Filomat 26:3 (2012), 615–621
DOI 10.2298/FIL1203615L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Root product of lattices

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Abstract. In this paper a new product of lattices, the root product, is defined, and here are given some basic properties of this product. By multiplying two lattices a new lattice L is obtained. The lattice L posses better properties in terms of dimension and determinant.

1. Notions and notations

Let us recall some notions and notations. Let $\leq$ denote the *ordering relation* on H. If they exist, the *least* and the *greatest* element of H will be denoted by 0 and 1, respectively. As usual, if H is a lattice, then the corresponding *meet* and *join* operations on H will be denoted by $\wedge$ and $\vee$, respectively. Since this paper deals with more than one ordered set, usually the notation $\leq_H$ will be used to indicate that it is ordering relation on H. Similarly, the notations 0_H, 1_H, $\wedge_H$ and $\vee_H$ will be used.

Let H and K be ordered sets. The *linear sum* of H and K, in notation $H + K$, is the set $H \cup K$ with the ordering relation preserving the orders in H and K, with addition that $h \leq k$, for all $h \in H$ and $k \in K$. Also, let us recall that ordering relation on the *direct product* $H \times K$ is defined by $(h_1, k_1) \leq (h_2, k_2)$ if and only if $h_1 \leq h_2$ and $k_1 \leq k_2$.

A *filter* of a lattice L is a subset $F \neq \emptyset$ of L such that $x \in F$ and $x \leq y$ imply $y \in F$ for all $x, y \in L$, and for all $x, y \in F$, $x \wedge y \in F$. For $a \in L$, the set $[a) = \{x \in L \mid a \leq x\}$ is the *principal filter generated by a*. If L is a finite lattice than every filter of L is principal filter.

An element a of a lattice L *covers* $b \in L$, which will be denoted by $b \prec a$, if $b < a$ and $c \in L$ such that $b \leq c \leq a$ implies $c = b$ or $c = a$.

For all non-defined notions and notations we refer to books [1]–[6], [8] and [11].

2. Root of the lattice

Let L be a lattice with greatest element 1. An element $a \in L$, $a \neq 1$ is *meet-irreducible* if any $b, c \in L$ such that $a = b \wedge c$ implies $a = b$ or $a = c$. The set of all meet-irreducible elements of L will be denoted by $\mathscr{I}(L)$. It is well-known fact that in a finite lattice L every element can be represented as meet of meet-irreducible elements of L. More information about that can be found in [15].

2010 *Mathematics Subject Classification*. Primary 06B35
Keywords. Lattices, meet-irreducible elements, root of a lattice, root products
Received: 04 May 2011; Received in revised form: 12 December 2011; Accepted: 23 December 2011
Communicated by Miroslav Ćirić
Research of the third author is supported by Ministry Education and Science, Republic of Serbia, Grant No. 174013
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Let L be a finite lattice with identity 1 and let $\mathscr{I}(L) = \{a_1, a_2, \ldots, a_n\}$ be a set of all meet-irreducible elements of L. Let $\mathscr{R}(L)$ be a subset of L consisting of identity 1 and all $a_k \in \mathscr{I}(L)$ such that $[a_k)$ is a chain.

Theorem 2.1. *Let L be a finite lattice. Then $(\mathscr{R}(L), \vee)$ is a subsemilattice of the semilattice $(L, \vee)$.*

Proof. Let $r_1, r_2 \in \mathcal{R}(L)$ be arbitrary elements. If $r_1 = 1_L$ or $r_2 = 1_L$, then we have that $r_1 \vee_L r_2 = 1_L \in \mathcal{R}(L)$. Otherwise, if $r_1 \neq 1_L$ and $r_2 \neq 1_L$, then $[r_1)$ and $[r_2)$ are chains, and $r_1 \vee_L r_2 \in [r_1) \cap [r_2)$, so $[r_1 \vee_L r_2)$ is also a chain.

