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Some new classes of (m, n)-hyperrings

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Abstract. The notion of (m, n)-ary hyperring was introduced by Davvaz at the 10$^{\text{th}}$ AHA congress [9], as the strong distributive structure. In this article we generalize it, by introducing the notion of (m, n)-ary hyperring with inclusive distributivity. We present construction of (m, n)-ary hyperrings associated with binary relations on semigroup. We also state the condition under which there exists (m, n)-ary hyperring of multiendomorphisms for a starting m-ary hypergroup (H, f). Finaly, we analyze connections between the obtained classes of (m, n)-ary hyperrings.

1. Introduction

The hyperstructure theory was introduced by F. Marty at the 8$^{\text{th}}$ Congress of Scandinavian Mathematicians held in 1934. A semihypergroup $(H, \circ)$ is a nonemty set H equipped with a hyperoperation $\circ$, that is a map $\circ : H \times H \to P^*(H)$, where $P^*(H)$ denotes the family of all nonempty subsets of H, and for all $(x, y, z) \in H^3 : x \circ (y \circ z) = (x \circ y) \circ z$. A semihypergroup is called a hypergroup in the sense of Marty [16] if for every $a \in H : a \circ H = H \circ a = H$. In the above definitions, if $A, B \in P^*(H)$, then $A \circ B$ is given by:

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b$$

$x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

A comprehensive review of the theory of hyperstrucutres appears in Corsini [4], Corsini and Leoreanu [7] and Vougiouklis [20]. Since 1934, the hyperstructure theory has had applications to several areas of both pure and applied mathematics. Abouth 70 years later, a suitable generalization of a hypergroup, called an n-ary hypergroup was introduced and studied by Davvaz and Vougiouklis in [12]. Davvaz et al. [11] considered a class of algebraic hypersystems which represent a generalization of semigroups, hypersemiroups and n-ary semigroups. The properties of this class were investigated in [10] and [11]. The notion of (m, n)-ary hyperring was introduced by Davvaz [9] as a triple (R, f, g) such that (R, f) is an m-ary hypergroup, (R, g) is an n-ary hypersemigroup and g is distributive over f in the sense of equality. In this article, by an (m, n)-ary hyperring we mean more general structure in the following sense: we let g to be distributive over f in the sense of inclusion. A subclass of the (m, n)-hyperrings, called Krasner (m, n)-hyperrings was studied by Mirvakili and Davvaz in [17]. Anvariyeh, Mirvakili and Davvaz [1], considered (m, n)-ary hypermodules on (m, n)-ary hyperring.

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If $(H, \oplus)$ is a commutative binary hypergroup and $F(H)$ the set of multiendomorphisms of H i.e. $F(H) = \{h : H \to P^*(H) | (\forall x, y \in H)\ h(x \oplus y) \subseteq h(x) \oplus h(y)\}$ then for all pairs $f, g \in F(H)$ we set:

$$f \oplus_F g = \Big\{h \in F(H) \,\Big|\, (\forall x \in H) h(x) \subseteq f(x) \oplus g(x)\Big\}$$
$$f \odot_F g = \Big\{h \in F(H) | (\forall x \in H) h(x) \subseteq f\big(g(x)\big)\Big\}.$$

It is known that the structure $\big(F(H), \oplus_F, \odot_F\big)$ is a binary hyperring (see Corsini [4], Example 422). In Section 3 of this aritcle, we determine condition under which we can construct the (m, n)-ary hyperring of multiendomorphisms of m-ary hypergroup (H, f). We show that we can associate a hyperring of multiendomorphisms with hypergroup (H, f) which is not necessary commutative.

The association between hyperstructures and binary relations had been studied by many authors, for example see Chvalina [2,3], Rosenberg [18], Corsini [5,6], Corsini and Leoreanu [8], and Spartalis [19]. Connections of n-ary hypergroups with binary relations was studied by Leoreanu and Davvaz in [15]. In Section 4 of this article, we obtain a class of strong distributive (m, n)-ary hyperrings associated with binary relations on semigroup. We investigate their morphisms and we also, establish connection between the constructed (m, n)-ary hyperring (H, f, g) and the hyperring of multiendomorphisms of m-hypergroup (H, f).

2. Preliminaries

The notion of (m, n)-ary hyperring was introduced by Davvaz [9]. In this section we generalize it, by introducing the notion of (m, n)-ary hyperring with inclusive distributivity and we give several examples of these structures.

We recall the following elementary background from [9].

A mapping $f : H \times \cdots \times H \to P^*(H)$, where H appears n times and $P^*(H)$ denotes the set of all non-empty subsets of H, is called an n-ary hyperoperation and n is called the arity of this hyperoperation. If f is an n-ary hyperoperation defined on H, then (H, f) is called an n-ary hypergroupoid. We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \dots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol.

In this convention $f(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_n)$ may be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. Similarly, for non-empty subsets $A_1, \dots, A_n$ of H we define:

$$f(A_1^n) = f(A_1, ..., A_n) = \cup\{f(x_1^n) | x_i \in A_i, i = 1, ..., n\}.$$

An n-ary hyperoperation f is called associative if:

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for every $i, j \in \{1, \dots, n\}$ and all $x_1, x_2, \dots, x_{2n-1} \in H$. An n-ary hypergroupoid with the associative hyperoperation is called an n-ary hipersemigroup. An n-ary hypersemigroup (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \le i \le n$, is called an n-ary hypergroup. This condition can be formulated by:

$$f(a_1^{i-1}, H, a_{i+1}^n) = H.$$

An n-ary hypergroupoid (H, f) is commutative if for all $\delta \in S_n$ and for every $a_1^n \in H$ we have $f(a_1, ..., a_n) = f(a_{\delta(1)}, \dots, a_{\delta(n)})$.

We introduce the following definition of (m, n)-ary hyperring.

