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On the diameter of the graph $\Gamma_{Ann(M)}(R)$

David F. Anderson[a], Shaban Ghalandarzadeh[b], Sara Shirinkam[b], Parastoo Malakooti Rad[c]

[a]*Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1320, USA*
[b]*Department of Mathematics, K. N. Toosi University of Technology, P. O. Box 16315-1618, Tehran, Iran.*
[c] *Faculty of Electronic and Computer and IT, Islamic Azad university, Qazvin Branch, Qazvin, Iran*

Abstract. For a commutative ring R with identity, the ideal-based zero- divisor graph, denoted by $\Gamma_I(R)$, is the graph whose vertices are $\{x \in R \setminus I \mid xy \in I$ for some $y \in R \setminus I\}$, and two distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, we investigate an annihilator ideal-based zero-divisor graph, denoted by $\Gamma_{Ann(M)}(R)$, by replacing the ideal I with the annihilator ideal $Ann(M)$ for an R-module M. We also study the relationship between the diameter of $\Gamma_{Ann(M)}(R)$ and the minimal prime ideals of $Ann(M)$. In addition, we determine when $\Gamma_{Ann(M)}(R)$ is complete. In particular, we prove that for a reduced R-module M, $\Gamma_{Ann(M)}(R)$ is a complete graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $M \cong M_1 \times M_2$ for M_1 and M_2 nonzero $\mathbb{Z}_2$-modules.

1. Introduction

The zero divisor graph of a commutative ring was introduced by I. Beck in 1988 [8], and further studied by D. D. Anderson and M. Naseer in 1993 [1]. However, they let all the elements of R be vertices of the graph, and they were mainly interested in colorings. D. F. Anderson and P. S. Livingston in 1999 [2], introduced and studied the zero-divisor graph of a commutative ring with identity, whose vertices are the nonzero zero-divisors and $x - y$ is an edge whenever $xy = 0$. Since then, the concept of zero-divisor graphs has been studied extensively by many authors, including [3, 12, 14, 17, 18], and [19]. For recent developments on graphs of commutative rings, see [4–6, 11], and [13].

S. P. Redmond in 2003 [18], extended the zero-divisor graph of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. For a given ideal I of R, he defined an undirected graph $\Gamma_I(R)$, whose vertices are $\{a \in R \setminus I \mid ab \in I$ for some $b \in R \setminus I\}$, where distinct vertices a and b are adjacent if and only if $ab \in I$. He proved that this graph is connected with $diam(\Gamma_I(R)) \leq 3$. Moreover, the concept of the zero-divisor graph for a ring has been extended to module theory by Sh. Ghalandarzadeh and P. Malakooti Rad in 2009 [10]. They defined the torsion graph of an R-module M, whose vertices are the nonzero torsion elements of M such that two distinct vertices x, y are adjacent if and only if $(x : M)(y : M)M = 0$. For a reduced multiplication R-module M, they proved that, if $\Gamma(M)$ is complemented, then $S^{-1}M$ is von Neumann regular, where $S = R \setminus Z(M)$. In addition, the authors in [16] have investigated the relationship between the diameter of $\Gamma(M)$ and $\Gamma(R)$.

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Email addresses: anderson@math.utk.edu (David F. Anderson), ghalandarzadeh@kntu.ac.ir (Shaban Ghalandarzadeh), sshirinkam@dena.kntu.ac.ir (Sara Shirinkam), pmalakoti@gmail.com (Parastoo Malakooti Rad)

Let R be a commutative ring with nonzero identity and M be a unitary R-module. In this paper, we will investigate the annihilator ideal-based zero-divisor graph by replacing the ideal I with the ideal $Ann(M)$ for the R-module M. Here the annihilator ideal-based zero-divisor graph $\Gamma_{Ann(M)}(R)$ is a simple graph, whose vertices are the set $\{a \in R \setminus Ann(M) \mid abM = 0 \text{ for some } b \in R \setminus Ann(M)\}$, where distinct vertices a and b are adjacent if and only if $abM = 0$, defined by Sh. Ghalandarzadeh et al. in 2011 [11]. In the first section, our main purpose is to characterize the diameter of $\Gamma_{Ann(M)}(R)$ in terms of properties of the R-module M and ring R. In addition, we investigate the relationship between the diameter of $\Gamma_{Ann(M)}(R)$ and the minimal prime ideals of $Ann(M)$ over a multiplication R-module M. In the second section, we determine when $\Gamma_{Ann(M)}(R)$ is complete. Also, we prove that for a reduced R-module M, $\Gamma_{Ann(M)}(R)$ is a complete graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $M \cong M_1 \times M_2$ for M_1 and M_2 nonzero $\mathbb{Z}_2$-modules. This paper can be viewed as generalizing some results in [14] for $\Gamma(R)$ to $\Gamma_{Ann(M)}(R)$. Also, many of the results in this research have corresponding analogs in that study.