Let $b, c \in L$ be such that $r_1 \vee_L r_2 = b \wedge c$. Then $r_1 \vee_L r_2 \leqslant b, c$, and since $[r_1 \vee_L r_2)$ is a chain, we have that $b \leqslant c$ or $c \leqslant b$, and thus we have that $r_1 \vee_L r_2 = b$ or $r_1 \vee_L r_2 = c$. So, $r_1 \vee_L r_2$ is a meet-irreducible element of L, and therefore $r_1 \vee_L r_2 \in \mathcal{R}(L)$. □

For an arbitrary finite lattice L, the semilattice $(\mathscr{R}(L), \vee)$ will be called the *root of the lattice L*. Clearly, the following holds:

$$|\mathscr{R}(L)| \leq |\mathscr{I}(L)| + 1. \tag{1}$$

If L is a lattice such that its every meet-irreducible element is an element of a root of L, then in (1) the equality holds. For example, this is true if L is a chain, Boolean lattice, or a lattice whose every meet-irreducible element is covered by either its greatest element or some other of its meet-irreducible elements (see L_3 in Fig. 1).

Example 2.2. Let us observe lattices L_i, $i \in \{1, 2, 3\}$ in Fig. 1 and their corresponding roots $\mathscr{R}(L_i)$, $i \in \{1, 2, 3\}$. For $i \in \{1, 2, 3\}$ the elements of a root $\mathscr{R}(L_i)$ of the lattice L_i are represented by black circles. The element a of the lattice L_1 is meet-irreducible, but it is not an element of the root $\mathscr{R}(L_1)$. Also, elements b, c and d of the lattice L_2 are meet-irreducible, but they do not belong to the root $\mathscr{R}(L_2)$. All meet-irreducible elements of the lattice L_3 are elements of the root $\mathscr{R}(L_3)$.

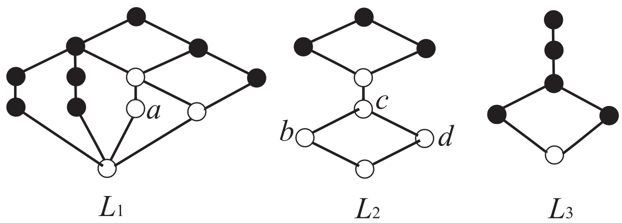

Fig. 1.

Theorem 2.3. *Let L be a finite lattice. Then $\mathscr{R}(L)$ is a chain if and only if L is a chain. In that case $\mathscr{R}(L) = L$.*

Proof. If L is a chain, than clearly $\mathscr{R}(L)$ is a chain. Conversely, let L be a lattice such that $\mathscr{R}(L)$ is a chain. The greatest element $1_L \in L$ belongs to the root $\mathscr{R}(L)$ by definition of the root. Let $1_L, r_1, \ldots, r_k, 0_{\mathscr{R}(L)}$ be all elements of the root $\mathscr{R}(L)$, such that $1_L \succ r_1 \succ \cdots \succ r_k \succ 0_{\mathscr{R}(L)}$. If there exists $a \in L$ such that $1_L \succ a \neq r_1$, then the principal filter $[a)$ is a chain, so $a \in \mathscr{R}(L)$, which is a contradiction. From this it follows that 1_L covers only element $r_1 \in \mathscr{R}(L)$. Similarly, if there exists an element $b \in L$ such that $r_j \succ b \neq r_{j+1}$ $(j = 1, 2, \ldots, k-1)$, then $b \in \mathscr{R}(L)$, which is also a contradiction. Thus, r_j covers only r_{j+1}. Analogously, r_k covers only the least element $0_{\mathscr{R}(L)}$ of the root $\mathscr{R}(L)$.

Let us also prove that the elements $1_L, r_1, \ldots, r_k, 0_{\mathscr{R}(L)}$ of the chain are the only elements of the lattice L, i.e., that $0_{\mathscr{R}(L)}$ is also the least element of the lattice L.

Let there exists an element $c \in L \setminus \mathscr{R}(L)$ such that $0_{\mathscr{R}(L)} \succ c$. If the element c is meet-irreducible, then $[c)$ is a chain and $c \in \mathscr{R}(L)$, which is a contradiction.

If the element c is meet-reducible, then besides the chain $c \prec 0_{\mathscr{R}(L)} \prec r_k \prec \cdots \prec r_1 \prec 1_L$ at least one more chain exists between elements c and 1_L. That chain is $c \prec a_1 \prec \cdots \prec a_i \prec r_j \prec \cdots \prec r_1 \prec 1_L$, for some $j = 1, 2, \ldots, k$, and $i \in N$, or $c \prec a_1 \prec \cdots \prec a_i \prec 1_L$, for some $a_1, \ldots, a_i \neq r_l$, $(l = 1, 2, \ldots, k,)$.

