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On weaker forms of relator Menger, relator Rothberger and relator Hurewicz properties

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Abstract. We introduce new selection principles in relator spaces using neighbourhoods and closures in the same manner as it was done in [1, 11, 20] in topological spaces and prove that these properties are weaker versions of the corresponding selection principles defined in [12]. Some properties of these selection principles are proved.

1. Introduction and definitions

In this paper we continue the investigation of selection principles in relator spaces started in [12]. In the second section we follow the idea of Bonanzinga, Cammaroto, Kočinac and Matveev (see [1]) who used neighbourhoods of sets to define selection principles which are weaker than the known properties of Menger, Rothberger and Hurewicz. In the third section as a motivation we used the papers [12, 20], where weaker properties than Menger were defined by closures of sets.

We first recall the basic facts about selection principles in topological spaces (for selection principles theory see the survey papers [15], [23], [28]).

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X. Then: (see [10, 22])

- $\mathcal{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n \in \mathcal{U}_n$ and $\{V_n : n \in \mathbb{N}\} \in \mathcal{B}$.
- $\mathcal{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an element of $\mathcal{B}$.

As usual, for a subset A of a topological space X and a collection $\mathcal{P}$ of subsets of X, we denote by $St(A,\mathcal{P})$ the union of all elements of $\mathcal{P}$ which have a nonempty intersection with A. In [13], Kočinac introduced star selection principles in the following way:

Definition 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a topological space X. Then:

- $\mathcal{S}_1^*(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{St(U_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

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- $\mathcal{S}^*_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}}\{St(V,\mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$.

- $\mathcal{SS}^*_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose points $x_n \in X$, $n \in \mathbb{N}$, such that $\{St(\{x_n\},\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

- $\mathcal{SS}^*_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that $\{St(F_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

For a space X we use the following notation:

- $\mathcal{O}$ denotes the collection of all open covers of X.

- Ω denotes the collection of all ω-covers of X; we say that an open cover $\mathcal{U}$ of X is an *ω-cover* (see [8]) if every finite subset of X is contained in some element of $\mathcal{U}$.

- Γ denotes the collection of all γ-covers of X; we say that an open cover $\mathcal{U}$ of X is a γ-cover ([8]) if it is infinite and each point of X belongs to all but finitely many elements of $\mathcal{U}$.

We say that a space X is:
R (*Rothberger*) if the selection hypothesis $\mathcal{S}_1(\mathcal{O},\mathcal{O})$ is true for X ([7],[21], [22]);
M (*Menger*) if the selection hypothesis $\mathcal{S}_{fin}(\mathcal{O},\mathcal{O})$ is true for X ([7], [9], [10], [18]);
H (*Hurewicz*) if the selection hypothesis $\mathcal{S}_{fin}(\mathcal{O},\Gamma)$ is true for X ([9], [16]);
SR (*star-Rothberger*) if the selection hypothesis $\mathcal{S}^*_1(\mathcal{O},\mathcal{O})$ is true for X ([13]);
SSR (*strongly star-Rothberger*) if the selection hypothesis $\mathcal{SS}^*_1(\mathcal{O},\mathcal{O})$ is true for X ([13]);
SM (*star-Menger*) if the selection hypothesis $\mathcal{S}^*_{fin}(\mathcal{O},\mathcal{O})$ is true for X ([13]);
SSM (*strongly star-Menger*) if the selection hypothesis $\mathcal{SS}^*_{fin}(\mathcal{O},\mathcal{O})$ is true for X ([13]);
SH (*star-Hurewicz*) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X one can choose finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \mathbb{N}$, such that for every $x \in X$, $x \in St(\cup\mathcal{V}_n,\mathcal{U}_n)$ for all but finitely many n ([2]);
SSH (*strongly star-Hurewicz*) if the selection hypothesis $\mathcal{SS}^*_{fin}(\mathcal{O},\Gamma)$ is true for X ([2]).

Száz in several papers on relator spaces (see [25-27]) showed that many topological structures can be derived from relator spaces.

Let us recall some basic facts on relations and relators.

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $X = Y$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$. The relations $\Delta_X = \{(x, x) : x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X, respectively.

If F is a relation on X to Y, $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F(A) = \bigcup_{x\in A} F(x)$ are called the images of x and A under F, respectively.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x\in X}\{x\} \times F(x)$. Therefore, the inverse F^{-1} can be defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$.

Further, if F is a relation on X to Y, and G is a relation on Y to Z, then the composition $G \circ F$ is defined such that $(G \circ F)(x) = G(F(x))$ for every $x \in X$. If F is a relation on X to Y and G is a relation on Z to W, then the product $F \times G$ is defined such that $(F \times G)(x, y) = F(x) \times G(y)$ for every $x \in X$ and $y \in Z$.

A relation R on X is called reflexive if $\Delta_X \subset R$.

A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a *relator* on X, and the ordered pair $X(\mathcal{R}) = (X,\mathcal{R})$ is called a *relator space*.

Let C denotes the family of covers of X and let Γ_C denotes the family of all $\mathcal{U} \in C$ such that every $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. For a given relation $R \in \mathcal{R}$, where $\mathcal{R}$ is a relator on X, we use the following notation:

- $\mathcal{U}_R = \{R(x) : x \in X\}$.

- $\omega(R) = \{R(F) : F \subset X \text{ finite}\}$.
- $\mathcal{C}_{\mathcal{R}} = \{\mathcal{U}_R : R \in \mathcal{R}\}$.
- $\Omega(\mathcal{R})=\{\omega(R) : R \in \mathcal{R}\}$.