Definition 2.1. An (m, n)-*ary hyperring* is an algebraic hyperstructure (R, f, g) which satisfies the following axioms:

1. (R, f) is an m-ary hypergroup.

2. (R, g) is an n-ary hypersemigroup.
3. The n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f i.e. for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R, 1 \le i \le n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) \subseteq f(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

(R, f, g) is called an n-ary hyperring if $n = m$.

The above definition contains the class of (m, n)-ary hyperrings in the sense of Davvaz. According to [9] an (m, n)-ary hyperring is an algebraic hyperstructure (R, f, g) which satisfies the conditions (1), (2) and

3″ for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R, 1 \le i \le n, g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n))$.

The (m, n)-ary hyperring in the sense of Davvaz will be called strong distributive (m, n)-ary hyperring.

Example 2.2. a) Let $(R, +, \cdot)$ be a ring and $\emptyset \ne P \subseteq R$ such that $RP = R$ and $Pz = zP$ for all $z \in R$. If we define an m-ary hyperoperation f and an n-ary hyperoperation g as follows:

$$\begin{aligned} f(x_1^m) &= x_1P + x_2P + ... + x_mP \\ g(x_1^n) &= x_1Px_2Px_3...x_{n-1}Px_n \end{aligned}$$

for any $x_1^m \in R$ and $x_1^n \in R$, then it can be verified that (H, f, g) is an (m, n)-ary hyperring. b) It is easy to see that if $(R, +, \cdot)$ is a ring with unity 1 and $P = \{1\}$ then (H, f, g) is a strong distributive (m, n)-ary hyperring. In this case, $f(x_1^m) = x_1 + ... + x_m$ and $g(x_1^n) = x_1 \cdot ... \cdot x_n$.

Example 2.3. Let $(R, +, \cdot)$ be a ring and I, J be ideals of a ring R. If we set:

$$f(x_1, x_2) = x_1 + x_2 + I$$
$$g(x_1, x_2) = x_1 \cdot x_2 + J$$

for all $x_1, x_2 \in R$, then (R, f, g) is $(2, 2)$-hyperring. If $I = J$, then obviously (R, f, g) is a strong distributive hyperring.

The following definition is a generalization of a suitable definition related to binary hyperrings.

Definition 2.4. Let (R_1, f_1, g_1) and (R_2, f_2, g_2) be (m, n)-ary hyperrings. A map $\varphi : R_1 \to R_2$ is called an *inclusion homomorphism* if the following conditions are satisfied:

1) $\varphi(f_1(a_1^m)) \subseteq f_2(\varphi(a_1), ..., \varphi(a_m))$ for all $a_1^m \in R_1$

2) $\varphi(g_1(a_1^n)) \subseteq g_2(\varphi(a_1), ..., \varphi(a_n))$ for all $a_1^n \in R_1$

A map φ is is called a *good* (or *strong*) *homomorphism* if in the conditions 1) and 2) the equality is valid.

We recall the following notion and result from [14], [15].

Let ρ be a binary relation on a non-empty set H. We define a partial n-ary hypergroupoid (H, f_p) as follows:

$$(\forall a \in H), f_\rho(\underbrace{a, ..., a}_{n \text{ times}}) = \{y | (a, y) \in \rho\}$$

and

$$(\forall a_1, a_2, ..., a_n \in H), f_\rho(a_1, a_2, ..., a_n) = f_\rho(\underbrace{a_1, ..., a_1}_{n \text{ times}}) \cup f_\rho(\underbrace{a_2, ..., a_2}_{n \text{ times}}) \cup ... \cup f_\rho(\underbrace{a_n, ..., a_n}_{n \text{ times}}).$$

By a partial n-ary hypergroupoid we mean a non-empty set H, endowed with a function from

$$\underbrace{H \times ... \times H}_{n \text{ times}}$$

to the set of subsets of H. Notice that (H, f_ρ) is an n-ary hypergroupoid if the domain of ρ is H.

An element $z \in H$ is called an *outer element* of ρ if there exists $y \in H$ such that $(y, z) \notin \rho^2$.

It is interesting to see when the above n-ary hypergroupoid (H, f_ρ) is an n-ary hypergroup.

Theorem 2.5. *Let ρ be a binary relation with full domain. The n-ary hypergroupoid (H, f_ρ) is an n-hypergroup if and only if the following conditions hold:*

1. *ρ has a full range;*
2. *$\rho \subseteq \rho^2$;*
3. *$(x, z) \in \rho^2 \Rightarrow (x, z) \in \rho$ for any outer element z of ρ.*

3. (m, n)-ary hyperring of multiendomorphisms

In this section we determine condition under which we can construct the (m, n)-ary hyperring of multiendomorphisms of m-ary hypergroup (H, f). We show that we can associate a hyperring of multiendomorphisms with hypergroup (H, f) which is not necessary commutative.

Let (H, f) be an m-ary hypergroup.

Before proving the next theorem we introduce the following notation:

$$a_{k1}^{km} = (a_{k1}, a_{k2}, ..., a_{km}), \quad a_{1k}^{mk} = (a_{1k}, a_{2k}, ..., a_{mk}),$$

for all $1 \leq k \leq m$.

If $h_i, h_{i+1}, ..., h_{m+i-1}$, is the sequence of multiendomorphsms of hypergroup (H, f), and $x \in H$, then we put:

$$f(h_i^{m+i-1}(x)) = f(h_i(x), ..., h_{m+i-1}(x))$$

for all $1 \leq i \leq m$.

If $h_1, ..., h_n$ are multiendomorphisms of hypergroup (H, f) and $x \in H$, then:

$$(h_1...h_n)(x) = (h_1 \circ ... \circ h_n)(x) = h_1(h_2(...(h_{n-1}(h_n(x)))$$

where we take

$$h_i(K) = \bigcup_{k \in K} h_i(k)$$

for any $K \subseteq H$ and $1 \leq i \leq n$.