In both cases, the principal filter $[a_i)$ $(i \in N)$ is a chain, so $a_i \in \mathscr{R}(L)$, which contradicts the fact that $\mathscr{R}(L)$ is a chain.

Thus $L \setminus \mathscr{R}(L) = \emptyset$, so $L = \mathscr{R}(L)$. □

3. $\otimes_r$-product and $\otimes_l$-product of lattices

Let $\mathscr{L}_f$ be the class of all finite lattices with at least two elements and let $\otimes_r$ and $\otimes_l$ be two binary operations on $\mathscr{L}_f$ defined as follows: for all $L_1, L_2 \in \mathscr{L}_f$,

$$L_1 \otimes_r L_2 = \{0\} + (L_1 \backslash \{0\}) \times \mathscr{R}(L_2),$$

$$L_1 \otimes_l L_2 = \{0\} + \mathscr{R}(L_1) \times (L_2 \backslash \{0\}).$$

In a very simple way we can prove the following proposition.

Proposition 3.1. *Let $L_1, L_2 \in \mathscr{L}_f$. Then the following hold.*

(*A*) *For every $r \in \mathcal{R}(L_2)$, the set $\{(l, r) \mid l \in L_1 \backslash \{0\}\} \cup \{0\}$ forms a sublattice of $L_1 \otimes_r L_2$ which is isomorphic to L_1.*

(*B*) *There exists a sublattice of $L_1 \otimes_r L_2$ with the greatest element $1_{L_1 \otimes_r L_2}$ and the least element $0_{L_1 \otimes_r L_2}$ which is isomorphic to L_1.*

(*C*) *For every $l \in L_1 \backslash \{0\}$, the set $\{(l, r) \mid r \in \mathcal{R}(L_2)\}$ forms a subsemilattice of $(L_1 \otimes_r L_2, \vee)$ which is isomorphic to $(\mathcal{R}(L_2), \vee)$.*

(*D*) *There exists a subsemilattice of $(L_1 \otimes_r L_2, \vee)$ with the greatest element $1_{L_1 \otimes_r L_2}$ which is isomorphic to $(\mathcal{R}(L_2), \vee)$.*

Proof. By the construction of the $\otimes_r$-product, it is easy to verify that assertions (A) and (C) hold. The assertion (B) follows by (A) if we take that $r = 1_{L_2}$ and the assertion (D) follows by (C) if we take $l = 1_{L_1}$. □

Clearly, $L_1 \otimes_r L_2$ is a finite lattice with at least two elements. In addition, $\otimes_r$ is not associative on $\mathscr{L}_f$ (see Example 3.9) and thus different order of operations $\otimes_r$ on lattices $L_1, \ldots, L_m \in \mathscr{L}_f$ gives different lattices, even with different number of elements.

For lattices $L_1, \ldots, L_m \in \mathscr{L}_f$, by $\mathscr{P}(L_1, L_2, \ldots, L_m)$ we will denote the subset of $\mathscr{L}_f$ consisting of all $\otimes_r$-products of lattices $L_1, \ldots, L_m$ in that order of appearance of lattices.

Theorem 3.2. *Let $L_1, L_2, \ldots, L_m \in \mathscr{L}_f$ and let $L \in \mathscr{P}(L_1, L_2, \ldots, L_m)$. Then*

$$\mathscr{R}(L) = \{(1, \ldots, 1, r_i, 1, \ldots, 1) \mid r_i \in \mathscr{R}(L_i) \backslash \{1\}, i = 1, 2, \ldots, m\} \cup \{(1, 1, \ldots, 1)\},$$

and

$$|\mathscr{R}(L)| = 1 + \sum_{i=1}^{m} (|\mathscr{R}(L_i)| - 1) = 1 - m + \sum_{i=1}^{m} |\mathscr{R}(L_i)|.$$

Proof. Let $L_1, L_2, \ldots, L_m \in \mathscr{L}_f$ and let $L \in \mathscr{P}(L_1, L_2, \ldots, L_m)$ be an arbitrary element.

Let $r = (1, \ldots, 1, r_i, 1, \ldots, 1)$ and let $a, b \in L$ be elements such that $r \leq a, b$. Then $a_j = 1, b_j = 1$ for $j \neq i$, and $r_i \leq a_i, b_i$. Since r_i is an element of the root $\mathscr{R}(L_i)$, i.e., $[r_i)$ is a chain, we have that $a_i \leq b_i$ or $b_i \leq a_i$, and hence $a \leq b$ or $b \leq a$, i.e., $[r)$ is a chain. So, every element of a form $(1, \ldots, 1, r_i, 1, \ldots, 1)$ is an element of the root $\mathscr{R}(L)$.