We can derive topological structures from relator spaces in the following way: If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we write:

$$int_{\mathcal{R}}(A) = \{x \in X : \exists R \in \mathcal{R} : R(x) \subset A\};$$
$$cl_{\mathcal{R}}(A) = \{x \in X : \forall R \in \mathcal{R} : R(x) \cap A \neq \emptyset\};$$
$$\mathcal{T}_{\mathcal{R}} = \{A \subset X : A \subset int_{\mathcal{R}}(A)\};$$
$$\mathcal{F}_{\mathcal{R}} = \{A \subset X : cl_{\mathcal{R}}(A) \subset A\}.$$
$$\mathcal{D}_{\mathcal{R}} = \{(A_n)_{n\in\mathbb{N}} : A_n \subset X, cl_{\mathcal{R}}(\textstyle\bigcup_{n\in\mathbb{N}} A_n) = X\}.$$

Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be relator spaces. We say that a map $f : X \longrightarrow Y$ is relator continuous if for every $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that for every $x \in X$ we have $f(R(x)) \subset S(f(x))$.

Recently, Kočinac introduced selection principles in relator spaces in the following way:

Definition 1.2. A relator space $(X, \mathcal{R})$ is:

- RR (*relator Rothberger*) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ one can choose $x_n \in X$, $n \in \mathbb{N}$, such that $\{R_n(x_n) : n \in \mathbb{N}\}$ is a cover of X;
- RM (*relator Menger*) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that $\{R_n(F_n) : n \in \mathbb{N}\}$ is a cover of X;
- RH (*relator Hurewicz*) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ one can choose finite $F_n \subset X$, $n \in \mathbb{N}$ such that every $x \in X$ belongs to $R_n(F_n)$ for all but finitely many $n \in \mathbb{N}$.

By a topological space we usually mean a Hausdorff topological space. The notation and terminology are as in [5].

2. Neighbourhood selection principles in relator spaces

In [1], neighbourhood star selection principles were defined.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. A space X satisfies:

- NSR$(\mathcal{A}, \mathcal{B})$ if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose $x_n \in X$, $n \in \mathbb{N}$, such that for every open O_n, $x_n \in O_n$, $n \in \mathbb{N}$, $\{St(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;
- NSM$(\mathcal{A}, \mathcal{B})$ if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, $\{St(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

In particular, a space X is *neighbourhood star-Rothberger* (NSR) (resp. *neighbourhood star-Menger* (NSM), *neighbourhood star-Hurewicz* (NSH)) if the selection hypothesis NSR$(\mathcal{O}, \mathcal{O})$ (resp. NSM$(\mathcal{O}, \mathcal{O})$, NSM$(\mathcal{O}, \Gamma)$) is true for X.

Notice that NSR and NSM spaces were considered in [14] under different names (nearly stongly star-Rothberger and nearly strongly star-Menger spaces).

We define neighbourhood selection principles in relator spaces in a similar way as it was done in [1].

Definition 2.2. A relator space $(X, \mathcal{R})$ is:

- NRR (*neighbourhood relator Rothberger*) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ one can choose $x_n \in X$, $n \in \mathbb{N}$, such that for every $O_n \in \mathcal{T}_\mathcal{R}$ containing x_n, $n \in \mathbb{N}$, $\{R_n(O_n) : n \in \mathbb{N}\}$ is a cover of X;
- NRM (*neighbourhood relator Menger*) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that for every $O_n \in \mathcal{T}_\mathcal{R}$ such that $F_n \subset O_n$, $n \in \mathbb{N}$, $\{R_n(O_n) : n \in \mathbb{N}\}$ is a cover of X;
- NRH (*neighbourhood relator Hurewicz*) if for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that for every $O_n \in \mathcal{T}_\mathcal{R}$ with $F_n \subset O_n$, $n \in \mathbb{N}$, each $x \in X$ belongs to $R_n(O_n)$ for all but finitely many $n \in \mathbb{N}$.

Let $(X, \mathcal{R})$ be a relator space. If we use the notation:

- $\mathcal{NC}_\mathcal{R} = \{R_\alpha(A_\alpha) : \alpha \in I, R_\alpha \in \mathcal{R},\ A_\alpha \subset X$ and for every $O_\alpha \supset A_\alpha, O_\alpha \in \mathcal{T}_\mathcal{R}, \{R_\alpha(O_\alpha) : \alpha \in I\} \in \mathcal{C}\}$,
- $\mathcal{N}\Gamma_\mathcal{R} = \{R_\alpha(A_\alpha) : \alpha \in I, R_\alpha \in \mathcal{R},\ A_\alpha \subset X$ and for every $O_\alpha \supset A_\alpha, O_\alpha \in \mathcal{T}_\mathcal{R}, \{R_\alpha(O_\alpha) : \alpha \in I\} \in \Gamma_\mathcal{C}\}$,

where I is an index set, then we have the following statements:

- $(X, \mathcal{R})$ is NRR iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_1(\mathcal{C}_\mathcal{R}, \mathcal{NC}_\mathcal{R})$;
- $(X, \mathcal{R})$ is NRM iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_1(\Omega(\mathcal{R}), \mathcal{NC}_\mathcal{R})$ iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_{fin}(\mathcal{C}_\mathcal{R}, \mathcal{NC}_\mathcal{R})$;
- $(X, \mathcal{R})$ is NRH iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_1(\Omega(\mathcal{R}), \mathcal{N}\Gamma_\mathcal{R})$ iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_{fin}(\mathcal{C}_\mathcal{R}, \mathcal{N}\Gamma_\mathcal{R})$.

We prove that these properties are preserved under relator continuous functions.

Theorem 2.3. *Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be relator spaces and let $f : X \to Y$ be a relator continuous surjection. If $(X, \mathcal{R})$ is* NRM, *then $(Y, \mathcal{S})$ is* NRM.

Proof. Let $(S_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{S}$. Then for every $n \in \mathbb{N}$ there exists $R_n \in \mathcal{R}$ such that for every $x \in X$, $f(R_n(x)) \subset S_n(f(x))$. Since $(X, \mathcal{R})$ is NRM, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for every $O_n \supset F_n$ with $O_n \in \mathcal{T}_\mathcal{R}$ we have $\bigcup R_n(O_n) = X$. Then we have that $\bigcup_{n\in\mathbb{N}} f(R_n(O_n)) = f(\bigcup_{n\in\mathbb{N}} R_n(O_n)) = f(X) = Y$. Note also that $f(R_n(O_n)) \subset S_n(f(O_n))$ for every $n \in \mathbb{N}$.