Let G be a simple graph and $V(G)$ denotes the set of vertices of G. Then G is a connected graph if there is a path between any two distinct vertices. A complete graph is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K^n. The distance $d(x, y)$ between connected vertices x, y is the length of a shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of G is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex.

A ring R is called reduced if $Nil(R) = 0$, and an R-module M is called a reduced module if $rm = 0$ implies that $rM \cap Rm = 0$, where $r \in R$ and $m \in M$. It is clear that M is a reduced module if and only if $r^2m = 0$ for $r \in R$, $m \in M$ implies that $rm = 0$. A proper submodule N of M is called a prime submodule of M, whenever $rm \in N$ implies that $m \in N$ or $r \in (N : M)$, where $r \in R$ and $m \in M$. A prime submodule N of M is called a minimal prime submodule of a submodule H of M, if it contains H and there is no smaller prime submodule with this property. A minimal prime submodule of the zero submodule is also known as a minimal prime submodule of the module M. We recall that an R-module M is said to be a multiplication module if for every submodule K of M, there exists an ideal I of R such that $K = IM$, [7]. By El-Bast and Smith ([9], Theorem 2.5), every non-zero multiplication R-module has a maximal submodule and so has a minimal prime submodule. The radical of an ideal I of a commutative ring R, denoted by $Rad(I)$, is defined as $Rad(I) = \{r \in R | r^n \in I \text{ for some positive integer } n\}$. If an ideal I of R is equal to its radical, then I is called a radical ideal.

Throughout this paper, $Nil(R)$ will be the ideal consisting of the nilpotent elements of R. Moreover, $Spec(M)$ will denote the set of the prime submodules of M, and $Nil(M) := \bigcap_{N \in Spec(M)} N$ will denote the nilradical of M. Also, by the proof of Lemma 3.7, step 1, in [10], one can check that a multiplication R-module M is reduced if and only if $Nil(M) = 0$. We shall often use $(x : M)$ and $(0 : M) = Ann(M)$ to denote the residual of Rx by M and the annihilator of an R-module M, respectively. The set $Z(M) := \{r \in R \mid rm = 0 \text{ for some } 0 \neq m \in M\}$ will denote the set of zero-divisors of M. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

2. The diameter of $\Gamma_{Ann(M)}(R)$

In this section, we investigate the relationship between the diameter of $\Gamma_{Ann(M)}(R)$ and the minimal prime ideals of $Ann(M)$ over a multiplication R-module M.

Lemma 2.1. *If M is reduced, then $I = Ann(M)$ is a radical ideal of R, and hence R/I is a reduced ring.*

Proof. Suppose that $r^n \in I$ for some $n \geq 1$, $r \in R$. Then $r^n m = 0$ for all $m \in M$, and thus $rm = 0$ for all $m \in M$ since M is reduced. Hence I is a radical ideal of R. □

The following example shows that the converse of the above lemma is not true.

Example 2.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}_4$. Then $Ann(M) = Ann(\mathbb{Z}) \cap Ann(\mathbb{Z}_4) = 0 \cap 4\mathbb{Z} = 0$ is a radical ideal of $\mathbb{Z}$. However, M is not reduced since $Ann((0, 1 + 4\mathbb{Z})) = 4\mathbb{Z}$ is not a radical ideal of $\mathbb{Z}$.

Lemma 2.3. *Let M be a reduced multiplication R-module and I be an ideal of R. If $I \subseteq P$ for some $P \in Min(Ann(M))$, then $I \subseteq Z(M)$.*