Theorem 3.1. *Let (H, f) be an m-ary hypergroup such that for all $a_{11}^{1m}, a_{21}^{2m}, ..., a_{m1}^{mm} \in H$ it holds:*

$$f(f(a_{11}^{1m}), f(a_{21}^{2m}), ..., f(a_{m1}^{mm})) = f(f(a_{11}^{m1}), f(a_{21}^{m2}), ..., f(a_{m1}^{mm})). \tag{1}$$

Let $F(H)$ be the set of multiendomorphisms of hypergroup (H, f) i.e.

$$F(H) = \{h : H \to P^*(H) | (\forall a_1^m \in H)\ h(f(a_1^m)) \subseteq f(h(a_1), ..., h(a_m))\}.$$

Define an m-ary hyperoperation $\oplus$ and an n-ary $(n \geq 2)$ hyperoperation $\odot$ on $F(H)$ as follows: For any $h_1^m \in F(H)$ set

$$\oplus(h_1^m) = \{h \in F(H) | (\forall x \in H) h(x) \subseteq f(h_1(x), ..., h_m(x))\}.$$

For any $h_1^n \in F(H)$ set

$$\odot(h_1^n) = \{h \in F(H) | (\forall x \in H) h(x) \subseteq (h_1 h_2 ... h_n)(x)\}.$$

The structure $(F(H), \oplus, \odot)$ is an (m, n)-ary hyperring.

Proof. For any $h_1^m \in F(H)$ it holds $\oplus(h_1^m) \neq \emptyset$, i.e. $\oplus$ is an m-ary hyperoperation. Indeed, let $h : H \to P^*(H)$ be a map defined by:

$$h(x) = f(h_1(x), ..., h_m(x)), \text{ for all } x \in H.$$

Then for every $a_1^m \in H$ it holds:

$$h(f(a_1^m)) = f(h_1(f(a_1^m)), ..., h_m(f(a_1^m))) \subseteq f(f(h_1(a_1), ..., h_1(a_m)), ..., f(h_m(a_1), ..., h_m(a_m))).$$

In what follows we shall denote the set $h_i(a_j)$ by A_{ij} and the sequence $A_{i1}, ..., A_{im}$ by A_{i1}^{im} for all $i, j \in \{1, ..., m\}$. So,

$$\begin{aligned} h(f(a_1^m)) \subseteq f(f(A_{11}^{1m}), ..., f(A_{m1}^{mm})) &= f(f(A_{11}^{m1}), ..., f(A_{1m}^{mm})) \\ &= f(f(h_1(a_1), ..., h_m(a_1)), ..., f(h_1(a_m), ..., h_m(a_m))) = f(h(a_1), ..., h(a_m)). \end{aligned}$$

Thus, $h \in \oplus(h_1^m)$.

Now, we prove that m-ary hyperoperation $\oplus$ is associative. Let, $i, j \in \{1, ..., m\}$ and $h_1^{2m-1} \in F(H)$. Set

$$\begin{aligned} L &= \bigoplus\left(h_1^{i-1}, \bigoplus(h_i^{m+i-1}), h_{m+i}^{2m-1}\right) = \bigcup\left\{\bigoplus(h_1^{i-1}, h', h_{m+i}^{2m-1}) \,\middle|\, h' \in \bigoplus(h_i^{m+i-1})\right\} \\ &= \bigcup\left\{\bigoplus(h_1^{i-1}, h', h_{m+i}^{2m-1}) \,\middle|\, h' \in F(H) \wedge (\forall x \in H)\, h'(x) \subseteq f(h_i^{m+i-1}(x))\right\}. \end{aligned}$$

Thus, if $h'' \in L$ then for all $x \in H$ it holds:

$$h''(x) \subseteq f(h_1^{i-1}(x), f(h_i^{m+i-1}(x)), h_{m+i}^{2m-1}(x)).$$

Conversely, if h'' is an element of $F(H)$ such that

$$h''(x) \subseteq f(h_1^{i-1}(x), f(h_i^{m+i-1}(x)), h_{m+i}^{2m-1}(x))$$

for all $x \in H$, and if we choose h' such that $h'(x) = f(h_i^{m+i-1}(x))$, for all $x \in H$, then $h' \in \oplus(h_i^{m+i-1})$ and $h'' \in \oplus(h_1^{i-1}, h', h_{m+i}^{2m-1})$ i.e. $h'' \in L$. So,

$$L = \{h'' \in F(H) | (\forall x \in H) h''(x) \subseteq f(h_1^{i-1}(x), f(h_i^{m+i-1}(x)), h_{m+i}^{2m-1}(x))\}.$$

On the other hand set:

$$D = \oplus(h_1^{j-1}, \oplus(h_j^{m+j-1}), h_{m+j}^{2m-1}).$$

Then,

$$D = \{h'' \in F(H) | (\forall x \in H) h''(x) \subseteq f(h_1^{j-1}(x), f(h_j^{m+j-1}(x)), h_{m+j}^{2m-1}(x))\}.$$

By the associativity of hiperoperation f, we obtain $L = D$.

Let $i \in \{1, ..., m\}$ and $h, h_1^{i-1}, h_{i+1}^m \in F(H)$. We prove that equation

$$h \in \oplus(h_1^{i-1}, h_i, h_{i+1}^m)$$

has a solution $h_i \in F(H)$. If we set $h_i(x) = H$ for all $x \in H$, then $h_i \in F(H)$ and for all $x \in H$ it holds:

$$f(h_1^{i-1}(x), h_i(x), h_{i+1}^m(x)) = H \supseteq h(x).$$

So, $h \in \oplus(h_1^{i-1}, h_i, h_{i+1}^m)$. Thus, $(F(H), \oplus)$ is an m-ary hypergroup.