We prove that the sequence $(f(F_n) : n \in \mathbb{N})$ of finite subsets of Y witnesses that $(Y, \mathcal{S})$ is NRM. Let $B_n \supset f(F_n)$, $B_n \in \mathcal{T}_\mathcal{S}$, and let $A_n = f^{-1}(B_n)$. We claim that $A_n \in \mathcal{T}_\mathcal{R}$. Indeed, if $x \in A_n$, then $y = f(x) \in B_n$. Since $B_n \in \mathcal{T}_\mathcal{S}$, there exists $S \in \mathcal{S}$ such that $S(y) \subset B_n$. By the assumption, we can find $R \in \mathcal{R}$ such that $f(R(x)) \subset S(y) \subset B_n$. That implies $R(x) \subset A_n$, so we proved that $A_n \in \mathcal{T}_\mathcal{R}$. We also have that $A_n \supset F_n$ and $f(R_n(A_n)) \subset S_n(B_n)$, so $\bigcup_{n\in\mathbb{N}} S_n(B_n) = Y$. □

It is a natural to ask how these properties are related to the corresponding properties in topological spaces.

Let $(X, \mathcal{T})$ be a topological space. For every $\mathcal{U} \in \mathcal{O}$ we can define a relation $R_\mathcal{U}$ on X in the following way: $R_\mathcal{U}(x) = St(\{x\}, \mathcal{U})$ for every $x \in X$. Then $\mathcal{R}^*_\mathcal{T} = \{R_\mathcal{U} : \mathcal{U} \in \mathcal{O}\}$ is a relator on X and the ordered pair $(X, \mathcal{R}^*_\mathcal{T})$ is a relator space. It is easy to prove the following statements:
(1): $(X, \mathcal{T})$ is SSR iff $(X, \mathcal{R}^*_\mathcal{T})$ is RR;
(2): $(X, \mathcal{T})$ is SSM iff $(X, \mathcal{R}^*_\mathcal{T})$ is RM;
(3): $(X, \mathcal{T})$ is SSH iff $(X, \mathcal{R}^*_\mathcal{T})$ is RH.

In order to prove similar statement in the case of neighbourhood selection principles, we need the following lemma:

Lemma 2.4. *If* $(X, \mathcal{T})$ *is a regular topological space, then* $\mathcal{T} = \mathcal{T}_{\mathcal{R}^*}$.

Proof. We will first prove that $\mathcal{T} \subset \mathcal{T}_{\mathcal{R}^*}$. Let $U \in \mathcal{T}$ and $x \in U$ is arbitrary. Since $(X, \mathcal{T})$ is regular, one can choose $V \in \mathcal{T}$ such that $x \in \overline{V} \subset U$. Then $\mathcal{U} = \{U, X \setminus \overline{V}\}$ is an open cover of X and $R_{\mathcal{U}}(x) = U$, so $R_{\mathcal{U}}$ witnesses that $U \in \mathcal{T}_{\mathcal{R}^*}$.

On the other hand, if $U \in \mathcal{T}_{\mathcal{R}^*}$ then for every $x \in U$ one can choose $R \in \mathcal{R}^*$ such that $R(x) \subset U$. By the definition of $\mathcal{R}^*$, there is an open cover $\mathcal{U}$ of X such that $R = R_{\mathcal{U}}$, so in fact $St(\{x\}, \mathcal{U}) \subset U$. Since $St(\{x\}, \mathcal{U})$ is an open set, then $U \in \mathcal{T}$. □

Theorem 2.5. *Let* $(X, \mathcal{T})$ *be a regular topological space. Then:*

(1) : $(X, \mathcal{T})$ *is* NSR *if and only if* $(X, \mathcal{R}^*_{\mathcal{T}})$ *is* NRR;

(2) : $(X, \mathcal{T})$ *is* NSM *if and only if* $(X, \mathcal{R}^*_{\mathcal{T}})$ *is* NRM;

(3) : $(X, \mathcal{T})$ *is* NSH *if and only if* $(X, \mathcal{R}^*_{\mathcal{T}})$ *is* NRH.

Proof. We prove only (2), because the proofs of the other two statements are similar. Let $(X, \mathcal{T})$ has the neighbourhood star-Menger property. We will show that $(X, \mathcal{R}^*_{\mathcal{T}})$ has the neighbourhood relator Menger property. Let $(R_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}^*_{\mathcal{T}}$. For every $n \in \mathbb{N}$, there exists an open cover $\mathcal{U}_n$ of X such that $R_n = R_{\mathcal{U}_n}$. By the assumption, one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, such that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, $\{St(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an open cover for X. Since $R_n(O_n) = St(O_n, \mathcal{U}_n)$ and $\mathcal{T} = \mathcal{T}_{\mathcal{R}^*}$ (by Lemma 2.4), it is obvious that $(F_n : n \in \mathbb{N})$ is the sequence we have been looking for. In the same manner we can prove the converse. □

Note that the implications ⇒ in the above theorem are always true.

Our next goal is to show that NRR, NRM, and NRH are in fact the weaker versions of RR, RM and RH, respectively.

First recall that (see [4, 6, 17]) a space X is *strongly star-compact* (resp., *strongly star-Lindelöf*), briefly SSC (resp., SSL) if for every open cover $\mathcal{U}$ of X there exists a finite (resp., countable) subset A of X such that $St(A, \mathcal{U}) = X$.

In [12] we defined the corresponding notions in relator spaces: A relator space $(X, \mathcal{R})$ is *relator compact* (resp., *relator Lindelöf*), briefly RC (resp., RL) if for every relation R from $\mathcal{R}$ there exists a finite (resp., countable) subset A of X such that $R(A) = X$. We say that a relator space $(X, \mathcal{R})$ is *σ-relator compact* (σ-RC) if X is the countable union of relator compact spaces.