Proof. Let $P \in Min(Ann(M))$ and $I \subseteq P$. Since M is a reduced R-module, M_P will be a reduced R_P-module. We show that M_P has exactly one maximal submodule. Suppose that M_P has two maximal submodules $S^{-1}H_1$ and $S^{-1}H_2$; so by Theorem 2.5 [9], there exist two maximal ideals $S^{-1}h_1$ and $S^{-1}h_2$, such that $S^{-1}H_1 = S^{-1}h_1S^{-1}M$ and $S^{-1}H_2 = S^{-1}h_2S^{-1}M$. Since R_P is a local ring, $S^{-1}h_1 = S^{-1}h_2 = S^{-1}P$ and $S^{-1}H_1 = S^{-1}H_2 = S^{-1}(PM)$. We know that $S^{-1}(PM)$ is a proper submodule of $S^{-1}M$; so $PM \neq M$. Also, if $S^{-1}H_0$ is a prime submodule of M_P, then by Corollary 2.11 [9], there is a prime ideal $S^{-1}h_0$ of $S^{-1}R$ such that $S^{-1}H_0 = S^{-1}h_0S^{-1}M$ and $Ann(S^{-1}M) \subseteq S^{-1}h_0$. Since R_P is a local ring, $S^{-1}h_0 \subseteq S^{-1}P$. One can easily check that $h_0 \subseteq P$ and $Ann(M) \subseteq h_0$. Since P is a minimal prime ideal of $Ann(M)$, $h_0 = P$ and $h_0M = PM$. So M_P has exactly one prime submodule. Therefore $Nil(M_P) = S^{-1}(PM)$. Since M_P is reduced, $Nil(M_P) = 0$. Thus $S^{-1}(PM) = 0$. On the other hand, $I \subseteq P$; hence $S^{-1}(IM) = 0$. Since $PM \neq M$, there is an $x \in M$ such that $x \notin PM$. Thus $(a/1)(x/1) = 0$ for all $a \in I$. Hence there exists an element $s \in R \setminus P$ such that $sax = 0$. We show that $sx \neq 0$. If $sx = 0$, then $s(x : M)M = 0$. So $s(x : M) \subseteq Ann(M) \subseteq P$, which is a contradiction since $s \notin P$ and $x \notin PM$. Consequently, $I \subseteq Z(M)$. □

Proposition 2.4. *Let M be a reduced R-module. Then $V(\Gamma_{Ann(M)}(R)) \bigcup Ann(M) = \bigcup_{P \in Min(Ann(M))} P$.*

Proof. Let $K := V(\Gamma_{Ann(M)}(R)) \bigcup Ann(M)$, and let $x \in \bigcup_{P\in Min(Ann(M))} P$. Then there exists a $P_0 \in Min(Ann(M))$ such that $x \in P_0$. First, suppose that $xM = 0$; so $x \in Ann(M)$. Next, assume that $xM \neq 0$. We claim that $\overline{P}_0 = P_0/Ann(M) \in Min(\overline{R})$, where $\overline{R} = R/Ann(M)$. Assume that $\overline{P}_0 \notin Min(\overline{R})$. Thus, there is a prime ideal $\overline{P}_1 = P_1/Ann(M)$ of $\overline{R}$ such that $\bar{P}_1 \subseteq \overline{P}_0$. Let $0 \neq y \in P_1$; hence $y + Ann(M) = \overline{y} \in \overline{P}_1$. Thus $\overline{y} = \overline{z}$ for some nonzero element $\overline{z}$ of $\overline{P}_0$. Therefore $y \in P_0$, and so $P_1 \subseteq P_0$. Hence $P_0 = P_1$. Consequently, $\overline{P}_0 \in Min(\overline{R})$. We know that $\overline{x} \in \overline{P}_0 \in Min(\overline{R})$. So $\overline{x} \in \bigcup_{\overline{P}\in Min(\bar{R})} \overline{P}$. Since M is reduced, $\overline{R}$ is a reduced ring by Lemma 2.1. Thus $\bigcup_{\overline{P}\in Min(\overline{R})} \bar{P} = Z(\overline{R})$, and so $\overline{x} \in Z(\overline{R})$. Thus $\overline{x}\,\overline{y} = 0$ for some $\overline{0} = \overline{y} \in \overline{R}$. So $xyM = 0$ and $yM \neq 0$. Hence $x \in K$. Therefore $\bigcup_{P\in Min(Ann(M))} P \subseteq K$.

