Filomat 26:3 (2012), 473–477
DOI 10.2298/FIL1203473D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on convergence in measure and selection principles

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Abstract. It is proved that some classes of sequences of measurable functions satisfy certain selection principles related to special modes of convergence (convergence in measure, almost everywhere convergence, almost uniform convergence, mean convergence).

1. Introduction

By $\mathbb{N}$, $\mathbb{R}$ and $\overline{\mathbb{R}}$ we denote the set of natural numbers, real numbers, and the extended real line $\mathbb{R} \cup \{-\infty, \infty\}$, respectively.

Throughout this note $(X, \mathcal{M}, \mu)$, or shortly X, denotes a measure space with a complete measure $\mu : \mathcal{M} \to \overline{\mathbb{R}}$ (and $\mathcal{M}$ a σ-algebra of subsets of X measurable with respect to μ). E is always an element in $\mathcal{M}$ such that $\mu(E) < \infty$. All functions are measurable and finite almost everywhere on E.

Our notation and terminology concerning measure spaces are standard and follow [2, 8, 9, 13–15].

The paper deals with the following problem. Let a sequence $(S_n : n \in \mathbb{N})$ of sequences of measurable functions, all converging (in a mode of convergence) to a function f, be given. Apply a selection procedure π to find a sequence s constructed by choosing elements from each S_n using π, and converging to f in the same or different mode of convergence.

We begin with the following definition of the selection principle we basically consider in this article.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sequences of measurable functions from a measure space $(X, \mathcal{M}, \mu)$ into $\mathbb{R}$ or $\overline{\mathbb{R}}$. Then:

The symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $b = (b_n : n \in \mathbb{N}) \in \mathcal{B}$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$.

For more information on selection principles (and corresponding infinitely long games) see the survey papers [10, 12] and references therein.

In a number of papers by the authors that appeared recently in the literature it was demonstrated that some classes $\mathcal{A}$ and $\mathcal{B}$ of sequences of positive real numbers have certain nice selection properties ([3–7]).

In this article our selections are related to special modes of convergence of sequences of measurable functions which converge in measure, or are (uniformly) almost everywhere convergent to a function.

2010 *Mathematics Subject Classification*. Primary 40A30; Secondary 28A99
Keywords. Selection principles, convergence in measure, almost everywhere convergence, almost uniform convergence, Egorov's theorem.
Received: 04 October 2011; Accepted: 12 October 2011
Communicated by Vladimir Rakočević
Research supported by MES RS.
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2. Definitions

Let $(X, \mathcal{M}, \mu)$ be a measure space. A function $f : E \to \overline{\mathbb{R}}$ is *almost everywhere finite on E* if $\mu(\{x \in E : f(x) = \infty \text{ or } f(x) = -\infty\}) = 0$.

A function $f : E \to \overline{\mathbb{R}}$ is a *measurable function* if for each $c \in \overline{\mathbb{R}}$, $\{x \in A : f(x) > c\} \in \mathcal{M}$ or, equivalently, $f^{\leftarrow}(B) \in \mathcal{M}$ for each $B \in \mathfrak{B}_{\overline{\mathbb{R}}}$, where $\mathfrak{B}_{\overline{\mathbb{R}}} = \{E \subset \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathfrak{B}_{\mathbb{R}}\}$ is the σ-algebra of Borel sets in $\overline{\mathbb{R}}$.

Let $E \in \mathcal{M}$ and let $f_n : E \to \overline{\mathbb{R}}$, $n \in \mathbb{N}$, be measurable and almost everywhere finite functions, and $f : E \to \overline{\mathbb{R}}$. Then:

1. $(f_n)_{n\in\mathbb{N}}$ converges *almost everywhere* to f on E, denoted $(f_n)_{n\in\mathbb{N}} \xrightarrow{a.e.} f$, if $\mu(\{x \in E : (f_n(x))_{n\in\mathbb{N}} \nrightarrow f(x)\}) = 0$.

2. $(f_n)_{n\in\mathbb{N}}$ converges *in measure* (or *μ-converges*) to f on E, denoted $(f_n)_{n\in\mathbb{N}} \xrightarrow{\mu} f$, if for each $\varepsilon > 0$ it holds $\lim_{n\to\infty} \mu(\{x \in E : f_n(x) \text{ finite } \forall\, n \in \mathbb{N} \text{ and } |f_n(x) - f(x)| \geq \varepsilon\}) = 0$.