The notion of neighbourhood star-Lindelöf property was introduced in [1] in the following way:

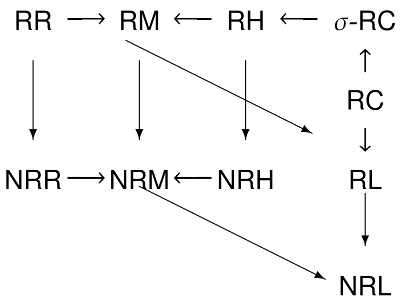
A space X is NSL (*neighbourhood star-Lindelöf*) if for every open cover $\mathcal{U}$ of X there exists a countable subset A of X such that for every neighbourhood U of A, $St(U, \mathcal{U}) = X$.

We now define the corresponding notion in relator spaces.

Definition 2.6. A relator space $(X, \mathcal{R})$ is NRL (*neighbourhood relator Lindelöf*) if for every relation R from $\mathcal{R}$ there exists a countable subset A of X such that for every $O \in \mathcal{T}_{\mathcal{R}}$, $O \supset A$ implies $R(O) = X$.

Note that a topological space $(X, \mathcal{T})$ is SSC (resp., SSL) iff a relator space $(X, \mathcal{R}^*_{\mathcal{T}})$ is RC (resp., RL), and if $(X, \mathcal{T})$ is regular, then $(X, \mathcal{T})$ is NSL iff $(X, \mathcal{R}^*_{\mathcal{T}})$ is NRL.

In the next diagram we present some implications that are obvious.



By using examples 2.4, 3.1, 3.2 and 3.3 from [1], we prove that the inverses of implications RL ⇒ NRL, RR ⇒ NRR, RM ⇒ NRM and RH ⇒ NRH do not hold.

Example 2.7. Consider $X = \mathbb{P}\times(\omega+1)$, where $\mathbb{P}$ denotes the set of irrational points, with the same topology as in Example 2.4 from [1]. A relator $\mathcal{R}^*_{\mathcal{T}}$ on X is defined in the same manner as earlier in the paper.

In [1], it was shown that $(X, \mathcal{T})$ is NSL and not SSL. Since $O \in \mathcal{T}_{\mathcal{R}^*}$ implies $O \in \mathcal{T}$, we conclude that the relator space $(X, \mathcal{R}^*_{\mathcal{T}})$ is NRL, but not RL.

Now we show that consistently, NRM, NRH and NRR do not imply RM, RH and RR, respectively. In fact, the examples are not even RL.

Recall first the definition of $\mathfrak{b}$, $\mathfrak{d}$ and $cov(\mathcal{M})$. For $f, g \in \mathbb{N}^{\mathbb{N}}$ put

$$f \leq^* g \text{ if } f(n) \leq g(n) \text{ for all but finitely many } n.$$

A subset B of $\mathbb{N}^{\mathbb{N}}$ is *bounded* if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for every $f \in B$. $D \subset \mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$ is denoted by $\mathfrak{b}$, and the minimal cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$ is denoted by $\mathfrak{d}$. A subset X of $\mathbb{N}^{\mathbb{N}}$ can be *guessed* by a function $g \in \mathbb{N}^{\mathbb{N}}$ if for every $f \in X$ the set $\{n \in \mathbb{N} : f(n) = g(n)\}$ is infinite. The minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ that cannot be guessed is denoted by $cov(\mathcal{M})$. (see [19])

Example 2.8. ($\omega_1 < \mathfrak{d}$) There is a NRM space which is not RL.

Example 2.9. ($\omega_1 < \mathfrak{b}$) There is a NRH space which is not RL.

Example 2.10. ($\omega_1 < cov(\mathcal{M})$) There is a NRR space which is not RL.

We will use the same space to prove these three assumptions. (see [1], Example 3.1, Example 3.2, Example 3.3)

Let S be a subset of $\mathbb{R}$ such that for every non-empty open $U \subset \mathbb{R}$, $|S \cap U| = \omega_1$ (then in particular, $|S| = \omega_1$). Consider $X_S = S \times (\omega + 1)$ topologized in the same manner as in [1].

In [1] it was proved that X_S with this topology is NSM (resp. NSH, NSR) under assumption $\omega_1 < \mathfrak{d}$ (resp. $\omega_1 < \mathfrak{b}$, $\omega_1 < cov(M)$), but X_S is not SSL. By [1] and Example 2.7, the relator space $(X_S, \mathcal{R}^*_{\mathcal{T}})$ is not RL. Since $O \in \mathcal{T}_{\mathcal{R}^*}$ implies $O \in \mathcal{T}$, we obtain that the relator space $(X_S, \mathcal{R}^*_{\mathcal{T}})$ is NRM (resp. NRH, NRR) under assumption $\omega_1 < \mathfrak{d}$ (resp. $\omega_1 < \mathfrak{b}$, $\omega_1 < cov(\mathcal{M})$).

Problem 2.11. *Do there exist* **ZFC** *examples of spaces as in Examples 2.8, 2.9 and 2.10?*

Now we study the NRM property in finite powers of spaces.

Theorem 2.12. *Let $(X, \mathcal{R})$ be a relator space. If $(X^n, \mathcal{R}^n)$ is* NRM *for every $n \in \mathbb{N}$, then for every sequence $(R_n : n \in \mathbb{N}) \subset \mathcal{R}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for every $O_n \supset F_n$, $O_n \in \mathcal{T}_{\mathcal{R}}$, $n \in \mathbb{N}$, $\{R_n(O_n) : n \in \mathbb{N}\}$ is an ω-cover for X.*