Now we show that $K \subseteq \bigcup_{P\in Min(Ann(M))} P$. Let $x \in K$. First, suppose that $xM = 0$. Thus $x \in \bigcup_{P\in Min(Ann(M))} P$. Next, assume that $xM \neq 0$. Thus x is a vertex of the graph since $x \in K$. Hence $xyM = 0$ for some $y \in R \setminus Ann(M)$. Thus $\overline{x} \in Z(\overline{R})$, where $\overline{R} = R/Ann(M)$ and $\overline{x} = x + Ann(M)$. Since M is reduced, $x \neq y$ and $\overline{R}$ is reduced by Lemma 2.1; so $\bigcup_{\overline{P}\in Min(Ann(M))} \overline{P} = Z(\overline{R})$. Hence $\overline{x} \in \overline{P}_0$ for some $\overline{P}_0 \in Min(\overline{R})$. Thus $x \in P_0$. We show that P_0 is a minimal prime ideal of R. If not, there exists a prime ideal P_1 of R such that $Ann(M) \subseteq P_1 \subseteq P_0$. So $\overline{P}_1 \subseteq \overline{P}_0 \in Min(\overline{R})$. Thus $\overline{P}_1 = \overline{P}_0$. Therefore, for all $z \in P_0$, we have $\overline{z} = \overline{P}_0 = \overline{P}_1$; so $z \in P_1$. Consequently, $P_0 = P_1$. Hence $P_0 \in Min(Ann(M))$, and so $K \in \bigcup_{P\in Min(Ann(M))} P$. □

Theorem 2.5. *Let M be a reduced multiplication R-module. If R has more than two minimal prime ideals of $Ann(M)$ and $R\alpha + R\beta \nsubseteq Z(M)$ for some $\alpha, \beta \in V(\Gamma_{Ann(M)}(R))$, then $diam(\Gamma_{Ann(M)}(R)) = 3$.*

Proof. Let α, β be two distinct vertices of $\Gamma_{Ann(M)}(R)$ with $R\alpha + R\beta \nsubseteq Z(M)$. First, suppose that $\alpha\beta M \neq 0$; so $d(\alpha, \beta) \neq 1$. If $d(\alpha, \beta) = 2$, then there exists a vertex γ such that $\alpha - \gamma - \beta$ is a path. Thus $\alpha\gamma M = 0 = \beta\gamma M$. Accordingly, $\gamma(R\alpha + R\beta)M = 0$. Since $\gamma M \neq 0$, $R\alpha + R\beta \nsubseteq Z(M)$, which is a contradiction. We shall now assume that $d(\alpha, \beta) \neq 2$. By Theorem 2.4 [18], $\Gamma_{Ann(M)}(R)$ is connected with $diam(\Gamma_{Ann(M)}(R)) \leq 3$. Therefore $d(\alpha, \beta) = 3$.

Next, assume that $\alpha\beta M = 0$. By Proposition 2.4 $\alpha, \beta \in \bigcup_{P\in Min(Ann(M))} P$. Also, by Lemma 2.3, α and β belong to two distinct minimal prime ideals of $Ann(M)$ since $R\alpha + R\beta \nsubseteq Z(M)$. Suppose that P,N and Q are distinct minimal prime ideals of $Ann(M)$ such that $\alpha \in P \setminus (Q \cup N)$ and $\alpha \in (Q \cap N) \setminus P$. Let $x \in (Q \cap P) \setminus N$. We show that $\alpha(\beta + \alpha x)M \neq 0$. If $\alpha(\beta + \alpha x)M = 0$, then for all $m \in M$, $\alpha(\beta m + \alpha xm) = \alpha^2xm = 0$. Hence $\alpha^2x \in Ann(M) \subseteq N$. We know that $x \notin N$ and N is a prime ideal of $Ann(M)$; so $\alpha \in N$, which is a contradiction. Therefore $\alpha(\beta + \alpha x)M \neq 0$. On the other hand, we have $\beta, x \in Q$. So $\beta + \alpha x \in Q \in Min(Ann(M))$. Thus $\beta + \alpha x \in \bigcup_{P\in Min(Ann(M))} P$. Since $\alpha(\beta + \alpha x)M \neq 0$, we have $\beta + \alpha x \notin Ann(M)$. By Proposition 2.4, $\beta + \alpha x$ is a vertex of the graph. Also, for all $y = R\alpha + R\beta$, we have $y = r\alpha + s\beta = r\alpha - s\alpha x + s\alpha x + s\beta = (r - sx)\alpha + s(\alpha x + \beta)$ for some $r, s \in R$. Thus $R\alpha + R\beta = R\alpha + R(\beta + \alpha x)$. So $R\alpha + R(\beta + \alpha x) \nsubseteq Z(M)$. Similarly to the above argument, we have $d(\alpha, \beta + \alpha x) = 3$. Consequently, $diam(\Gamma_{Ann(M)}(R)) = 3$. □

The following example shows that the condition $|Min(Ann(M))| > 2$ is not superfluous.

