{e-1}u + \frac{e^{41}}{e-1} & u \in (e^{10}, e^{11}] \\ \frac{e^{30}}{e-1}u + \frac{e^{41}}{e-1} & u \in (-e^{11}, -e^{10}] \\ 0 & |u| > e^{11} \end{cases},$$

$$g(x, y, t) = 3e^t \sin x \sin y - e^{4t}\sin^4 x \sin^4 y,$$

and

$$u(x, y, 1) = \varphi_0(x, y) \equiv e \sin x \sin y.$$

The exact solution of the latter equation is

$$u(x, y, t) = e^t \sin x \sin y$$

Especially

$$u\left(x, y, \frac{799}{800}\right) \equiv u(x, y) = \exp\left(\frac{799}{800}\right)\sin x \sin y.$$

Let $\varphi_\epsilon(x, y) \equiv \varphi(x, y) = (\epsilon + 1)e \sin x \sin y$.
We have

$$\|\varphi_\epsilon - \varphi\|_2 = \sqrt{\int_0^\pi \int_0^\pi \epsilon^2 e^2 \sin^2(x)\sin^2 y dx dy} = \epsilon e\frac{\pi}{2}$$

We find the regularized solution $u_\epsilon\left(x, y, \frac{799}{800}\right) \equiv u_\epsilon(x, y)$ having the following form

$$\begin{aligned} u_\epsilon(x, y) = v_m(x, y) &= w_{11,m}\sin x \sin y + w_{12,m}\sin x \sin 2y + w_{13,m}\sin x \sin 3y + \\ &+ w_{21,m}\sin 2x \sin y + w_{22,m}\sin 2x \sin 2y + w_{23,m}\sin 2x \sin 3y + \\ &+ w_{31,m}\sin 3x \sin y + w_{32,m}\sin 3x \sin 2y + w_{33,m}\sin 3x \sin 3y \end{aligned}$$

where

$$v_1(x,y) = (\epsilon+1)e\sin x\sin y$$
$$w_{11,1} = (\epsilon+1)e,$$
$$w_{12,1} = w_{13,1} = w_{21,1} = w_{22,1} = w_{23,1} = w_{31,1} = w_{32,1} = w_{33,1} = 0.$$

and

$$\begin{cases} a = \frac{1}{40000} \\ t_m = 1 - am \quad m = 1,2,...,500 \\ w_{ij,m+1} = \frac{e^{-t_{m+1}(i^2+j^2)}}{\epsilon + e^{-t_m(i^2+j^2)}} w_{ij,m} - \\ \quad - \frac{4}{\pi^2}\int\limits_{t_{m+1}}^{t_m} \frac{e^{-t_{m+1}(i^2+j^2)}}{e^{s/t_m} + e^{-s(i^2+j^2)}} \left(\int\limits_0^\pi\int\limits_0^\pi \left(v_m^4(x,y) + g(x,y,s)\right)\sin ix\sin jydxdy\right)ds, \ \ i,j = 1,2,3. \end{cases}$$

Let $a_\epsilon = \|u_\epsilon - u\|$ be the error between the regularized solution u_ϵ and the exact solution u.
Let $\epsilon = \epsilon_1 = 10^{-5}, \epsilon = \epsilon_2 = 10^{-7}, \epsilon = \epsilon_3 = 10^{-11}$, we have

ϵ	u_ϵ	a_ϵ
$\epsilon_1 = \frac{10^{-4}}{5}$	$2.513141464\sin x\sin y - 0.1629918493.10^{-4}\sin x\sin 3y -$ $-0.1629918493.10^{-4}\sin 3x\sin y + 0.2376014320.10^{-8}\sin 3x\sin 3y$	0.3168997337
$\epsilon_2 = 10^{-5}$	$2.605725660\sin x\sin y - 0.1848200081.10^{-4}\sin x\sin 3y -$ $-0.1848200076.10^{-4}\sin 3x\sin y + 0.2706307134.10^{-8}\sin 3x\sin 3y$	0.1714688215
$\epsilon_3 = 10^{-7}$	$2.700649206\sin x\sin y - 0.9954278845.10^{-4}\sin x\sin 3y -$ $-0.9954278660.10^{-4}\sin 3x\sin y + 0.3645200168.10^{-7}\sin 3x\sin 3y$	0.02236435269

5. Conclusion

The paper studies the a nonlocal boundary value problem for solving 2D nonlinear backward heat equation in a rectangular domain. Some error estimate were derived, and one numerical example was provided. The approach is based on transforming the problem into the Fourier domain. However, this method does not apply to more general domain due to its reliance on the Fourier method. Is there any alternative ways to derive similar estimates without resorting to the Fourier method. Otherwise the approach is of limited interest. In the future, we hope that the regularized problem of finding problem on general domain.

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