Conversely, let $\bar{r} = (\bar{r}_i)_{i=1}^m$ be an element of the root $\mathscr{R}(L)$. First, we will prove that for $i = 1, 2, \ldots, m$, $\bar{r}_i$ is an element of the root $\mathscr{R}(L_i)$. Let $a_i, b_i \in L_i$ be such that $\bar{r}_i \leq a_i, b_i$. Then, for elements $a = (1, \ldots, 1, a_i, 1, \ldots, 1)$ and $b = (1, \ldots, 1, b_i, 1, \ldots, 1)$ in L holds $\bar{r} \leq a, b$, and since $[\bar{r})$ is a chain, we have $a \leq b$ or $b \leq a$. Thus $a_i \leq b_i$ or $b_i \leq a_i$ in L_i, and $[\bar{r}_i)$ is a chain, so $\bar{r}_i$ is an element of the root $\mathscr{R}(L_i)$. Further, if for some $i \neq j$ we have that $\bar{r}_i \neq 1$ and $\bar{r}_j \neq 1$, then the elements $u = (1, \ldots, 1, \bar{r}_i, 1, \ldots, 1)$ and $v = (1, \ldots, 1, \bar{r}_j, 1, \ldots, 1)$ are incomparable in L and $\bar{r} \leq u, v$, which is in contradiction to the fact that $[\bar{r})$ is a chain. Thus $\bar{r}_k \neq 1$ for at most one k.

Further, it is clear that $|\mathscr{R}(L)| = 1 + \sum_{i=1}^{m} (|\mathscr{R}(L_i)| - 1) = 1 - m + \sum_{i=1}^{m} |\mathscr{R}(L_i)|$. □

Corollary 3.3. *For $L_1, L_2, \ldots, L_m \in \mathscr{L}_f$, all lattices in $\mathscr{P}(L_1, L_2, \ldots, L_m)$ have the same roots.*

Theorem 3.4. *Let $m \in \mathbb{N}$, let $L_1, L_2, \ldots, L_m \in \mathscr{L}_f$, and let $L_{Ar}^{(m)} = (\cdots((L_1 \otimes_r L_2) \otimes_r L_3) \otimes_r \cdots) \otimes_r L_m$. Then*

$$|L_{Ar}^{(m)}| = 1 + (|L_1| - 1) \cdot \prod_{i=2}^{m} |\mathscr{R}(L_i)|. \tag{2}$$

Proof. We have that

$$\begin{aligned} L_{Ar}^{(m)} &= (\cdots((L_1 \otimes_r L_2) \otimes_r L_3) \otimes_r \cdots) \otimes_r L_m \\ &= (\cdots((\{0\} + (L_1 \backslash \{0\} \times \mathscr{R}(L_2)) \otimes_r L_3) \otimes_r \cdots) \otimes_r L_m \\ &= (\cdots(\{0\} + ((\{0\} + (L_1 \backslash \{0\}) \times \mathscr{R}(L_2)) \backslash \{0\}) \times \mathscr{R}(L_3)) \otimes_r \cdots) \otimes_r L_m \\ &= (\cdots(\{0\} + (L_1 \backslash \{0\}) \times \mathscr{R}(L_2) \times \mathscr{R}(L_3)) \otimes_r \cdots) \otimes_r L_m \\ &= \{0\} + (L_1 \backslash \{0\}) \times \mathscr{R}(L_2) \times \mathscr{R}(L_3) \times \cdots \times \mathscr{R}(L_m). \end{aligned}$$

Thus, the number of elements of this $\otimes_r$-product is given by (2). □

Theorem 3.5. *Let $m \in \mathbb{N}$, let $L_1, L_2, \ldots, L_m \in \mathscr{L}_f$, and let $L_{Br}^{(m)} = L_1 \otimes_r (\cdots \otimes_r (L_{m-2} \otimes_r (L_{m-1} \otimes_r L_m)) \cdots)$. Then*

$$|L_{Br}^{(m)}| = 1 + (|L_1| - 1) \left(\sum_{i=2}^{m} |\mathscr{R}(L_i)| - m + 2 \right). \tag{3}$$