Now we prove that $(F(H), \odot)$ is an n-ary hypersemigroup. Let $h_1^n \in F(H)$. For all $x \in H$, $h_n(x) \neq \emptyset$. Hence,

$$(h_1 h_2 ... h_n)(x) \neq \emptyset.$$

Let $h : H \to P^*(H)$ be a map defined by $h(x) = (h_1 ... h_n)(x)$. We want to prove that $h \in \odot(h_1^n)$ i.e. that $\odot$ is an n-ary hyperoperation. For any $a_1^m \in H$ it holds:

$$\begin{aligned} h(f(a_1^m)) = (h_1 h_2 ... h_n)(f(a_1^m)) &= (h_1 h_2 ... h_{n-1})(h_n(f(a_1^m))) \subseteq (h_1 h_2 ... h_{n-1})(f(h_n(a_1), ..., h_n(a_m))) \\ &\subseteq (h_1 h_2 ... h_{n-2})(f(h_{n-1}(h_n(a_1)), ..., h_{n-1}(h_n(a_m))) \subseteq \cdots \subseteq \\ &\subseteq f\left[(h_1 h_2 ... h_n)(a_1), ..., (h_1 h_2 ... h_n)(a_m)\right] = f(h(a_1), ..., h(a_m)). \end{aligned}$$

So, $h \in \odot(h_1^n)$.

Let us prove that $\odot$ is associative. Let $i, j \in \{1, ..., n\}$ and $h_1^{2n-1} \in F(H)$. Set

$$L = \bigodot\left(h_1^{i-1}, \bigodot(h_i^{n+i-1}), h_{n+i}^{2n-1}\right)$$

and

$$D = \bigodot\left(h_1^{j-1}, \bigodot(h_j^{n+j-1}), h_{n+j}^{2n-1}\right).$$

Then

$$L = \bigcup\left\{ \bigodot\left(h_1^{i-1}, h', h_{n+i}^{2n-1}\right) \,\middle|\, h' \in F(H) \wedge (\forall x \in H)\, h'(x) \subseteq (h_i ... h_{n+i-1}(x)) \right\}.$$

So, if $h'' \in L$ then $h''(x) \subseteq (h_1 ... h_{2n-1})(x)$, for all $x \in H$. On the other hand if $h'' \in F(H)$ and $h''(x) \subseteq (h_1 ... h_{2n-1})(x)$ for all $x \in H$, then we choose $h' \in F(H)$ such that $h'(x) = (h_i ... h_{n+i-1})(x)$ and consequently we obtain $h'' \in \odot(h_1^{i-1}, h', h_{n+i}^{2n-1})$ where $h' \in \odot(h_i^{n+i-1})$. Thus, $h'' \in L$. So,

$$L = \left\{ h'' \in F(H) \,\middle|\, (\forall x \in H) h''(x) \subseteq (h_1 ... h_{2n-1})(x) \right\}.$$

Similarly,

$$D = \left\{ h'' \in F(H) \,\middle|\, (\forall x \in H) h''(x) \subseteq (h_1 ... h_{2n-1})(x) \right\}.$$

Thus, $L = D$.

Now we prove that the n-ary hyperoperation $\odot$ is distributive with respect to the m-ary hyperoperation $\oplus$. Let $h_1^{i-1}, h_{i+1}^n, g_1^m \in F(H), 1 \le i \le n$. Set

$$\begin{aligned} L = \bigodot\left(h_1^{i-1}, \bigoplus(g_1^m), h_{i+1}^n\right) &= \bigcup\left\{ \bigodot(h_1^{i-1}, h', h_{i+1}^n) \,\middle|\, h' \in \oplus(g_1^m) \right\} \\ &= \bigcup\left\{ \bigodot(h_1^{i-1}, h', h_{i+1}^n) \,\middle|\, h' \in F(H) \wedge (\forall x \in H) h'(x) \subseteq f(g_1(x), ..., g_m(x)) \right\}. \end{aligned}$$

So, if $k \in L$ then for all $x \in H$, it holds:

$$\begin{aligned} k(x) &\subseteq (h_1 ... h_{i-1})(f((g_1 h_{i+1} ... h_n)(x), ..., (g_m h_{i+1} ... h_n)(x))) \\ &\subseteq (h_1 ... h_{i-2})(f((h_{i-1} g_1 h_{i+1} ... h_n)(x), ..., (h_{i-1} g_m h_{i+1} ... h_n)(x))) \\ &\subseteq \cdots \subseteq f((h_1 ... h_{i-1} g_1 h_{i+1} ... h_n)(x), ..., (h_1 ... h_{i-1} g_m h_{i+1} ... h_n)(x)). \end{aligned}$$

On the other hand,

$$D = \bigoplus\left(\bigodot(h_1^{i-1}, g_1, h_{i+1}^n), ..., \bigodot(h_1^{i-1}, g_m, h_{i+1}^n)\right) = \bigcup\left\{ \bigoplus(k_1, ..., k_m) \,\middle|\, k_j \in \odot(h_1^{i-1}, g_j, h_{i+1}^n), j \in \{1, 2, ..., m\} \right\}.$$

Let $k \in L$. Choose $k_1, ..., k_m \in F(H)$ such that for all $j \in \{1, 2..., m\}$

$$k_j(x) = (h_1 ... h_{i-1} g_j h_{i+1} ... h_n)(x), \text{ for all } x \in H.$$

Then $k_j \in \odot(h_1^{i-1}, g_j, h_{i+1}^n)$ and $k \in \oplus(k_1, ..., k_m)$. Thus, $k \in D$. So, $L \subseteq D$. $\square$

Remark 3.2. If (H, f) is an m-ary hypergroup that satisfies condition (1) then for any $n \geq 2$, there exsits (m, n)-ary hyperring $(F(H), \oplus, \odot)$. The structure $(F(H), \oplus, \odot)$ will be called (m, n)-ary hyperring of multiendomorphisms of m-ary hypergroup (H, f).