Example 2.6. Let $R = \mathbb{Z}\times\mathbb{Z} = M$. One can easily check that M is a reduced multiplication $\mathbb{Z}\times\mathbb{Z}$-module and $Ann(M) = \{0\}$. Thus $\Gamma_{Ann(M)}(R) = \Gamma(R)$. Also, we have $R\alpha + R\beta \nsubseteq Z(M)$ for $\alpha = (1, 0), \beta = (0, 1) \in V(\Gamma_{Ann(M)}(R))$ and $Min(Ann(M)) = \{0 \times \mathbb{Z}, \mathbb{Z} \times 0\}$. As one sees in Fig. 1, $\Gamma_{Ann(M)}(R)$ is a complete bipartite graph, and $diam(\Gamma_{Ann(M)}(R)) \neq 3$. So the condition $|Min(Ann(M))| > 2$ is not superfluous.

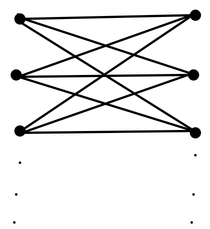

Figure 1: $\Gamma_{Ann(M)}(R)$, where $R = \mathbb{Z} \times \mathbb{Z}$ and $M = \mathbb{Z} \times \mathbb{Z}$.

Theorem 2.7. *Let M be a reduced multiplication R-module and $R\alpha + R\beta \nsubseteq Z(M)$ for some $\alpha, \beta \in V(\Gamma_{Ann(M)}(R))$. Then $diam(\Gamma_{Ann(M)}(R)) \leq 2$ if and only if R has exactly two minimal prime ideals of $Ann(M)$.*

Proof. Suppose that $diam(\Gamma_{Ann(M)}(R)) \leq 2$ and $R\alpha + R\beta \nsubseteq Z(M)$ for some $\alpha, \beta \in V(\Gamma_{Ann(M)}(R))$. By Proposition 2.4, $\alpha, \beta \in \bigcup_{P\in Min(Ann(M))} P$. Since for some $\alpha, \beta \in V(\Gamma_{Ann(M)}(R))$, $R\alpha + R\beta \nsubseteq Z(M)$, by Lemma 2.3, there are at least two distinct minimal prime ideals P and Q of $Ann(M)$ such that $\alpha \in P \setminus Q$, $\beta \in Q \setminus P$. By Theorem 2.5, if R has more than two minimal prime ideals of $Ann(M)$, then $diam(\Gamma_{Ann(M)}(R)) = 3$. So R has exactly two minimal prime ideals of $Ann(M)$.
Conversely, suppose that P and Q are the only two minimal prime ideals of $Ann(M)$. By Proposition 2.4, $V(\Gamma_{Ann(M)}(R)) \bigcup Ann(M) = P \cup Q$. First, assume that α, β are two vertices of the graph such that $\alpha \in P \setminus Q$ and $\beta \in Q \setminus P$. We show that $\bigcap_{N\in Min(M)} N = PM \bigcap QM$. Let N_0 be a minimal prime submodule of M. By Corollary 2.11 [9], $N_0 = P_0M$, where P_0 is a prime ideal of R and $Ann(M) \subseteq P_0$. If P_0 is a minimal prime ideal of $Ann(M)$, then $N_0 = PM$ or $N_0 = QM$. Otherwise, $Ann(M) \subseteq P \subseteq P_0$ or $Ann(M) \subseteq Q \subseteq P_0$. Since N_0 is a minimal prime submodule of M, $N_0 = PM$ or $N_0 = QM$. Thus $\bigcap_{N\in Min(M)} N = PM \bigcap QM$. By Theorem 2.4 [16], $PM \bigcap QM = Nil(M)$. Since $\alpha M \subseteq PM$ and $\beta M \subseteq QM$, $\alpha\beta M \subseteq PM \bigcap QM$. Also, since M is reduced, $Nil(M) = 0$. Hence $\alpha\beta M = 0$. Thus $d(\alpha, \beta) = 1$. Now let r, s be two distinct vertices of the graph. If $r \in P \setminus Q$ and $s \in Q \setminus P$, then by the above argument, $d(r, s) = 1$. Assume that $r, s \in P$; so $r\beta \in P$. Also, since $\beta \in Q \setminus P$, we have $r\beta \in Q$. Thus $r\beta M \subseteq PM \bigcap QM = Nil(M) = 0$. Similarly, $s\beta M = 0$. Also, if $r, s \in Q$, then similarly to the above argument, we have $r\alpha M = 0 = s\alpha M$. Therefore $d(r, s) = 2$. It follows that $diam(\Gamma_{Ann(M)}(R)) \leq 2$. □

