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A new application of generalized power increasing sequences

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Abstract. In this paper, an application of quasi-σ-power increasing sequences has been generalized for quasi-f-power increasing sequences. We have also obtained some new results.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by t_n^α nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), that is

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = O(n^\alpha), \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha; \delta \mid_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if (see [5])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^\alpha \mid^k < \infty. \tag{3}$$

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see[1]). A positive sequence $X = (X_n)$ is said to be a quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ holds for all $n \geq m \geq 1$ (see [6]). It should be noted that every almost increasing sequence is a quasi-σ-power increasing sequence for any nonnegative σ, but the converse may not be true as can be seen by taking an example, say $X_n = n^{-\sigma}$ for $\sigma > 0$. A sequence (λ_n) is said to be of bounded variation, denote by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} \mid \lambda_n - \lambda_{n+1} \mid < \infty$. In [3], we have proved the following theorem dealing with an application of a quasi-σ-power increasing sequences.

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Theorem 1.1. *Let (X_n) be a quasi-σ-power increasing sequence for some σ $(0 < \sigma < 1)$. Suppose also that there exist sequences (β_n) and (λ_n), such that*

$$| \Delta\lambda_n | \leq \beta_n \tag{4}$$

$$\beta_n \to 0 \quad as \quad n \to \infty \tag{5}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty \tag{6}$$

$$| \lambda_n | X_n = O(1) \quad as \quad n \to \infty. \tag{7}$$

If the sequence (u_n^α) defined by (see [7])

$$u_n^\alpha = | t_n^\alpha |, \quad \alpha = 1 \tag{8}$$

$$u_n^\alpha = \max_{1 \leq v \leq n} | t_v^\alpha |, \quad 0 < \alpha < 1 \tag{9}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\delta k-1} (u_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

then the series $\sum a_n\lambda_n$ is summable $| C, \alpha; \delta |_k$, $k \geq 1$ and $0 \leq \delta < \alpha \leq 1$.

Remark 1.2. It may be noted that the condition "$(\lambda_n) \in \mathcal{BV}$ " should be added in the statement of Theorem 1.1.

2. The main result

The aim of this paper is to generalize Theorem 1.1 using a new class of power increasing sequences. For this purpose, we need the concept of a quasi-f-power increasing sequence. A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence, if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$, holds for $n \geq m \geq 1$, where $f = (f_n) = [n^\sigma(\log n)^\gamma, \gamma \geq 0, 0 < \sigma < 1]$ (see [8]). It should be noted that if we take γ=0, then we get a quasi-σ-power increasing sequence.

Now, we shall prove the following more general theorem.

Theorem 2.1. *Let $(\lambda_n) \in \mathcal{BV}$ and (X_n) be a quasi-f-power increasing sequence. If the conditions from (4) to (7) and (10) are satisfied, then the series $\sum a_n\lambda_n$ is summable $| C, \alpha; \delta |_k$, $k \geq 1$ and $0 \leq \delta < \alpha \leq 1$.*

We need the following lemmas for the proof of our theorem.

Lemma 2.2. *Except for the condition $(\lambda_n) \in \mathcal{BV}$, under the conditions on (X_n), (β_n) and (λ_n) as expressed in the statement of the theorem, the following conditions hold*

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{11}$$

$$nX_n\beta_n = O(1), \tag{12}$$

Proof. Since $\beta_n \to 0$, then we have $\Delta\beta_n \to 0$, and hence

$$\begin{aligned}
\sum_{n=1}^{\infty} \beta_n X_n &\leq \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} |\Delta\beta_v| = \sum_{v=1}^{\infty} |\Delta\beta_v| \sum_{n=1}^{v} X_n \\
&= \sum_{v=1}^{\infty} |\Delta\beta_v| \sum_{n=1}^{v} n^{\sigma}(\log n)^{\gamma} X_n n^{-\sigma}(\log n)^{-\gamma} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\sigma}(\log v)^{\gamma} X_v \sum_{n=1}^{v} n^{-\sigma}(\log n)^{-\gamma} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\sigma}(\log v)^{\gamma} X_v \sum_{n=1}^{v} n^{\epsilon}(\log n)^{-\gamma} n^{-\sigma-\epsilon}, \quad 0 < \epsilon < \sigma + \epsilon < 1 \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\sigma} X_v (\log v)^{\gamma} v^{\epsilon} (\log v)^{-\gamma} \sum_{n=1}^{v} n^{-\sigma-\epsilon} \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\sigma+\epsilon} X_v \int_0^v x^{-\sigma-\epsilon} dx \\
&= O(1) \sum_{v=1}^{\infty} |\Delta\beta_v| v^{\sigma+\epsilon} X_v v^{1-\sigma-\epsilon} \\
&= O(1) \sum_{v=1}^{\infty} v|\Delta\beta_v| X_v = O(1).
\end{aligned}$$