Remark 3.3. If (H, f) is a commutative binary hypergroup, then (H, f) satisfies condition (1) of previous theorem. Thus, the binary hyperring of multiendomorphisms of commutative binary hypergroup (H, f) is a special case of (m, n)-ary hyperring constructed in Theorem 3.1. But, the following example shows that there also exist noncommutative hypergroups, that satisfy condition (1), implying that we can associate a hyperring of multiendomorphisms with noncommutative hypergroup.

Example 3.4. If $H = \{x, y, z\}$ and f is defined by the following table:

f	x	y	z
x	H	H	H
y	H	H	$\{x, y\}$
z	H	$\{z, x\}$	H

then (H, f) is a noncommutative binary hypergroup which satisfies condition (1).

4. (m, n)-ary hyperrings associated with binary relations

In this section we construct a class of (m, n)-ary hyperrings associated with binary relations on semigroup. Then, we investigate their morphisms and we also, establish connection between the constructed (m, n)-ary hyperring (H, f, g) and the hyperring of multiendomorphisms of m-hypergroup (H, f).

Theorem 4.1. *Let $(H, \cdot)$ be a semigroup equipped with binary relations ρ_1 and ρ_2 such that $\rho_1 \subseteq \rho_2$. Let ρ_i $(i = 1, 2)$ be a reflexive and transitive relation such that for all $a, b, x \in H$,*

$$(a, b) \in \rho_i \text{ implies } (a \cdot x, b \cdot x) \in \rho_i \text{ and } (x \cdot a, x \cdot b) \in \rho_i. \tag{2}$$

We define an m-ary hyperoperation f and an n-ary hyperoperation g on H, as follows:

$$f(a_1^m) = \left\{ z \,\middle|\, (a_1, z) \in \rho_1 \vee (a_2, z) \in \rho_1 \vee ... \vee (a_m, z) \in \rho_1 \right\}$$

for any $a_1^m \in H$, and $g(a_1^n) = \left\{ z \,\middle|\, a_1 \cdot a_2 \cdot ... \cdot a_n \, \rho_2 \, z \right\}$ for any $a_1^n \in H$. The structure (H, f, g) is a strong distributive (m, n)-ary hyperring.

Proof. Since ρ_1 is reflexive and transitive relation, then by Theorem 2.5, (H, f) is an m-ary hypergroup.

Now we prove that (H, g) is an n-ary hypersemigroup. Since ρ_2 is reflexive, then for any $a_1^n \in H$ it holds $g(a_1^n) \neq \emptyset$ i.e. g is an n-ary hyperoperation. Let $i, j \in \{1, ..., n\}$ and $a_1^{2n-1} \in H$.

Set

$$L = g\left(a_1^{i-1}, g(a_i^{n+i-1}), a_{n+i}^{2n-1}\right) = \bigcup \left\{ g(a_1^{i-1}, z, a_{n+i}^{2n-1}) \,\middle|\, z \in g(a_i^{n+i-1}) \right\}$$

and

$$D = g\left(a_1^{j-1}, g(a_j^{n+j-1}), a_{n+j}^{2n-1}\right) = \bigcup \left\{ g(a_1^{j-1}, \delta, a_{n+j}^{2n-1}) \,\middle|\, \delta \in g(a_j^{n+j-1}) \right\}.$$

Suppose $w \in L$. Then there exists $z \in g(a_i^{n+i-1})$ such that $w \in g(a_1^{i-1}, z, a_{n+i}^{2n-1})$. Thus, $(a_i \cdot ... \cdot a_{n+i-1}, z) \in \rho_2$ and $(a_1 \cdot ... \cdot a_{i-1} \cdot z \cdot a_{n+i} \cdot ... \cdot a_{2n-1}, w) \in \rho_2$. By the condition (2) we have

$$(a_1 \cdot ... \cdot a_{i-1} \cdot a_i \cdot ... \cdot a_{n+i-1} \cdot a_{n+i} \cdot ... \cdot a_{2n-1}, a_1 \cdot ... \cdot a_{i-1} \cdot z \cdot a_{n+i} \cdot ... \cdot a_{2n-1}) \in \rho_2,$$

while $(a_1 \cdot ... \cdot a_{i-1} \cdot z \cdot a_{n+i} \cdot ... \cdot a_{2n-1}, w) \in \rho_2$. Since ρ_2 is transitive, then $(a_1 \cdot ... \cdot a_{2n-1}, w) \in \rho_2$.

Therefore if we set $\delta = a_j \cdot ... \cdot a_{n+j-1}$ then $\delta \in g(a_j^{n+j-1})$ and $w \in g(a_1^{j-1}, \delta, a_{n+j}^{2n-1})$, i.e. $w \in D$. So, $L \subseteq D$. Similarly, we obtain $D \subseteq L$.

Now we prove that n-ary hyperoperation g is strong distributive with respect to the m-ary hyperoperation f. Let $i \in \{1, ..., n\}$ and $a_1^{i-1}, a_{i+1}^n, x_1^m \in H$. Set

$$L = g\Big(a_1^{i-1}, f(x_1^m), a_{i+1}^n\Big) = \bigcup \Big\{g(a_1^{i-1}, w, a_{i+1}^n) \,\Big|\, w \in f(x_1^m)\Big\}$$

and

$$D = f\Big(g(a_1^{i-1}, x_1, a_{i+1}^n), ..., g(a_1^{i-1}, x_m, a_{i+1}^n)\Big) = \bigcup \Big\{f(\delta_1, ..., \delta_m) \,\Big|\, \delta_1 \in g(a_1^{i-1}, x_1, a_{i+1}^n), ..., \delta_m \in g(a_1^{i-1}, x_m, a_{i+1}^n)\Big\}.$$

If $y \in L$, then there exists $w \in f(x_1^m)$ such that $y \in g(a_1^{i-1}, w, a_{i+1}^n)$. Thus, there exists $k \in \{1, ..., m\}$ such that $(x_k, w) \in \rho_1 \subseteq \rho_2$ and $(a_1 \cdot ... \cdot a_{i-1} \cdot w \cdot a_{i+1} \cdot ... \cdot a_n, y) \in \rho_2$. By condition (2) we obtain

$$(a_1 \cdot ... \cdot a_{i-1} \cdot x_k, \cdot a_{i+1} \cdot ... \cdot a_n, a_1 \cdot ... \cdot a_{i-1} \cdot w \cdot a_{i+1} \cdot ... \cdot a_n) \in \rho_2,$$

while $(a_1 \cdot ... \cdot a_{i-1} \cdot w \cdot a_{i+1} \cdot ... \cdot a_n, y) \in \rho_2$. Since ρ_2 is transitive we obtain $y \in g(a_1^{i-1}, x_k, a_{i+1}^n)$.