Proof. Let $(R_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}$ and let $\mathbb{N} = N_1 \cup N_2 \cup \ldots$ be a partition of $\mathbb{N}$ into infinite (pairwise disjoint) sets. For every $k \in \mathbb{N}$ and $m \in N_k$ let $S_m = (R_m)^k$. Then $(S_m : m \in N_k)$ is the sequence of relations from $\mathcal{R}^k$. Since $(X^k, \mathcal{R}^k)$ is NRM, one can choose finite subsets $A_m \subset X^k$, $m \in N_k$, such that for every $O_m \supset A_m$ with $O_m \in \mathcal{T}_{\mathcal{R}^k}$ and $m \in N_k$, $\{S_m(O_m) : m \in N_k\}$ is a cover for X^k. For every $m \in N_k$, let F_m be a finite subset of X such that $(F_m)^k \supset A_m$. Consider the sequence $(F_n : n \in \mathbb{N})$. Let $(T_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{T}_{\mathcal{R}}$ such that $F_n \subset T_n$ for every $n \in \mathbb{N}$. We claim that $\{R_n(T_n) : n \in \mathbb{N}\}$ is an ω-cover for X. Let $F = \{x_1, x_2, ..., x_p\}$ be a finite subset of X. Then $\langle x_1, x_2, ..., x_p\rangle \in X^p$, so there exists $n \in N_p$ such that $\langle x_1, x_2, ..., x_p\rangle \in S_n({T_n}^p)$, so that we have $F \subset R_n(T_n)$. □

Theorem 2.13. *If relator spaces $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ are* NRH*, then the relator space $(X \times Y, \mathcal{R} \times \mathcal{S})$ is also* NRH*.*

Proof. Let $(T_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R} \times \mathcal{S}$. For every $n \in \mathbb{N}$, $T_n = R_n \times S_n$, where $R_n \in \mathcal{R}$, $S_n \in \mathcal{S}$. We can choose a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for every sequence $(U_n : n \in \mathbb{N})$ of elements of $\mathcal{T}_\mathcal{R}$ with $U_n \supset F_n$ for every $n \in \mathbb{N}$, every point $x \in X$ belongs to all but finitely many $R_n(U_n)$. We can also find a sequence $(G_n : n \in \mathbb{N})$ of finite subsets of Y such that for every sequence $(V_n : n \in \mathbb{N})$ of elements of $\mathcal{T}_\mathcal{S}$ with $G_n \subset V_n$ for every $n \in \mathbb{N}$, every point $y \in Y$ belongs to all but finitely many $S_n(V_n)$. We show that the sequence $(A_n : n \in \mathbb{N})$ of finite subsets of $X \times Y$, where $A_n = F_n \times G_n$ for every $n \in \mathbb{N}$, witnesses that the relator space $(X \times Y, \mathcal{R} \times \mathcal{S})$ is NRH. Let $O_n \supset A_n$, $O_n \in \mathcal{T}_{\mathcal{R}\times\mathcal{S}}$. Then there exist $U_n \subset X$ and $V_n \subset Y$ such that $O_n = U_n \times V_n$, where $U_n \supset F_n$ and $V_n \supset G_n$. We show that $U_n \in \mathcal{T}_\mathcal{R}$ and $V_n \in \mathcal{T}_\mathcal{S}$. Let $(x, y) \in O_n$. Then there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $(R \times S)(x, y) \subset O_n$. That implies $R(x) \subset U_n$ and $S(y) \subset V_n$.

Let $(x, y) \in X \times Y$. There exist $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ so that x belongs to $R_n(U_n)$ for each $n > n_1$ and y belongs to $S_n(V_n)$ for each $n > n_2$. Put $n_0 = max\{n_1, n_2\}$. Now we have that $(x, y) \in R_n(U_n) \times S_n(V_n)$ for each $n > n_0$, i.e. $(x, y) \in T_n(O_n)$ for every $n > n_0$. □

In a similar way we can prove the following statement:

Theorem 2.14. *If a relator space* $(X, \mathcal{R})$ *is* NRM *and a relator space* $(Y, \mathcal{S})$ *is* NRH, *then the relator space* $(X\times Y, \mathcal{R}\times\mathcal{S})$ *is* NRM*.*

3. Closure selection principles in relator spaces

In [11], we used the closures of open sets to define properties similar to the well-known properties of Menger and star-Menger. In this section we will consider the corresponding properties in relator spaces. We will assume that every relation from the relator is reflexive.

First we recall the definitions of almost Menger and almost star-Menger property and naturally introduce the notion of almost strongly star-Menger property.

Definition 3.1. A topological space $(X, \mathcal{T})$ is:

- AM (*almost Menger*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup\{\mathcal{V}'_n : n \in \mathbf{N}\}$ is a cover of X, where $\mathcal{V}'_n = \{\overline{V} : V \in \mathcal{V}_n\}$ ([14]; compare with [29]);
- ASM (*almost star-Menger*) if for each sequence $(\mathcal{U}_n : n \in \mathbf{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbf{N})$ such that for each $n \in \mathbf{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\overline{St(\cup\mathcal{V}_n, \mathcal{U}_n)} : n \in \mathbf{N}\}$ is a cover of X ([11]);
- ASSM (*almost strongly star-Menger*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\overline{St(F_n, \mathcal{U}_n)} : n \in \mathbb{N}\}$ is a cover of X;

We naturally define the following notions in relator spaces.

Definition 3.2. A relator space $(X, \mathcal{R})$ is:

- ARM (*Almost relator Menger*) if for each sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{cl_\mathcal{R}(R_n(F_n)) : n \in \mathbb{N}\}$ is a cover of X;
- ARR (*Almost relator Rothberger*) if for each sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $\{cl_\mathcal{R}(R_n(x_n)) : n \in \mathbb{N}\}$ is a cover of X;
- ARH (*Almost relator Hurewicz*) if for each sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that every point $x \in X$ belongs to all but finitely many $cl_\mathcal{R}(R_n(F_n))$.

If we use the notation:

- $cl(\omega(R)) = \{cl_{\mathcal{R}}(R(F)) : F \subset X \text{ finite}\}$,
- $cl(\Omega(\mathcal{R}))= \{cl(\omega(R)) : R \in \mathcal{R}\}$,
- $cl(\mathcal{U}_R) = \{cl_{\mathcal{R}}(R(x)) : x \in X\}$,
- $cl(C_{\mathcal{R}}) = \{cl(\mathcal{U}_R) : R \in \mathcal{R}\}$,

then for a relator space $(X, \mathcal{R})$ we have the following statements:

- $(X, \mathcal{R})$ is ARM iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_1(cl(\Omega(\mathcal{R})), C)$ iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_{fin}(cl(C_{\mathcal{R}}), C)$;
- $(X, \mathcal{R})$ is ARR iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_1(cl(C_{\mathcal{R}}), C)$;
- $(X, \mathcal{R})$ is ARH iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_1(cl(\Omega(\mathcal{R})),\Gamma_C)$ iff $(X, \mathcal{R})$ satisfies the selection hypothesis $\mathcal{S}_{fin}(cl(C_{\mathcal{R}}), \Gamma_C)$.