3. $(f_n)_{n\in\mathbb{N}}$ converges *almost uniformly* to f on E, denoted $(f_n)_{n\in\mathbb{N}} \xrightarrow{a.u.} f$, if for each $\varepsilon > 0$ there is a measurable subset $E_\varepsilon \subset E$ with $\mu(E_\varepsilon) < \varepsilon$ such that $(f_n(x))_{n\in\mathbb{N}} \rightrightarrows f(x)$ on $E \setminus E_\varepsilon$.

Notice the fact that in each of the above three kinds of convergence the function f is measurable and almost everywhere finite on E.

Clearly, almost uniform convergence on E implies almost everywhere convergence (without assumption $\mu(E) < \infty$).

If $\mu(E) < \infty$ and $f, f_n : E \to \overline{\mathbb{R}}$ for each $n \in \mathbb{N}$, then according to the well-known theorems in measure theory, the following holds:

$$\begin{array}{ccc} \text{a.e. convergence} \Leftrightarrow & \text{a.u. convergence} & \Rightarrow \mu \text{ convergence} \\ \Uparrow & \Uparrow & \\ \text{pointwise convergence} \Leftarrow & \text{uniform convergence} & \end{array}$$

Notation

Let $(X, \mathcal{M}, \mu)$ be a measure space, $E \in \mathcal{M}$, $\mu(E) < \infty$, and f a function measurable and finite a.e. on E. Then:

$\Sigma_f^{a.e.}(E) = \{(f_n)_{n\in\mathbb{N}} : (f_n)_{n\in\mathbb{N}} \xrightarrow{a.e.} f \text{ on } E\} = \{(f_n)_{n\in\mathbb{N}} : (f_n)_{n\in\mathbb{N}} \xrightarrow{a.u.} f \text{ on } E\}$.

$\Sigma_f^{\mu}(E) = \{(f_n)_{n\in\mathbb{N}} : (f_n)_{n\in\mathbb{N}} \xrightarrow{\mu} f \text{ on } E\}$.

3. Results

Throughout this section, as we mentioned in Introduction, $(X, \mathcal{M}, \mu)$ will be always a measure space, and E an element in $\mathcal{M}$ such that $\mu(E) < \infty$.

We prove first a theorem which is, in a sense, a selective version of the celebrated Egorov theorem [2, Theorem 2.2.1].

Theorem 3.1. *The selection principle* $\mathsf{S}_1(\Sigma_f^{a.e.}(E), \Sigma_f^{a.e.}(E))$ *is satisfied.*

Proof. Let $(S_n = (f_n^m)_{m\in\mathbb{N}} : n \in \mathbb{N})$ be a sequence of elements from $\Sigma_f^{a.e.}(E)$. Let $\delta > 0$ $(\delta \leq \mu(E))$. For each $n \in \mathbb{N}$ we have $(f_n^m)_{m\in\mathbb{N}} \xrightarrow{a.u.} f$ on E, and thus for each $n \in \mathbb{N}$ there is a measurable set $E_n \subset E$ such that $\mu(E_n) < \frac{\delta}{2^n}$ and $(f_n^m)_{m\in\mathbb{N}}$ uniformly converges to f on $E \setminus E_n$. Let $E_\delta = \bigcup_{n\in\mathbb{N}} E_n$. Then E_δ is measurable, $\mu(E_\delta) > 0$, and

$$\mu(E_\delta) \leq \sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta.$$

Also, for each $n \in \mathbb{N}$, $(f_n^m)_{m\in\mathbb{N}}$ uniformly converges to f on $E \setminus E_\delta$, so that there are $m_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that $|f_n^m(x) - f(x)| < \frac{1}{2^n}$ for each $x \in E \setminus E_\delta$ and each $m \geq m_n$.

Construct now the sequence $(g_n)_{n\in\mathbb{N}}$ of functions defined on $E \setminus E_\delta$ in the following way:

$$g_n(x) = f_n^{m_n^*}(x), \quad \text{for some } m_n^* \geq m_n, \ n \in \mathbb{N}.$$

In this way, we have constructed a sequence of functions defined on E (we can redefine functions g_n on the set E_δ) which uniformly converges to f on the set $E \setminus E_\delta$.

