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A note on some sequence spaces of weighted means

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Abstract. We show that the sequence spaces a_0^r, a_c^r and a_∞^r are equal to the sets of all sequences whose Cesàro means of order 1 converge to 0, converge and are bounded. As a consequence of this, we are able to considerably simplify the known results and their proofs in [1, 2], and to add the characterisations of some more classes of matrix transformations.

1. Introduction, notations and known results

Aydın and Başar defined the sequence spaces a_0^r, a_c^r, $a_0^r(\Delta)$ and $a_c^r(\Delta)$ for $0 < r < 1$ in [1, 2]. They determined some Schauder bases for their spaces, found the α–, β– and γ–duals, and characterised some classes of matrix transformations on them. Furthermore, various classes of compact matrix operators on the spaces a_0^r, a_c^r, $a_0^r(\Delta)$ and $a_c^r(\Delta)$ were characterised in [3, 4, 7]. We include the sets a_∞^r and $a_\infty^r(\Delta)$ in our studies, and show that the sets a_0^r, a_c^r and a_∞^r are equal to the matrix domains of the Cesàro matrix of order 1 in the sets c_0, c and ℓ_∞ of null, convergent and bounded sequences. Applying this result and using known results on the spaces of generalised weighted means established in [6] and [8], we are able to considerably simplify the results and their proofs in [1] and [2], and add the characterisations of some more classes of matrix transformations; in particular, the sets $a_0^r(\Delta)$, $a_c^r(\Delta)$ and $a_\infty^r(\Delta)$ reduce to simple special cases of the spaces s_α^0, s_α and $s_\alpha^{(c)}$ in [5].

Now we recall the most important notations, definitions and results needed in this paper.

A sequence $(b_n)_{n=0}^\infty$ in a linear metric space X is called a Schauder basis if, for each $x \in X$, there exists a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \lambda_n b_n$.

By ω and ϕ we denote the set of all complex sequences $x = (x_k)_{k=0}^\infty$ and all finite sequences. We write bs and cs for the sets of all bounded and convergent series; also let $\ell_p = \{x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty\}$ for $1 \leq p < \infty$. As usual, e and $e^{(n)}$ $(n = 0, 1, \dots)$ are the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A subspace X of ω is said to be a BK space if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 0, 1, \dots)$ where $P_n(x) = x_n$ for all $x \in X$. A BK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \lim_{m\to\infty} x^{[m]}$ where $x^{[m]} = \sum_{n=0}^m x_n e^{(n)}$ is the m^{th} section of the sequence x.

If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a : \omega : a \cdot x \in Y \text{ for all } x \in X\}$ for the multiplier space of X and Y; in

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particular, we use the notations $x^\alpha = x^{-1} * \ell_1$, $x^\beta = x^{-1} * cs$ and $x^\gamma = x^{-1} * bs$, and $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ for the α–, β– and γ–duals of X.

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers and any sequence x, we write $A_n = (a_{nk})_{k=0}^\infty$ for the sequence in the n^{th} row of A, $A_n x = \sum_{k=0}^\infty a_{nk} x_k$ $(n = 0, 1, \dots)$ and $Ax = (A_n x)_{n=0}^\infty$, provided $A_n \in x^\beta$ for all n. If X and Y are subsets of ω, then $X_A = \{a \in \omega : Ax \in X\}$ denotes the matrix domain of A in X and (X, Y) is the class of all infinite matrices that map X into Y; so $A \in (X, Y)$ if and only if $X \subset Y_A$. A matrix A is said to be regular, if $A \in (c, c)$ and $\lim_{n\to\infty} A_n x = \lim_{k\to\infty} x_k$ for all $x \in c$.

An infinite matrix $T = (t_{nk})_{n,k=0}^\infty$ is said to be a triangle if $t_{nk} = 0$ $(k > n)$ and $t_{nn} \neq 0$ for all n. We write $\mathcal{U}$ for the set of all sequences u with $u_k \neq 0$ for all k; if $u \in \mathcal{U}$ then $1/u = (1/u_k)_{k=0}^\infty$. Let $\mathbf{n} + \mathbf{1} = (n+1)_{n=0}^\infty$. We define the matrices Σ, Δ, Δ^+ and $C^{(1)}$ by $\Sigma_{nk} = 1$ $(0 \le k \le n)$, $\Sigma_{nk} = 0$ $(n > k)$, $\Delta_{nn} = \Delta^+_{nn} = 1$, $\Delta_{n-1,n} = \Delta^+_{n,n+1} = -1$, $\Delta_{n,k} = \Delta^+_{nk} = 0$ (otherwise) and $C^{(1)}_{nk} = (1/(n+1))\Sigma_{nk}$ for all $n, k = 0, 1, \dots$.