So, if we choose $\delta_1, ..., \delta_m$ such that $\delta_l \in g(a_1^{i-1}, x_l, a_{i+1}^n)$ for $l \in \{1, 2, ..., m\} \backslash \{k\}$ and $\delta_k = y$, then $y \in f(\delta_1, ..., \delta_m)$, i.e., $y \in D$.

Suppose now $y \in D$. Then there exist $\delta_1 \in g(a_1^{i-1}, x_1, a_{i+1}^n), ..., \delta_m \in g(a_1^{i-1}, x_m, a_{i+1}^n)$ such that $y \in f(\delta_1, ..., \delta m)$. Hence, there exists $k \in \{1, ..., m\}$ such that $(\delta_k, y) \in \rho_1 \subseteq \rho_2$ while $(a_1 \cdot ... \cdot a_{i-1} \cdot x_k \cdot a_{i+1} \cdot ... \cdot a_n, \delta_k) \in \rho_2$. Since ρ_2 is transitive we obtain $(a_1 \cdot ... \cdot a_{i-1} \cdot x_k \cdot a_{i+1} \cdot ... \cdot a_n, y) \in \rho_2$ i.e. $y \in g(a_1^{i-1}, x_k, a_{i+1}^n)$. As $x_k \in f(x_1^m)$, we have $y \in L$.

Therefore, $D = L$. □

Throughout the following text the quadruple $(H, \cdot, \rho_1, \rho_2)$ will denote a semigroup $(H, \cdot)$ equipped with binary relations ρ_1 and ρ_2 such that ρ_1 and ρ_2 satisfy the conditions of Theorem 4.1. By an (m, n)-ary hyperring associated with $(H, \cdot, \rho_1, \rho_2)$ we mean an (m, n)- ary hyperring (H, f, g) constructed in Theorem 4.1.

Theorem 4.2. *Let (H, f, g) be an (m, n)-ary hyperring associated with $(H, \cdot, \rho_1, \rho_2)$ and $(F(H), \oplus, \odot)$ be an (m, n)-ary hyperring of multiendomorphisms of the m-ary hypergroup (H, f).*

If we define a mapping $\varphi : (H, f, g) \to (F(H), \oplus, \odot)$ by $\varphi(a) = h_a$, for all $a \in H$, where $h_a : H \to P^*(H)$ is defined by:

$$h_a(x) = f(\underbrace{a, ..., a}_{m-1 \text{ times}}, x), \text{ for all } x \in H,$$

then the following holds:

1. $\varphi(f(a_1^m)) \subseteq \bigoplus(\varphi(a_1), ..., \varphi(a_m))$, *for all* $a_1^m \in H$.
2. *If*

$$(a \cdot b, w) \in \rho_2 \Rightarrow (a, w) \in \rho_1 \vee (b, w) \in \rho_1 \tag{3}$$

for any triple of elements $a, b, w \in H$, then

$$\varphi(g(a_1^n)) \subseteq \bigodot(\varphi(a_1), ..., \varphi(a_n))$$

for any $a_1^n \in H$.

3. *If ρ_1 is an order, then φ is injective.*

Proof. First notice that for any $a_{11}^{1m}, a_{21}^{2m}, ..., a_{m1}^{mm} \in H$, it holds:

$$f\Big(f(a_{11}^{1m}), ..., f(a_{m1}^{mm})\Big) = \Big\{z \,\Big|\, \exists k, l \in \{1, ..., m\}, (a_{kl}, z) \in \rho_1\Big\} = f\Big(f(a_{11}^{m1}), ..., f(a_{1m}^{mm})\Big).$$

Thus, by Theorem 3.1, there exists an (m, n)-ary hyperring $(F(H), \oplus, \odot)$.

Now we verify that $h_a \in F(H)$, for any $a \in H$. Let $a_1^m \in H$. Set

$$L = h_a\Big(f(a_1, ..., a_m)\Big) = \bigcup \Big\{h_a(x) \,\Big|\, x \in f(a_1, ..., a_m)\Big\} = f\Big(\underbrace{a, ..., a}_{m-1}, f(a_1, ..., a_m)\Big).$$

and

$$D = f\Big(h_a(a_1), ..., h_a(a_m)\Big) = \bigcup \Big\{f(x_1, ..., x_m) \,\Big|\, x_j \in f(\underbrace{a, ..., a}_{m-1}, a_j), j = 1, ..., m\Big\}.$$

Let $z \in L$. We have the following possibilities:

(i) If $(a, z) \in \rho_1$, we put $x_1 = x_2 = ... = x_m = a$ and then $z \in f(x_1, ..., x_m)$ and

$$x_j \in f(\underbrace{a, ..., a}_{m-1}, a_j),$$

for all $j \in \{1, ..., m\}$. So, $z \in D$.

(ii) If there exists $u \in f(a_1, ..., a_m)$ such that $(u, z) \in \rho_1$, then, there exists $i \in \{1, ..., m\}$ such that $(a_i, u) \in \rho_1$ and $(u, z) \in \rho_1$. By transitivity of ρ_1, we have $(a_i, z) \in \rho_1$.If we put $x_i = z$ and $x_1 = ... = x_{i-1} = x_{i+1} = ... = x_m = a$, then $z \in f(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_m)$ and

$$x_j \in f(\underbrace{a, ..., a}_{m-1}, a_j) \text{ for all } j \in \{1, ..., m\}.$$

So, $z \in D$.