Proof. We have that

$$\begin{aligned} L_{Br}^{(m)} &= L_1 \otimes_r (\cdots \otimes_r (L_{m-2} \otimes_r (L_{m-1} \otimes_r L_m)) \cdots) \\ &= \{0\} + (L_1 \backslash \{0\}) \times \mathscr{R}\left(L_2 \otimes_r (\cdots \otimes_r (L_{m-2} \otimes_r (L_{m-1} \otimes_r L_n)) \cdots)\right) \end{aligned}$$

By Theorem 3.2, we have that

$$|\mathscr{R}\left(L_2 \otimes_r (\cdots \otimes_r (L_{m-2} \otimes_r (L_{m-1} \otimes_r L_m)) \cdots)\right)| = \sum_{i=2}^{m} |\mathscr{R}(L_i)| - m + 2,$$

so the equality (3) holds. □

Corollary 3.6. $|L_{Br}^{(m)}| \leq |L_{Ar}^{(m)}|$

Proof. By Bernoulli's inequality it follows

$$\prod_{i=2}^{m} |\mathscr{R}(L_i)| = \prod_{i=2}^{m} (1 + |\mathscr{R}(L_i)| - 1) \geq 1 + \sum_{i=2}^{m} (|\mathscr{R}(L_i)| - 1) = \sum_{i=2}^{m} |\mathscr{R}(L_i)| - m + 2,$$

and thus

$$1 + (|L_1| - 1) \cdot \prod_{i=2}^{m} |\mathscr{R}(L_i)| \geq 1 + (|L_1| - 1) \left(\sum_{i=2}^{m} |\mathscr{R}(L_i)| - m + 2 \right),$$

i.e., $|L_{Br}^{(m)}| \leq |L_{Ar}^{(m)}|$. □

If all $\otimes_r$-products in which lattices $L_1, L_2, \ldots, L_m$ occur in that order are observed, then cardinality of $\otimes_r$-product depends on the parentheses, but the number of elements of the corresponding roots are always the same (it is given by Theorem 3.2). For some application, $\otimes_r$-product that has the greatest cardinality is interesting for observation, and by Theorems 3.4, 3.5 and Corollary 3.6 it follows that it is the $\otimes_r$-product $L_{Ar}^{(m)} = (\cdots((L_1 \otimes_r L_2) \otimes_r L_3) \otimes_r \cdots) \otimes_r L_m$.

By duality, for $\otimes_l$-product the following holds.

Theorem 3.7. *Let $m \in \mathbb{N}$ and let $L_1, L_2, \dots, L_m \in \mathscr{L}_f$. Furthermore, let $L_{Al}^{(m)} = (\cdots((L_1 \otimes_l L_2) \otimes_l L_3) \otimes_l \cdots) \otimes_l L_m$ and $L_{Bl}^{(m)} = L_1 \otimes_l (\cdots \otimes_l (L_{m-2} \otimes_l (L_{m-1} \otimes_l L_m)) \cdots)$. Then*

(i) $|L_{Al}^{(m)}| = 1 + (|L_m| - 1)\left(\sum_{i=1}^{m-1} |\mathscr{R}(L_i)| - m + 2\right)$;

(ii) $|L_{Bl}^{(m)}| = 1 + (|L_m| - 1) \cdot \prod_{i=1}^{m-1} |\mathscr{R}(L_i)|$;

(iii) $|L_{Al}^{(m)}| \leq |L_{Br}^{(m)}|$.

Corollary 3.8. *Let $m \in \mathbb{N}$ and let $L_1, L_2, \dots, L_m \in \mathscr{L}_f$ be such that $L_1 = L_2 = \cdots = L_m$. Then*

$$|L_{Ar}^{(m)}| = |L_{Bl}^{(m)}| \geq |L_{Al}^{(m)}| = |L_{Br}^{(m)}|.$$

Proof. This follows from Theorems 3.2, 3.4, 3.5 and Corollary 3.3. □

Example 3.9. Let L_1, L_2 and L_3 be lattices given in Fig. 2. Then the lattice $L_1 \otimes_r L_2$ is given in Fig. 3., and the lattice $(L_1 \otimes_r L_2) \otimes_r L_3$ is given in Fig. 4. The elements of the corresponding roots are denoted respectively by black circles. In Fig. 5. the lattice $L_2 \otimes_r L_3$ is given, and in Fig. 6. the lattice $L_1 \otimes_r (L_2 \otimes_r L_3)$ is given. The number of root-elements of the lattices given in Figs. 4. and 6. is the same and it is exactly

$$|\mathscr{R}(L_1)| + |\mathscr{R}(L_2)| + |\mathscr{R}(L_3)| - 2 = 9.$$

Clearly, the lattice in Fig. 4. has more elements than the lattice in Fig. 6.