Theorem 2.8. *Let M be a multiplication R-module with $Nil(M) \neq 0$. If there are $\alpha, \beta \in V(\Gamma_{Ann(M)}(R))$ such that $R\alpha + R\beta \nsubseteq Z(M)$, then $diam(\Gamma_{Ann(M)}(R)) = 3$.*

Proof. $\alpha, \beta \in V(\Gamma_{Ann(M)}(R))$ such that $R\alpha + R\beta \nsubseteq Z(M)$. So $d(\alpha, \beta) \neq 2$. Suppose that $\alpha\beta M \neq 0$. Hence $d(\alpha, \beta) \neq 1$. Thus by Theorem 2.4 [18], $diam(\Gamma_{Ann(M)}(R)) = 3$.

Next, let $\alpha\beta M = 0$. So $d(\alpha, \beta) = 1$. Since $Nil(M) \neq 0$, there exists a nonzero element $x \in Nil(M)$. Hence $(x : M)M \neq 0$; so $qm_0 \neq 0$ for some nonzero $q \in (x : M)$, $m_0 \in M$. Consider the pair α and $\beta + \alpha q$. One can easily show that $R\alpha + R\beta = R\alpha + R(\beta + \alpha q)$ and $\beta + \alpha q$ is a vertex of the graph. Therefore $R\alpha + R(\beta + \alpha q) \nsubseteq Z(M)$. If $\alpha^2 qm_0 = 0 = \beta^2 qm_0$, then $q(R\alpha^2 + R\beta^2)m_0 = 0$. On the other hand, $(R\alpha^2 + R\beta^2)m_0 = (R\alpha + R\beta)^2 m_0$ since $\alpha\beta M = 0$. So $q(R\alpha + R\beta)^2 m_0 = 0$. Hence $R\alpha + R\beta \subseteq Z(M)$ since $q(R\alpha + R\beta)m_0 \neq 0$, which is a contradiction. Thus without loss of generality, we may assume $\alpha^2 qm_0 \neq 0$. Hence $\alpha(\beta + \alpha q)M \neq 0$. Consequently, $d(\alpha, \beta + \alpha q) \neq 1$. Also, $d(\alpha, \beta + \alpha q) \neq 2$ since $R\alpha + R(\beta + \alpha q) \nsubseteq Z(M)$. So $diam(\Gamma_{Ann(M)}(R)) = 3$. □

3. Complete graphs

In this section, we determine when $\Gamma_{Ann(M)}(R)$ is complete. We will need the following characterization from Theorem 2.8 [2], of when $\Gamma(R)$ is complete.

Theorem 3.1. *Let R be a commutative ring. Then $\Gamma(R)$ is a complete graph if and only if either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$.*

Theorem 3.2. *Let M be a reduced R-module. Then $\Gamma_{Ann(M)}(R)$ is a complete (nonempty) graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $M \cong M_1 \times M_2$ for M_1 and M_2 nonzero $\mathbb{Z}_2$-modules.*

Proof. Suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $M \cong M_1 \times M_2$ for nonzero $\mathbb{Z}_2$-modules M_1 and M_2. Then $I = Ann(M) = 0$; so $\Gamma_I(R) = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2) = K^2$ is a complete, nonempty graph. Conversely, let M be a reduced R-module and $I = Ann(M)$. Then I is a radical ideal of R, and hence R/I is a reduced ring by Lemma 2.1 Assume that $\Gamma_I(R)$ is nonempty and complete. Let α be a vertex of $\Gamma_I(R)$. Then α^2 is also a vertex since I is a radical ideal of R. If $\alpha^2 \neq \alpha$, then $\alpha^3 = \alpha^2 \cdot \alpha \in I$ since $\Gamma_I(R)$ is complete. But then $\alpha \in I$ since I is a radical ideal of R, which is a contradiction. Thus $\alpha^2 = \alpha$ for every vertex α of $\Gamma_I(R)$. Hence $R = R\alpha \oplus R(1 - \alpha)$. So we may assume that $R = R_1 \times R_2$ with $\alpha = (1, 0)$ a vertex of $\Gamma_I(R)$. Moreover, $M = M_1 \times M_2$ for R_i-modules M_i, $(i = 1, 2)$. Since $\alpha = (1, 0)$ is a vertex of $\Gamma_I(R)$, $(1, 0) \notin I$. Thus $(1, 0)(M_1 \times M_2) = M_1 \times 0$ is nonzero; so $M_1 \neq 0$. Moreover, $I = Ann(M) = Ann(M_1) \times Ann(M_2)$.