It is clear that RM implies ARM, RR implies ARR, RH implies ARH. It is natural to ask is the converse true in every of these three cases, and also to find out the connection with the corresponding properties in topological spaces.

The following example shows that the implications RM $\Rightarrow$ ARM and RH $\Rightarrow$ ARH can not be inverted. It also shows that there exists ARM (resp. ARH) space which is not NRM (NRH).

Example 3.3. Let $\mathbb{R}$ be the set of real numbers and let D be a countable and dense subset of $\mathbb{R}$. We denote $X = \mathbb{R}$ and for every $\epsilon \in \mathbb{R}$ we define a relation R_ϵ such that $R_\epsilon(x) = \{x\} \cup (D \cap (x - \epsilon, x + \epsilon))$ (see [24, Example 68]). Then $(X, \mathcal{R})$ is a relator space, where $\mathcal{R} = \{R_\epsilon : \epsilon \in \mathbb{R}\}$.

In order to prove that the relator space $(X, \mathcal{R})$ is not RM, we will prove that it is not RL. Let A be a countable subset of X and $\epsilon \in \mathbb{R}$. Then $R_\epsilon(A) \subset A \cup D$. The set $A \cup D$ is countable and $\mathbb{R}$ is uncountable, so $R_\varepsilon(A)$ can not cover X.

Notice that $(X, \mathcal{R})$ is not even NRM. Indeed, let $(R_{\varepsilon_n} : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}$. For every $n \in \mathbb{N}$ we pick arbitrary finite subsets F_n of X. For every $x \in F_n$, $R_{\varepsilon_n}(x) \in \mathcal{T}_{\mathcal{R}}$. Denote $O_n = R_{\varepsilon_n}(F_n)$. Then $R_{\varepsilon_n}(O_n) \subset F_n \cup D$. If $B = \bigcup_{n\in\mathbb{N}} F_n$, then $\bigcup_{n\in\mathbb{N}} R_{\varepsilon_n}(O_n) \subset B \cup D$. Since $B \cup D$ is countable and $\mathbb{R}$ is uncountable, $(X, \mathcal{R})$ is not NRM.

We prove that $(X, \mathcal{R})$ is ARM. Moreover, we prove that $(X, \mathcal{R})$ is ARH. Let $(R_{\varepsilon_n} : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}$. For every $x \in X$ and every $\epsilon \in \mathbb{R}$, $cl_{\mathcal{R}}(R_\varepsilon(x)) = cl_{\mathcal{D}}(D_\epsilon(x))$, where $\mathcal{D} = \{D_\epsilon : \epsilon \in \mathbb{R}\}$ and $D_\epsilon(x) = (x - \epsilon, x + \varepsilon)$ for every $x \in \mathbb{R}$ (These closures are equal because the set D is dense in $\mathbb{R}$). We show that the space $(\mathbb{R}, \mathcal{D})$ is RH. Indeed, for every $n \in \mathbb{N}$, the space $([-n, n], \mathcal{D})$ is relator compact, so we can find finite subsets F_n of $\mathbb{R}$, $n \in \mathbb{N}$, such that $[-n, n] \subset D_{\varepsilon_n}(F_n)$. Let $x \in X$. Then there exists $n_0 \in \mathbb{N}$ such that $x \in [-n_0, n_0]$, so for every $n \geq n_0$, $x \in D_{\varepsilon_n}(F_n)$. Since $(\mathbb{R}, \mathcal{D})$ is RH, the space $(X, \mathcal{R})$ is ARH.

Problem 3.4. *Is there an* ARR *space which is not* RR*?*

Problem 3.5. *Is there an* NRM *(resp.* NRR, NRH*) space which is not* ARM *(*ARR, ARH*)?*

Theorem 3.6. *If a relator space $(X, \mathcal{R})$ is* ARM *and for every $R \in \mathcal{R}$ and every $x \in X$, $R(x)$ intersects only finitely many $R(y)$, $y \in X$, then X is countable.*

Proof. Let $(R_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}$. Since $(X, \mathcal{R})$ is ARM, we can choose finite subsets F_n, $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} cl_{\mathcal{R}}(R_n(F_n)) = X$. Put $A_n = \{x \in X : R_n(x) \cap R_n(F_n) \neq \emptyset\}$. By the assumption, the set A_n is finite for every $n \in \mathbb{N}$. We prove that $\bigcup_{n\in\mathbb{N}} A_n = X$. Let $x \in X$. Then there exists $n \in \mathbb{N}$ such that $x \in cl_{\mathcal{R}}(R_n(F_n))$. For every $R \in \mathcal{R}$, $R(x) \cap R_n(F_n) \neq \emptyset$, so $R_n(x) \cap R_n(F_n) \neq \emptyset$. That implies $x \in A_n$. $\square$

Lemma 3.7. *If a topological space $(X, \mathcal{T})$ is regular, then for every subset A of X, $\overline{A} = cl_{\mathcal{R}^*_{\mathcal{T}}}(A)$.*

The proof is similar to the proof of Lemma 2.4.

The following is straightforward:

Theorem 3.8. *If $(X, \mathcal{T})$ is regular then $(X, \mathcal{T})$ is* ASSM *if and only if the relator space $(X, \mathcal{R}^*_{\mathcal{T}})$ is* ARM.

In [11], we showed that the almost Menger property is preserved under almost continuous functions. We shall see that the similar statement holds in relator spaces. First we define the notion of almost relator continuous function.