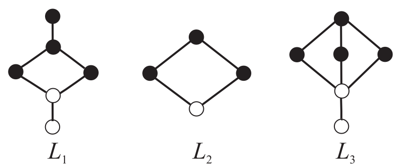

Fig. 2.

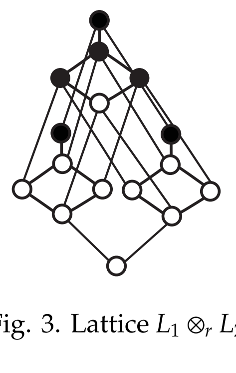

Fig. 3. Lattice $L_1 \otimes_r L_2$

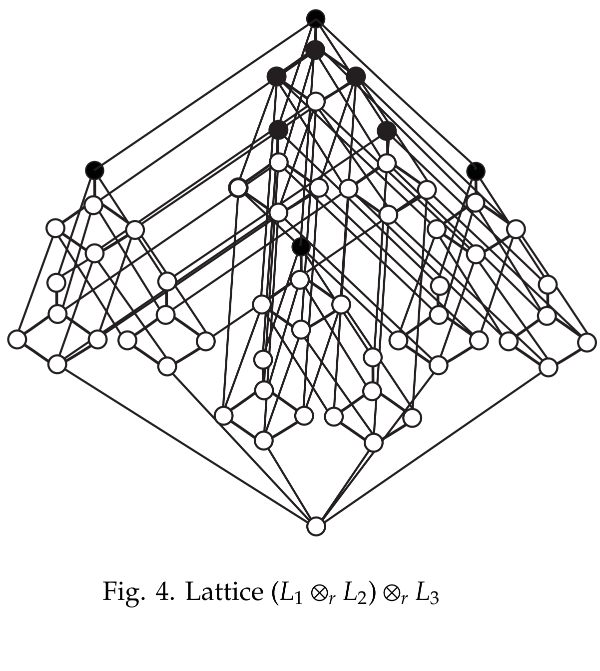

Fig. 4. Lattice $(L_1 \otimes_r L_2) \otimes_r L_3$

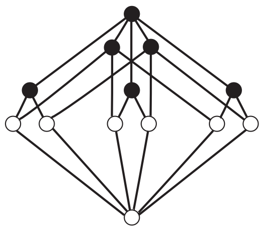

Fig. 5. Lattice $L_2 \otimes_r L_3$

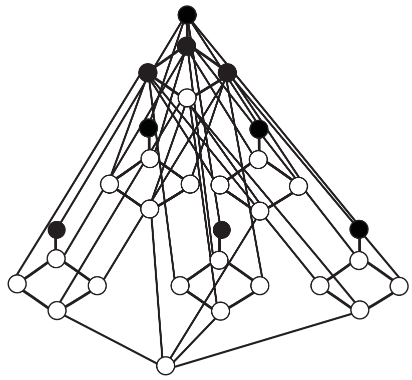

Fig. 6. Lattice $L_1 \otimes_r (L_2 \otimes_r L_3)$

Example 3.10. Let L_1 and L_2 be the lattices given in Fig. 7. Then $|L_1 \otimes_r L_2| = 29$ and $|L_2 \otimes_r L_1| = 21$, and hence $L_1 \otimes_r L_2 \neq L_2 \otimes_r L_1$.

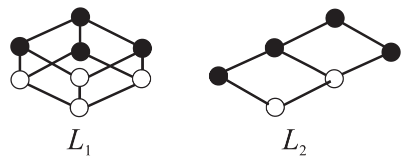

Fig. 7.

Corollary 3.11. *The $\otimes_r$-product is not commutative on $\mathscr{L}_f$.*

A lattice $L \in \mathscr{L}_f$ is $\otimes_r$*-simple* if $L \backslash \{0\} = \mathscr{R}(L)$. The class of all $\otimes_r$-simple lattices will be denoted by $\mathscr{L}_s$.