We prove that the sequence $(g_n : n \in \mathbb{N})$ is a required selector for $(S_n : n \in \mathbb{N})$. Let $\varepsilon > 0$. Consider two cases:

Case 1: $\varepsilon \geq \delta$.

Then, by the construction of functions g_n, $n \in \mathbb{N}$, we conclude that E_δ is the subset of E with $\mu(E_\delta) < \varepsilon$ such that $(g_n)_{n\in\mathbb{N}}$ uniformly converges to f on $E \setminus E_\delta$.

Case 2: $\varepsilon < \delta$.

There is $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} \frac{\delta}{2^n} < \varepsilon$. Set $E_\varepsilon = \bigcup_{n=n_0}^{\infty} E_n$. Then $\mu(E_\epsilon) < \varepsilon$ (and $\mu(E_\varepsilon) > 0$), and for each $n \geq n_0$ it holds $|g_n(x) - f(x)| < 2^{-n}$ for each $x \in E \setminus E_\varepsilon$. This means that $(g_n)_{n\in\mathbb{N}} \xrightarrow{a.u.} f$ on E. This completes the proof. □

Remark 3.2. Notice that this theorem remains true if f_n^m, $n, m \in \mathbb{N}$, and f are functions from E into a separable metric space Y, and f_n^m's are measurable with respect to $\mathcal{M}$ and the Borel σ-algebra $\mathcal{B}(Y)$ (see [2, Theorem 7.1.12]).

Remark 3.3. Theorem 3.1 holds also for the sets E with infinite measure provided there is a μ-integrable function φ such that $|f_n^m| \leq \varphi$ for all $m, n \in \mathbb{N}$ (see [2, 2.12.45]).

¿From Theorem 3.1 we obtain the following corollary.

Corollary 3.4. *The selection principle* $\mathsf{S}_1(\Sigma_f^{a.e.}(E), \Sigma_f^{\mu}(E))$ *is satisfied.*

Also, we have the following result.

Theorem 3.5. *The selection principle* $\mathsf{S}_1(\Sigma_f^{\mu}(E), \Sigma_f^{a.e.}(E))$ *is satisfied.*

Proof. Let $(S_n : n \in \mathbb{N})$, $S_n = (f_n^m)_{m\in\mathbb{N}}$, be a sequence of elements from $\Sigma_f^{\mu}(E)$. By the Riesz theorem (see [2, Theorem 2.2.5], [15, Theorem 6.24]) for each $n \in \mathbb{N}$ the sequence S_n contains a subsequence s_n converging to f almost everywhere. Apply now Theorem 3.1 to the sequence $(s_n : n \in \mathbb{N})$ to conclude that there are functions $h_n \in s_n$, hence $h_n \in S_n$, $n \in \mathbb{N}$, such that $(h_n)_{n\in\mathbb{N}} \xrightarrow{a.e.} f$, i.e. the sequence $(h_n : n \in \mathbb{N})$ witnesses for $(S_n : n \in \mathbb{N})$ that $\mathsf{S}_1(\Sigma_f^{\mu}(E), \Sigma_f^{a.e.}(E))$ is true. □

The next theorem shows that for μ-convergence we have something more.

First, we define the following selection principle (see [11, 12] for general case, and [1] for special case when $\mathcal{A}$ and $\mathcal{B}$ are both the collection Σ_x of sequences in a topological space converging to a point x in the space).

Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Then the symbol $\alpha_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(a_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $b \in \mathcal{B}$ such that for each $n \in \mathbb{N}$ the set $a_n \setminus b$ is finite. (A space X satisfying $\alpha_1(\Sigma_x, \Sigma_x)$ for each $x \in X$ is called an α_1-*space*.)