Again, we have that

$$\begin{aligned}
n\beta_n X_n &= nX_n \sum_{v=n}^{\infty} \Delta\beta_v \leq nX_n \sum_{v=n}^{\infty} |\Delta\beta_v| \\
&= n^{1-\sigma}(\log n)^{-\gamma} n^{\sigma}(\log n)^{\gamma} X_n \sum_{v=n}^{\infty} |\Delta\beta_v| \\
&\leq n^{1-\sigma}(\log n)^{-\gamma} \sum_{v=n}^{\infty} v^{\sigma}(\log v)^{\gamma} X_v |\Delta\beta_v| \\
&\leq \sum_{v=n}^{\infty} v^{1-\sigma}(\log v)^{-\gamma} X_v v^{\sigma}(\log v)^{\gamma} |\Delta\beta_v| \\
&= \sum_{v=1}^{\infty} v X_v |\Delta\beta_v| = O(1).
\end{aligned}$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. ([4]) *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1\leq m\leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right| . \tag{13}$$

Proof Theorem 2.1. Let (T_n^{α}) be the n-th (C, α), with $0 < \alpha \leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (1), we have

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{14}$$

Applying Abel's transformation first and then using Lemma 2.3, we have that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p+\frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v,$$

$$\begin{aligned}
\mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\mid\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\mid+\frac{\mid\lambda_n\mid}{A_n^\alpha}\mid\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\mid\\
&\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha u_v^\alpha\mid\Delta\lambda_v\mid+\mid\lambda_n\mid u_n^\alpha\\
&= T_{n,1}^\alpha+T_{n,2}^\alpha.
\end{aligned}$$

Since

$$\mid T_{n,1}^\alpha+T_{n,2}^\alpha\mid^k\leq 2^k(\mid T_{n,1}^\alpha\mid^k+\mid T_{n,2}^\alpha\mid^k),$$

to complete the proof of the theorem, by (3), it is enough to show that

$$\sum_{n=1}^{\infty}n^{\delta k-1}\mid T_{n,r}^\alpha\mid^k<\infty,\quad for\quad r=1,2.$$

Now, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}n^{\delta k-1}\mid T_{n,1}^\alpha\mid^k &\leq \sum_{n=2}^{m+1}n^{\delta k-1}(A_n^\alpha)^{-k}\{\sum_{v=1}^{n-1}(A_v^\alpha)^k(u_v^\alpha)^k\mid\Delta\lambda_v\mid\}\times\{\sum_{v=1}^{n-1}\mid\Delta\lambda_v\mid\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}n^{\delta k-\alpha k-1}\{\sum_{v=1}^{n-1}v^{\alpha k}(u_v^\alpha)^k\beta_v\}\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(u_v^\alpha)^k\beta_v\sum_{n=v+1}^{m+1}\frac{1}{n^{1+\alpha k-\delta k}}\\
&= O(1)\sum_{v=1}^{m}v^{\alpha k}(u_v^\alpha)^k\beta_v\int_v^\infty\frac{dx}{x^{1+\alpha k-\delta k}}\\
&= O(1)\sum_{v=1}^{m}v^{\delta k}(u_v^\alpha)^k\beta_v=O(1)\sum_{v=1}^{m}v\beta_v v^{\delta k-1}(u_v^\alpha)^k\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}r^{\delta k-1}(u_r^\alpha)^k+O(1)m\beta_m\sum_{v=1}^{m}v^{\delta k-1}(u_v^\alpha)^k\\
&= O(1)\sum_{v=1}^{m-1}\mid\Delta(v\beta_v)\mid X_v+O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1}\mid(v+1)\Delta\beta_v-\beta_v\mid X_v+O(1)m\beta_m X_m\\
&= O(1)\sum_{v=1}^{m-1}v\mid\Delta\beta_v\mid X_v+O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)m\beta_m X_m\\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2.2. Finally, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid T_{n,2}^{\alpha} \mid^{k} &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\delta k-1} (u_n^{\alpha})^k \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid n^{\delta k-1} (u_n^{\alpha})^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\delta k-1} (u_v^{\alpha})^k + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\delta k-1} (u_n^{\alpha})^k \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 2.2. This completes the proof of the theorem.

It should be noted that, if we take $\delta = 0$ (resp. $\alpha = 1$), then we get a new result for $\mid C, \alpha \mid_k$ (resp. $\mid C, 1; \delta \mid_k$) summability. Also, if we take (X_n) as an almost increasing sequence, then we obtain a result of Bor [2] (in this case the condition $(\lambda_n) \in \mathcal{BV}$ is not needed). If we take $\gamma = 0$, then we get Theorem 1.1.

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