Thus, $h_a \in F(H)$.

(1) Let $a_1^m \in H$. Set:

$$L = \varphi\Big(f(a_1^m)\Big) = \Big\{h_w \,\Big|\, (a_1, w) \in \rho_1 \vee ... \vee (a_m, w) \in \rho_1\Big\}$$

and

$$D = \bigoplus \Big(\varphi(a_1), ..., \varphi(a_m)\Big) = \Big\{h \in F(H) \,\Big|\, (\forall x \in H)\, h(x) \subseteq f\Big(h_{a_1}(x), ..., h_{a_m}(x)\Big)\Big\}.$$

Let $h_w \in L$ and $x \in H$. Then

$$h_w(x) = f(\underbrace{w, ..., w}_{m-1}, x) = \Big\{z \,\Big|\, (w, z) \in \rho_1 \vee (x, z) \in \rho_1\Big\}.$$

Suppose $z \in h_w(x)$. We have two possibilities:

(i) If $(w, z) \in \rho_1$, since $h_w \in L$, then there exists $j \in \{1, ..., m\}$ such that $(a_j, w) \in \rho_1$, and by the transitivity of ρ_1 we have $(a_j, z) \in \rho_1$, i.e., $z \in f(a_j, ..., a_j, x) = h_{a_j}(x)$.
Since $h_{a_j}(x) \subseteq f(h_{a_1}(x), ..., h_{a_m}(x))$, then $z \in f\Big(h_{a_1}(x), ..., h_{a_m}(x)\Big)$.

(ii) If $(x, z) \in \rho_1$, then $z \in f(x, ..., x)$. Since $x \in h_{a_1}(x), ..., x \in h_{a_m}(x)$, then $z \in f\Big(h_{a_1}(x), ..., h_{a_m}(x)\Big)$.

So, $h_w(x) \subseteq f\Big(h_{a_1}(x), ..., h_{a_m}(x)\Big)$, for all $x \in H$ i.e. $h_w \in D$. Thus $L \subseteq D$.

(2) Let $a_1^n \in H$. First, notice that for any $x \in H$ and $i \in \{1, ..., n\}$ it holds $h_{a_i}(x) \subseteq (h_{a_1}...h_{a_i})(x)$ and $h_{a_i}(x) \subseteq (h_{a_i}...h_{a_n})(x)$.

Indeed, since $y \in h_{a_j}(y)$ for all $y \in H$ and $1 \leq j \leq n$, then $h_{a_i}(x) \subseteq h_{a_{i-1}}\Big(h_{a_i}(x)\Big)$ and $h_{a_{i-1}}\Big(h_{a_i}(x)\Big) \subseteq h_{a_{i-2}}\Big(h_{a_{i-1}}\Big(h_{a_i}(x)\Big)\Big)$.

Thus, $h_{a_i}(x) \subseteq (h_{a_{i-2}}h_{a_{i-1}}h_{a_i})(x)$. So, after finite number of steps we obtain:

$$h_{a_i}(x) \subseteq (h_{a_1}...h_{a_i})(x). \tag{4}$$

For the second inclusion we proceed in a similar way.

As $x \in h_{a_n}(x)$ then $h_{a_{n-1}}(x) \subseteq h_{a_{n-1}}\Big(h_{a_n}(x)\Big)$. Thus, $x \in (h_{a_{n-1}}h_{a_n})(x)$ implying that $h_{a_{n-2}}(x) \subseteq (h_{a_{n-2}}h_{a_{n-1}}h_{a_n})(x)$. After finite number of steps we obtain

$$h_{a_i}(x) \subseteq (h_{a_i}...h_{a_n})(x). \tag{5}$$

From (4) and (5) it follows $h_{a_i}(x) \subseteq h_{a_1}\Big(...\Big(h_{a_i}(x)\Big)\Big) \subseteq (h_{a_1}...h_{a_n})(x)$.

Now, set

$$L = \varphi\Big(g(a_1^n)\Big) = \Big\{ h_b \,\Big|\, (a_1 \cdot ... \cdot a_n, b) \in \rho_2 \Big\}$$

and

$$D = \bigodot\Big(\varphi(a_1), ..., \varphi(a_n)\Big) = \Big\{ h \in F(H) \,\Big|\, (\forall x \in H)\, h(x) \subseteq (h_{a_1}...h_{a_n})(x) \Big\}.$$

Let $h_b \in L$ and $x \in H$. Then

$$h_b(x) = f(\underbrace{b, ..., b}_{m-1}, x) = \Big\{ z \,\Big|\, (b, z) \in \rho_1 \vee (x, z) \in \rho_1 \Big\}.$$

If $z \in h_b(x)$ we have the following possibilities:

(i) If $(b, z) \in \rho_1$, since $h_b \in L$, then $(a_1 \cdot ... \cdot a_n, b) \in \rho_2$. As $\rho_1 \subseteq \rho_2$, by transitivity of ρ_2 we have $(a_1 \cdot ... \cdot a_n, z) \in \rho_2$. By the condition (3), there exists $i \in \{1, ..., n\}$ such that $(a_i, z) \in \rho_1$ i.e. $z \in h_{a_i}(x) \subseteq (h_{a_1}...h_{a_n})(x)$.

(ii) If $(x, z) \in \rho_1$, then $z \in h_{a_1}(x) \subseteq (h_{a_1}...h_{a_n})(x)$.

Thus, $h_b(x) \subseteq (h_{a_1}...h_{a_n})(x)$, for all $x \in H$ i.e. $h_b \in D$.