Theorem 3.6. *The selection principle* $\alpha_1(\Sigma_f^{\mu}(E), \Sigma_f^{\mu}(E))$ *is true.*

Proof. Let $(S_n : n \in \mathbb{N})$, $S_n = (f_n^m)_{m\in\mathbb{N}}$, be a sequence of elements from $\Sigma_f^\mu(E)$ and let $\varepsilon > 0$. For all $m, n \in \mathbb{N}$ let

$$E_n^m = \{x \in E : |f_n^m(x) - f(x)| \geq \frac{\varepsilon}{n}\}.$$

In this way, for each $n \in \mathbb{N}$ the sequence $s_n = (\mu(E_n^m))_{m\in\mathbb{N}}$ of real numbers corresponds to the sequence S_n of functions. By definition of μ-convergence we have actually the sequence $(s_n : n \in \mathbb{N})$ of real sequences each converging to 0. Since $\mathbb{R}$ (as each first countable space) satisfies the α_1 property, there is a sequence s in $\mathbb{R}$ converging to 0 and such that for each n the set $s_n \setminus s$ is finite. Associate to s the sequence S of corresponding functions from sequences S_n, $n \in \mathbb{N}$. Then $S_n \setminus S$ is finite for each $n \in \mathbb{N}$ and $S \xrightarrow{\mu} f$, i.e. $\alpha_1(\Sigma_f^\mu(E), \Sigma_f^\mu(E))$ is satisfied. □

If we take from the sequence s in the proof of the previous theorem a subsequence $t = (t_k)_{k\in\mathbb{N}}$ such that $t_k \in S_k$ for each $k \in \mathbb{N}$, we obtain the following corollary.

Corollary 3.7. *The selection principle* $\mathsf{S}_1(\Sigma_f^\mu(E), \Sigma_f^\mu(E))$ *is true.*

Recall now another mode of convergence. A sequence $(f_n)_{n\in\mathbb{N}}$ of *integrable* functions defined on a subset E of a measure space $(X, \mathcal{M}, \mu)$ is said to *converge in mean* (or that it is *mean convergent*) to a function f, denoted $(f_n)_{n\in\mathbb{N}} \xrightarrow{L_1} f$, if

$$\lim_{n\to\infty} \|f_n - f\|_1 = 0,$$

where $\|f\|_1 = \int_E f d\mu$ is the L^1-norm.

Denote by $\Sigma_f^{L_1}(E)$ the set of all sequences of integrable functions which mean converge to a function f.

It is known that mean convergence implies convergence in measure (due to the Chebyshev inequality). Thus from Corollary 3.7 we obtain:

Corollary 3.8. *The selection principle* $\mathsf{S}_1(\Sigma_f^{L_1}(E), \Sigma_f^\mu(E))$ *is true.*

On the other hand, Theorem 3.1 and the Lebesgue dominated convergence theorem [2, Theorem 2.8.1] imply the following theorem.

Theorem 3.9. *Let* $(S_n = (f_n^m)_{m\in\mathbb{N}} : n \in \mathbb{N})$ *be a sequence of elements of* $\Sigma_f^{a.e.}(E)$ *such that there is a* μ*-integrable function* φ *with* $|f_n^m| \leq \varphi$ *for all* $n, m \in \mathbb{N}$. *Then there is a sequence* $s = (f_n^{m_n} : n \in \mathbb{N})$ *which mean converges to* f.

If $\mathcal{A}$ and $\mathcal{B}$ are as above, then the symbol $\alpha_4(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(a_n : n \in \mathbb{N})$ of elements from $\mathcal{A}$ there is a sequence $b \in \mathcal{B}$ such that $a_n \cap b \neq \emptyset$ for infinitely many $n \in \mathbb{N}$ (see, for example, [12])

By Corollary 3.8 for each sequence $(S_n : n \in \mathbb{N})$ of elements of $\Sigma_f^{L_1}(E)$, there are elements $g_n \in S_n$, $n \in \mathbb{N}$, such that $(g_n)_{n\in\mathbb{N}}$ converges in measure to f. By the Riesz theorem there is a subsequence $(g_{n_k}) = (h_k)$ of (g_n) converging almost everywhere to f. But this means that for infinitely many n we have chosen $h_n \in S_n$ so that (h_n) almost everywhere converges to f. Therefore, we have the following result:

Theorem 3.10. *The selection principles* $\alpha_4(\Sigma_f^{L_1}(E), \Sigma_f^{a.e}(E))$ *is satisfied.*

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