Definition 3.9. Let $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ be relator spaces. We say that a function $f : (X, \mathcal{R}) \to (Y, \mathcal{S})$ is *almost relator continuous* if for every $S \in \mathcal{S}$ there exists $R \in \mathcal{R}$ such that for every $x \in X$, $f(cl_{\mathcal{R}}(R(x))) \subset S(f(x))$.

Theorem 3.10. *Let $(X, \mathcal{R})$ be* ARM *and $(Y, \mathcal{S})$ be a relator space. If $f : (X, \mathcal{R}) \to (Y, \mathcal{S})$ is an almost relator continuous surjection, then $(Y, \mathcal{S})$ is* RM.

Proof. Let $(S_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{S}$. Since f is almost relator continuous, we can pick a relation $R_n \in \mathcal{R}$ for every $n \in \mathbb{N}$ such that $f(cl_{\mathcal{R}}(R_n(x))) \subset S_n(f(x))$ for every $x \in X$. By the assumption, we can find finite $F_n \subset X$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} cl_{\mathcal{R}}(R_n(F_n)) = X$. Now we have that $Y = f(\bigcup_{n \in \mathbb{N}} cl_{\mathcal{R}}(R_n(F_n))) = \bigcup_{n \in \mathbb{N}} f(cl_{\mathcal{R}}(R_n(F_n))) \subset \bigcup_{n \in \mathbb{N}} S_n(f(F_n))$. So, the sequence $(f(F_n) : n \in \mathbb{N})$ of finite subsets of Y witnesses that $(Y, \mathcal{S})$ is RM. □

Now we study the ARM property in finite powers.

Theorem 3.11. *Let $(X, \mathcal{R})$ be a relator space. If all finite powers of $(X, \mathcal{R})$ are* ARM*, then for every sequence $(R_n : n \in \mathbb{N})$ of relations from $\mathcal{R}$, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X, such that $\{cl_{\mathcal{R}}(R_n(F_n)) : n \in \mathbb{N}\}$ is an ω-cover of X.*

Proof. Let $(R_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}$ and let $\mathbb{N} = N_1 \cup N_2 \cup ...$ be a partition of $\mathbb{N}$ into infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $m \in N_k$ let $S_m = (R_m)^k$. Then $(S_m : m \in N_k)$ is a sequence of relations from $\mathcal{R}^k$. Since $(X^k, \mathcal{R}^k)$ is ARM, then one can choose finite subsets A_m of X^k, $m \in N_k$, such that $\{cl_{\mathcal{R}^k}(S_m(A_m)) : m \in N_k\}$ is a cover of X^k. For every $m \in N_k$, let F_m be a finite subset of X such that $F_m^k \supset A_m$. Consider the sequence of all $F_m, m \in N_k$, $k \in \mathbb{N}$, chosen in this way and denote it $(F_n : n \in \mathbb{N})$. We claim that $\{cl_{\mathcal{R}}(R_n(F_n)) : n \in \mathbb{N}\}$ is an ω-cover of X. Let $F = \{x_1, x_2, ..., x_p\}$ be a finite subset of X. Then $\langle x_1, x_2, ..., x_p \rangle \in X^p$. There is $n \in N_p$ such that $\langle x_1, x_2, ..., x_p \rangle \in cl_{\mathcal{R}^p}(R_n^p(F_n^p))$, so that we have $F \subset cl_{\mathcal{R}}(R_n(F_n))$. □

The following two statements and their proofs are similar to the statements of Theorems 2.13 and 2.14.

Theorem 3.12. *The product of two* ARH *spaces is* ARH.

Theorem 3.13. *If a relator space $(X, \mathcal{R})$ is* ARM *and a relator space $(Y, \mathcal{S})$ is* ARH*, then the relator space $(X \times Y, \mathcal{R} \times \mathcal{S})$ is* ARM.

Recently, in [20], the notion of weakly Menger space was introduced in the following way:

A topological space is *weakly Menger* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is dense in X.

Now we define the corresponding property in relator spaces.

Definition 3.14. A relator space $(X, \mathcal{R})$ is *weakly relator Menger* (WRM) if for every sequence $(R_n : n \in \mathbb{N})$ of elements of $\mathcal{R}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $cl_{\mathcal{R}}(\bigcup_{n \in \mathbb{N}} R_n(F_n)) = X$.

Let $(X, \mathcal{R})$ be a relator space. Then the following statements are equivalent:

- (1): $(X, \mathcal{R})$ is WRM;
- (2): The selection hypothesis $\mathcal{S}_1(\Omega(\mathcal{R}), \mathcal{D}_{\mathcal{R}})$ is true for X;
- (3): The selection hypothesis $\mathcal{S}_{fin}(C_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}})$ is true for X.

It is clear that every almost Menger space is weakly Menger. In [20], it was proved that there exists a weakly Menger space which is not almost Menger. Naturally, we are interested in the relationship between ARM and WRM spaces.

Theorem 3.15. *Let $(X, \mathcal{R})$ be a relator space. If $(X, \mathcal{R})$ is* ARM*, then $(X, \mathcal{R})$ is* WRM.

Proof. Let $(R_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}$. Since $(X, \mathcal{R})$ is ARM, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\bigcup_{n\in\mathbb{N}} cl_{\mathcal{R}}(R_n(F_n)) = X$. We will show that $cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} R_n(F_n)) = X$. Let $x \in X$. Then there exists $n \in \mathbb{N}$ such that $x \in cl_{\mathcal{R}}(R_n(F_n))$. So, for every $R \in \mathcal{R}$, $R(x) \cap R_n(F_n) \neq \emptyset$. That implies $R(x) \cap \bigcup_{n\in\mathbb{N}} R_n(F_n) \neq \emptyset$ for every $R \in \mathcal{R}$, so $x \in cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} R_n(F_n))$. □

We show that the inverse does not hold.