Theorem 3.12. *Up to an isomorphism, the $\otimes_r$-product is commutative on $\mathscr{L}_s$.*

Proof. Let $L_1, L_2 \in \mathscr{L}_s$. Then $L_1 \otimes_r L_2 = \{0\} + (L_1 \backslash \{0\}) \times \mathscr{R}(L_2) \cong \{0\} + (L_2 \backslash \{0\}) \times \mathscr{R}(L_1) = L_2 \otimes_r L_1$, and therefore, $L_1 \otimes_r L_2 \cong L_2 \otimes_r L_1$. □

The collection of all filters on a finite poset X, ordered dually to inclusion, is a finite distributive lattice L; its poset of meet-irreducibles is isomorphic to X. The converse is given by Birkhoff's theorem [2], as follows. Every finite distributive lattice is isomorphic to the lattice of all filters of the poset of its meet-irreducible elements, ordered dually to inclusion. As is known, the same poset of meet-irreducibles determine also some other, non-distributive lattices in which it is the poset of meet-irreducibles. In [15], conditions under which an arbitrary finite lattice has the same (up to isomorphism) poset of meet-irreducibles as that distributive lattice, are given.

For $n \in \mathbb{N}$, the Boolean lattice $\mathbf{2}^n$ is a lattice of greatest cardinality among all lattices with n meet-irreducible elements. From this it follows that among all latices whose corresponding root has $n+1$ elements, $\mathbf{2}^n$ is a lattice of greatest cardinality. In that case, $|\mathbf{2}^n \otimes_r L_2| = (2^n - 1) \cdot |\mathscr{R}(L_2)| + 1$ and

$$|\mathscr{R}(\mathbf{2}^n \otimes_r L_2)| = 1 + (|\mathscr{R}(\mathbf{2}^n)| - 1) + (|\mathscr{R}(L_2)| - 1) = 1 + n + 1 - 1 + |\mathscr{R}(L_2)| - 1 = n + |\mathscr{R}(L_2)|.$$

Thus the following assertions hold.

Corollary 3.13. *Let $\mathbf{2}^n$ be a Boolean lattice ($n \in \mathbb{N}$) and let $L \in \mathscr{L}_f$. Then $\mathscr{R}(\mathbf{2}^n \otimes_r L) = \mathscr{I}(\mathbf{2}^n \otimes_r L) \cup \{1\}$ and $|\mathscr{I}(\mathbf{2}^n \otimes_r L)| = n - 1 + |\mathscr{R}(L_2)|$.*

Corollary 3.14. *Let $L_1, L_2 \in \mathscr{L}_f$. Let n be the number of meet-irreducible elements of the lattice L_1. Then the lattice $L_1 \otimes_r L_2$, treated as a function of L_1, has the greatest cardinality in the case that L_1 is a Boolean lattice $\mathbf{2}^n$ and $|\mathscr{I}(L)| = n + |\mathscr{R}(L_2)| - 1$.*

Corollary 3.15. *Let $\mathbf{2}^n$ be a Boolean lattice ($n \in \mathbb{N}$) and for every $i \in \{1, 2, \ldots, m\}$ let $L_i = \mathbf{2}^n$. Then*

(i) $|L^m_{Ar}| = 1 + (2^n - 1)(n + 1)^m$,

(ii) $|L^m_{Br}| = 1 + (2^n - 1)(1 + n(m - 1))$,

and $|\mathscr{I}(L^m_n)| = nm$.

Proof. From Theorems 3.4 and 3.5 follows (i) and (ii), respectively. Let L^m_n be L^m_{Ar} or L^m_{Br}. We will calculate the number of meet-irreducible elements of L^m_n. From Theorem 3.2 follows that it is the number of root-elements of L^m_n is $|\mathscr{R}(L^m_n)| = 1 + \sum_{i=1}^{m}(|\mathscr{R}(L_i)| - 1) = 1 + m((n + 1) - 1) = 1 + nm$. Then, by Theorem 3.13 follows that $|\mathscr{I}(L^m_n)| = nm$. □

Concluding remarks: In this paper we gave a new construction of a lattice starting from a given family of lattices. By the given algorithm, one can construct a lattice of large cardinality starting from quite small and simple lattices. The obtained lattice can be used as a co-domain of fuzzy sets whose cuts are presented as binary words. Connections between coding theory and lattice valued fuzzy sets can be found in [10],[12]–[17]. Using our root product in coding theory could provide more code words without considerably increasing the code length.

Acknowledgement

The authors are grateful to the reviewer and area editor for valuable suggestions which significantly improved the quality of the paper.

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