(3) Let ρ_1 be an order on H. Suppose $a, b \in H$ and $\varphi(a) = \varphi(b)$ i.e. $h_a = h_b$.

Then, $h_a(a) = h_b(a)$ and $h_a(b) = h_b(b)$. Thus,

$$f(\underbrace{a, ..., a}_{m}) = f(\underbrace{b, ..., b}_{m-1}, a) \text{ and } f(\underbrace{a, ..., a}_{m-1}, b) = f(\underbrace{b, ... b}_{m}).$$

Since,

$$f(\underbrace{a, ..., a}_{m-1}, b) = f(\underbrace{b, ..., b}_{m-1}, a),$$

then $f(a, ..., a) = f(b, ..., b)$.

From $a \in f(a, ..., a)$ it follows $a \in f(b, ..., b)$, i.e., $(b, a) \in \rho_1$. Similarly, it is proved $(a, b) \in \rho_1$. As ρ_1 is an order, we obtain $a = b$. □

Example 4.3. Notice that $(N, \cdot, \leq, \leq)$ satisfies the conditions of Theorem 4.1. Thus, there exists an (m, n)-ary hyperring (N, f, g) associated with $(N, \cdot, \leq, \leq)$.

For all $k_1^m \in N$ and $k_1^n \in N$ we have

$$f(k_1^m) = \left\{ k \in N \,\middle|\, \min\{k_1, ..., k_m\} \leq k \right\} \text{ and } g(k_1^n) = \left\{ k \in N \,\middle|\, k_1 \cdot ... \cdot k_n \leq k \right\}.$$

It is easy to see that $(N, \cdot, \leq, \leq)$ satisfies the conditions of Theorem 4.2. So, there exists an inclusion monomorphism of (N, f, g) into the $(F(N), \oplus, \odot)$.

Definition 4.4. Let the triples (H_1, ρ_1, ρ_2) and $(H_2, \delta_1, \delta_2)$ denote the nonempty set H_1 equipped with binary relations ρ_1, ρ_2 and nonempty set H_2 with binary relations δ_1, δ_2.

(a) The map $\alpha : H_1 \to H_2$ is said to be *isotone* if

$$x \, \rho_i \, y \;\Rightarrow\; \alpha(x) \, \delta_i \, \alpha(y),$$

for all $x, y \in H_1$ and $i \in \{1, 2\}$.

(b) The map $\alpha : H_1 \to H_2$ is said to be *strongly isotone* if

$$\alpha(x) \, \delta_i \, y \;\Leftrightarrow\; (\exists x' \in H_1) \, x \, \rho_i \, x' \;\wedge\; \alpha(x') = y,$$

for all $(x, y) \in H_1 \times H_2$ and $i \in \{1, 2\}$.

Theorem 4.5. *Let (H_1, f_1, g_1) be an (m, n)-ary hyperring associated with $(H_1, \cdot, \rho_1, \rho_2)$ and (H_2, f_2, g_2) be an (m, n)-ary hyperring associated with $(H_2, \cdot, \delta_1, \delta_2)$.*

(1) *If $\alpha : (H_1, \cdot) \to (H_2, \cdot)$ is an isotone homomorphism of semigroups $(H_1, \cdot)$ and $(H_2, \cdot)$ then $\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)$ is an inclusion homomorphism.*

(2) *If $\alpha : (H_1, \cdot) \to (H_2, \cdot)$ is a strongly isotone homomorphism, then $\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)$ is a strong homomorphism.*

Proof. (1) Let $\alpha : (H_1, \cdot) \to (H_2, \cdot)$ be an isotone homomorphism and $x_1^m \in H$. If $w \in \alpha(f_1(x_1^m))$, then there exists $z \in H_1$ such that $w = \alpha(z)$ and $(x_i, z) \in \rho_1$ for some $i \in \{1, ..., m\}$.

Since, α is isotone then $(\alpha(x_i), \alpha(z) = w) \in \delta_1$ and so $w \in f_2(\alpha(x_1), ..., \alpha(x_m))$. Thus $\alpha(f_1(x_1^m)) \subseteq f_2(\alpha(x_1), ..., \alpha(x_m))$.

Now, let $y_1^n \in H$. To prove that $\alpha\big(g_1(y_1^n)\big) \subseteq g_2\big(\alpha(y_1), ..., \alpha(y_n)\big)$ we can proceed similarly as in the proof of Theorem 3.2. (ii) in [19].

Therefore, $\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)$ ia an inclusion homomorphism.

(2) Let $\alpha : (H_1, \cdot) \to (H_2, \cdot)$ be a stronly isotone homomorphism. Since α is isotone, then by (1) we obtain that $\alpha : (H_1, f_1, g_1) \to (H_2, f_2, g_2)$ is an inclusion homomorphism.

Thus, for any $x_1^m \in H_1$, it holds $\alpha\big(f_1(x_1^m)\big) \subseteq f_2\big(\alpha(x_1), ..., \alpha(x_m)\big)$.

Suppose $w \in f_2(\alpha(x_1), ..., \alpha(x_m))$. Then $(\alpha(x_i), w) \in \delta_1$ for some $i \in \{1, ..., m\}$. Since α is strongly isotone, then there exists $z \in H_1$ such that $(x_i, z) \in \rho_1$ and $\alpha(z) = w$. Thus, $w = \alpha(z) \in \alpha(f_1(x_1^m))$. Therefore,

$$f_2\big(\alpha(x_1), ..., \alpha(x_m)\big) \subseteq \alpha\big(f_1(x_1^m)\big).$$

Thus, $\alpha(f_1(x_1^m)) = f_2(\alpha(x_1), ..., \alpha(x_m))$.

Now, let $y_1^n \in H_1$. In similar way as in the proof of Theorem 3.3. (i) in [19], we prove that

$$\alpha(g_1(y_1^n)) = g_2(\alpha(y_1), ..., \alpha(y_n)).$$

This completes the proof. □

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