Since $(1, 0)$ is a vertex of $\Gamma_I(R)$, $(a, 0) = (1, 0)(a, b) \in I$ for some $(a, b) \in R_1 \times R_2 \setminus I$. Thus $(a, 0)(M_1 \times M_2) = 0$; so $aM_1 = 0$. Hence $bM_2 \neq 0$ since $(a, b) \notin I$. Thus $M_2 \neq 0$, and so $(0, 1)(M_1 \times M_2) \neq 0$. Therefore $(0, 1) \notin I$, but $(1, 0)(0, 1) = (0, 0) \in I$; so $(0, 1)$ is also a vertex of $\Gamma_I(R)$.

We next show that $(c, 0)$ is a vertex of $\Gamma_I(R)$ if and only if $c = 1$. Suppose that $(c, 0)$ is a vertex for some $c \in R_1 \setminus \{0, 1\}$. Thus $(c, 0)$ and $(1, 0)$ are distinct vertices of $\Gamma_I(R)$, and thus are adjacent since $\Gamma_I(R)$ is complete. Hence $(c, 0) = (c, 0)(1, 0) \in I$, a contradiction. Similarly, $(0, d)$ is a vertex of $\Gamma_I(R)$ if and only if $d = 1$. We show that (c, d) is a vertex of $\Gamma_I(R)$ if and only if $(c, d) = (1, 0)$ or $(c, d) = (0, 1)$. Suppose that (c, d) is a vertex of $\Gamma_I(R)$ distinct from both $(1, 0)$ and $(0, 1)$. Then $(c, 0) = (c, d)(1, 0) \in I$ and $(0, d) = (c, d)(0, 1) \in I$; and hence $(c, d) = (c, 0) + (0, d) \in I$, a contradiction. Thus $|\Gamma_I(R)| = 2$; so $\Gamma_I(R) = K^2$. By Corollary 2.7 [18], either $(i)|I| = 1$ and $\Gamma(R/I) = K^2$, or $(ii)|I| = 2$ and $\Gamma(R/I) = K^1$.

(i) Suppose that $|I| = 1$ and $\Gamma(R/I) = K^2$. Then $I = 0$; so $\Gamma_I(R) = \Gamma(R) = K^2$. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$ by Example 2.1 [2]. However, $R/I \cong R$ is a reduced ring by Lemma 2.1; so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $M \cong M_1 \times M_2$ and $I = 0$, we must have both M_1 and M_2 nonzero.

(ii) Suppose that $|I| = 2$ and $\Gamma(R/I) = K^1$. Thus $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$ by Example 2.1 [2]. However, R/I must be a reduced ring by Lemma 2.1; so neither of these cases is possible. Thus $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $M \cong M_1 \times M_2$, where each M_i is a nonzero $\mathbb{Z}_2$-module $(i = 1, 2)$.

This completes the proof. □

The next example shows that the above theorem fails if we do not assume that M is a reduced R-module.

Example 3.3. Let $n \geq 2$ be an integer. By Theorem 6.1 [18], there is a ring R with a nonzero ideal I such that $\Gamma_I(R) = K^n$. Specifically, let $R = \mathbb{Z}_4 \times \mathbb{Z}_n$ and $I = 0 \times \mathbb{Z}_n$. Then $R/I \cong \mathbb{Z}_4$; so $\Gamma(R/I) = K^1$. Thus $\Gamma_I(R) = K^n$. So for $M = R/I$, we have $Ann(M) = I$. Hence $\Gamma_{Ann(M)}(R) = K^n$. For $n = 1$, let $R = M = \mathbb{Z}_4$. Then $I = Ann(M) = 0$; so $\Gamma_{Ann(M)}(R) = \Gamma(\mathbb{Z}_4) = K^1$. So for every $n \geq 1$, there is a ring R and an R-module M such that $\Gamma_{Ann(M)}(R) = K^n$.