Example 3.16. There exists a relator space which is WRM and not ARM.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{I}$ the set of irrational numbers. For every $x \in \mathbb{I}$ enumerate all sequences of rational numbers converging to x in the Euclidean topology as $\{x_\alpha : \alpha < \mathbf{c}\}$. We construct a relator $\mathcal{D}$ on $\mathbb{R}$ in the following way: for every $\alpha < \mathbf{c}$, we define a relation D_α such that $D_\alpha(r) = \{r\}$ for every $r \in \mathbb{Q}$ and $D_\alpha(x) = \{x\} \cup \{x_{\alpha,i} : i \in \mathbb{N}\}$ for every $x \in \mathbb{I}$ and we put $\mathcal{D} = \{D_\alpha : \alpha < \mathbf{c}\}$ (see [24, Example 65]).

The relator space $(\mathbb{R}, \mathcal{D})$ is WRM because for every sequence $(D_n : n \in \mathbb{N})$ of relations from $\mathcal{D}$, we can choose finite subsets F_n of $\mathbb{R}$ such that $\bigcup_{n\in\mathbb{N}} D_n(F_n) = \mathbb{Q}$ and $cl_{\mathcal{D}}(\mathbb{Q}) = \mathbb{R}$.

Let us show that the relator space $(\mathbb{R}, \mathcal{D})$ is not ARM. Notice that $cl_{\mathcal{D}}(D(x)) = D(x)$ for every $D \in \mathcal{D}$ and every $x \in \mathbb{R}$. Let $(D_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{D}$. Since $\mathbb{I}$ is uncountable and for every $D \in \mathcal{D}$ and every $x \in \mathbb{I}$ only $D(x)$ contains x, for every sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $\mathbb{R}$ there exists $x \in \mathbb{I}$ such that $x \notin \bigcup_{n\in\mathbb{N}} cl_{\mathcal{D}}(D_n(F_n))$.

We say that a relator space $(X, \mathcal{R})$ is *almost relator Lindelöf* (ARL) if for every relation $R \in \mathcal{R}$ there exists a countable subset A of X such that $cl_{\mathcal{R}}(A) = X$.

Theorem 3.17. *If a relator space $(X, \mathcal{R})$ is* WRM*, then $(X, \mathcal{R})$ is* ARL.

Proof. Let $R \in \mathcal{R}$. Then there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} R(F_n)) = X$. If we put $A = \bigcup_{n\in\mathbb{N}} F_n$, then A is countable and $cl_{\mathcal{R}}(A) = X$. □

The property of WRM is preserved under relator continuous functions.

Theorem 3.18. *If a relator space $(X, \mathcal{R})$ is* WRM *and $f : (X, \mathcal{R}) \to (Y, \mathcal{S})$ is a relator continuous surjection, then $(Y, \mathcal{S})$ is also* WRM.

Proof. Let $(S_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{S}$. Since f is relator continuous, for every $n \in \mathbb{N}$ there exists $R_n \in \mathcal{R}$ such that $f(R_n(x)) \subset S_n(f(x))$ for every $x \in X$. Then we can find finite subsets F_n of X such that $cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} R_n(F_n)) = X$. We prove that the sequence $(f(F_n) : n \in \mathbb{N})$ witnesses that the relator space $(Y, \mathcal{S})$ is WRM.

Let $y \in Y$. Then there exists $x \in X$ such that $y = f(x)$. We have that $x \in cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} R_n(F_n))$. So, for every $R \in \mathcal{R}$ there exists $n \in \mathbb{N}$ such that $R(x) \cap R_n(F_n) \neq \emptyset$. That implies $f(R(x)) \cap f(R_n(F_n)) \neq \emptyset$ for some $n \in \mathbb{N}$. If we pick arbitrary $S \in \mathcal{S}$, then we can find $R \in \mathcal{R}$ such that $f(R(x)) \subset S(f(x))$ for every $x \in X$. So, we conclude that $S(f(x)) \cap S_n(f(F_n)) \neq \emptyset$ for some $n \in \mathbb{N}$, i.e. we have that $S(y) \cap \bigcup_{n\in\mathbb{N}} S_n(f(F_n)) \neq \emptyset$. That implies $y \in cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} S_n(f(F_n)))$. □

We now consider the property of WRM in finite powers of spaces.

We say that a relator space $(X,\mathcal{R})$ is *almost relator compact* (ARC) if for every $R \in \mathcal{R}$ there exists a finite subset F of X such that $cl_{\mathcal{R}}(R(F)) = X$.

Theorem 3.19. *Let $(X,\mathcal{R})$ and $(Y,\mathcal{S})$ be relator spaces. If $(X,\mathcal{R})$ is* WRM *and $(Y,\mathcal{S})$ is* ARC*, then the product $(X\times Y,\mathcal{R}\times\mathcal{S})$ is* WRM*.*

Proof. Let $(T_n : n \in \mathbb{N})$ be a sequence of relations from $\mathcal{R}\times\mathcal{S}$. For every $n \in \mathbb{N}$, $T_n = R_n \times S_n$, where $R_n \in \mathcal{R}$, $S_n \in \mathcal{S}$. There exist sequences $(F_n : n \in \mathbb{N})$ and $(G_n : n \in \mathbb{N})$ of finite subsets of X and Y respectively, such that $cl_{\mathcal{R}}(\bigcup_{n\in\mathbb{N}} R_n(F_n)) = X$ and $cl_{\mathcal{S}}(S_n(G_n)) = Y$ for each $n \in \mathbb{N}$. We show that $cl_{\mathcal{R}\times\mathcal{S}}(\bigcup_{n\in\mathbb{N}} T_n(F_n\times G_n)) = X\times Y$. Let $(x, y) \in X \times Y$. Then for every $R \in \mathcal{R}$, $R(x) \cap (\bigcup_{n\in\mathbb{N}} R_n(F_n)) \neq \emptyset$ and for every $S \in \mathcal{S}$ and every $n \in \mathbb{N}$, $S(y) \cap S_n(G_n) \neq \emptyset$. So, we can find $n \in \mathbb{N}$ such that for every $T = R\times S \in \mathcal{R}\times\mathcal{S}$, $T(x, y) \cap T_n(F_n \times G_n) \neq \emptyset$. □

Problem 3.20. *Is the product of two* WRM *spaces also a* WRM *space?*

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