Theorem 3.4. *Let M be a reduced R-module. If $\Gamma(R)$ is complete, then either $\Gamma_{Ann(M)}(R)$ is complete or the vertex sets of $\Gamma_{Ann(M)}(R)$ and $\Gamma(R)$ are disjoint.*

Proof. Since $\Gamma(R)$ is complete, by Theorem 3.1, either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $xy = 0$ for all $x, y \in Z(R)$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then by Theorem 3.2, $\Gamma_{Ann(M)}(R)$ is complete. Suppose that $xy = 0$ for all $x, y \in Z(R)$. Let $x \in V(\Gamma(R)) \bigcap V(\Gamma_{Ann(M)}(R))$. Then $x^2 = 0$; hence $x^2M = 0$. Since M is reduced, $xM = 0$, which is a contradiction. Consequently, the vertex sets of $\Gamma_{Ann(M)}(R)$ and $\Gamma(R)$ are disjoint. □

Corollary 3.5. *Let M be a reduced R-module. If* $\Gamma(R)$ *is complete, then either* $\Gamma_{Ann(M)}(R)$ *is complete or* $Z(R)M = 0$.

Corollary 3.6. *Let R be a ring and M be an R-module.*

(1) $diam(\Gamma_{Ann(M)}(R)) = 0$ *if and only if* $Nil(M) \neq 0$, *R is isomorphic to either* $\mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$, *and* $Ann(M) = 0$.

(2) $diam(\Gamma_{Ann(M)}(R)) = 1$ *if and only if either (i) M is reduced,* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $M \cong M_1 \times M_2$ *for* M_1 *and* M_2 *nonzero* $\mathbb{Z}_2$*-modules, or (ii)* $Nil(M) \neq 0$ *and* $xyM = 0$ *for each distinct pair of vertices x and y.*

Let R be a ring and M be a multiplication R-module. Suppose that there exist two distinct vertices α *and* β *such that* $R\alpha + R\beta \nsubseteq Z(M)$.

(3) $diam(\Gamma_{Ann(M)}(R)) = 2$ *if and only if either (i) R has exactly two minimal prime ideals of* $Ann(M)$, *M is reduced, and* $\Gamma_{Ann(M)}(R)$ *has at least three vertices, or (ii) for each distinct pair of vertices* α *and* β*, there exists a vertex which is adjacent to both* α *and* β.

(4) $diam(\Gamma_{Ann(M)}(R)) = 3$ *if and only if either (i) R has more than two minimal prime ideals of* $Ann(M)$ *and M is reduced, or (ii)* $Nil(M) \neq 0$.

Proof. (1) By Example 2.1 [2] and by Corollary 2.7 [18].

(2) By Theorem 3.2.

(3) Suppose that $diam(\Gamma_{Ann(M)}(R)) = 2$ and there exist two distinct vertices α and β such that $R\alpha + R\beta \nsubseteq Z(M)$. If M is reduced, then R has exactly two minimal prime ideals of $Ann(M)$, by Theorem 2.7

Conversely, suppose that (i) holds. By Theorem 2.7, $diam(\Gamma_{Ann(M)}(R)) \leq 2$. Assume that $diam(\Gamma_{Ann(M)}(R)) = 0$. Thus by (1), $Nil(M) \neq 0$, which is a contradiction. Suppose that $diam(\Gamma_{Ann(M)}(R)) = 1$. Hence by (2), either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, or $Nil(M) \neq 0$. Since $\Gamma_{Ann(M)}(R)$ has at least three vertices and M is reduced, we have a contradiction. Therefore $diam(\Gamma_{Ann(M)}(R)) = 2$.

(4) Suppose that $diam(\Gamma_{Ann(M)}(R)) = 3$, M is reduced, and R has exactly two minimal prime ideals of $Ann(M)$. By Theorem 2.7, $diam(\Gamma_{Ann(M)}(R)) \leq 2$, which is a contradiction. Now assume that P is the only minimal prime ideal of $Ann(M)$. By Proposition 2.4, for all vertices α and β we have $\alpha, \beta \in P$. Thus $R\alpha + R\beta \subseteq Z(M)$ by Lemma 2.3, which is a contradiction. Therefore R has more than two minimal prime ideals of $Ann(M)$.

Conversely, if $Nil(M) \neq 0$, then $diam(\Gamma_{Ann(M)}(R)) = 3$ by Theorem 2.8. Now assume that M is reduced and R has more than two minimal prime ideals of $Ann(M)$. Then $diam(\Gamma_{Ann(M)}(R)) = 3$ by Theorem 2.5. □

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