Let $u, v \in \mathcal{U}$ and X be a subset of ω. The sets $W(u, v; X) = v^{-1} * (u^{-1} * X)_\Sigma$ of generalised weighted means were defined and studied in [6] and [8]. In particular, $W(1/(\mathbf{n}+\mathbf{1}), e, c_0) = (c_0)_{C^{(1)}}$, $W(1/(\mathbf{n}+\mathbf{1}), e, c) = c_{C^{(1)}}$ and $W(1/(\mathbf{n}+\mathbf{1}), e, \ell_\infty) = (\ell_\infty)_{C^{(1)}}$ are the spaces of all sequences that are summable to 0, summable, and bounded by the Cesàro method $C^{(1)}$ of order 1; we write $C_0 = (c_0)_{C^{(1)}}$, $C = c_{C^{(1)}}$ and $C_\infty = (\ell_\infty)_{C^{(1)}}$, for short.

Let $0 < r < 1$ and $A^{(r)} = (a^{(r)}_{nk})_{n,k=0}^\infty$ be the triangle with $a^r_{nk} = (1 + r^k)/(n+1)$ $(0 \le k \le n;\ n = 0, 1, \dots)$. Aydın and Başar defined the spaces $a^r_0 = (c_0)_{A^{(r)}}$ and $a^r_c = c_{A^{(r)}}$ in [1]. We also define $a^r_\infty = (\ell_\infty)_{A^{(r)}}$, write $\tilde{\mathbf{r}} = (1 + r^k)_{k=0}^\infty$, and observe $a^r_0 = \tilde{\mathbf{r}}^{-1} * C_0 = W(1/(\mathbf{n}+\mathbf{1}), \tilde{\mathbf{r}}, c_0)$, $a^r_c = \tilde{\mathbf{r}}^{-1} * C = W(1/(\mathbf{n}+\mathbf{1}), \tilde{\mathbf{r}}, c)$ and $a^r_\infty = \tilde{\mathbf{r}}^{-1} * C_\infty = W(1/(\mathbf{n}+\mathbf{1}), \tilde{\mathbf{r}}, \ell_\infty)$. The spaces $a^r_0(\Delta) = (a^r_0)_\Delta$ and $a^r_c(\Delta) = (a^r_c)_\Delta$ were studied by the same authors in [2]; we will also consider the space $a^r_\infty(\Delta) = (a^r_\infty)_\Delta$.

We remark that since the matrices $A^{(r)}$ and Δ are triangles, ℓ_∞, c and c_0 are BK spaces with respect to their natural norms defined by $\|x\|_\infty = \sup_k |x_k|$ ([10, p. 55]), c_0 is a closed subspace of c and c is a closed subspace of ℓ_∞ ([10, Corollary 4.2.4]), the spaces a^r_∞, a^r_c and a^r_0 are BK spaces with their natural norms defined by $\|x\|_{a^r_\infty} = \|A^{(r)}x\|_\infty = \sup_n |A^r_n x|$ by [10, Theorem 4.3.12], a^r_0 is a closed subspace of a^r_c, and a^r_c is a closed subspace of a^r_∞ by [10, Theorem 4.3.14]; similarly $a^r_\infty(\Delta)$, $a^r_c(\Delta)$ and $a^r_0(\Delta)$ are BK spaces with their natural norms defined by $\|x\|_{a^r_\infty(\Delta)} = \|\Delta x\|_{a^r_\infty}$, $a^r_0(\Delta)$ is a closed subspace of $a^r_c(\Delta)$, and $a^r_c(\Delta)$ is a closed subspace of $a^r_\infty(\Delta)$. These results contain [1, Theorem 2.1] and [2, Theorem 2.1].

Schauder bases for a^r_0 and a^r_c were determined in [1, Theorem 3.1 (a) and (b)], and for $a^r_0(\Delta)$ and $a^r_c(\Delta)$ in [2, Theorem 3.1 (a) and (b)]. We observe that, since c_0 has AK and $(e, e^{(0)}, e^{(1)}, \dots)$ is a Schauder basis for c, the statements in [1, Theorem 3.1 (a) and (b)] are an immediate consequence of the first part of [6, Theorem 2.2], and an application of the second part of [6, Theorem 2.2] to the bases of a^r_0 and a^r_c yields the statements in [2, Theorem 3.1 (a) and (b)]. We remark that, since matrix multiplication is associative for triangles by [10, Corollary 1.4.5], putting $B^{(r)} = A^{(r)} \cdot \Delta$, we obtain $a^r_0(\Delta) = (c_0)_{B^{(r)}}$ and $a^r_c(\Delta) = c_{B^{(r)}}$ and [2, Theorem 3.1 (a) and (b)] would also be an immediate consequence of [7, Corollary 2.3 (a) and (b)].

2. The main results

First, we determine simpler Schauder bases for the spaces a^r_0, a^r_c, $a^r_0(\Delta)$ and $a^r_c(\Delta)$.

If y is any sequence, we write $\sigma_n(y) = C^{(1)}_n y$ for the n^{th} $C^{(1)}$ mean of y.

Theorem 2.1. *Let $0 < r < 1$. Then a^r_0 has AK. Every sequence $x = (x_k)_{k=0}^\infty \in a^r_c$ has a unique representation*

$$x = \xi \cdot e + \sum_{k=0}^\infty (x_k - \xi) e^{(k)} \text{ where } \xi = \lim_{n\to\infty} \sigma_n(x \cdot \tilde{\mathbf{r}}). \tag{1}$$

Proof. Since $a^r_0 = \tilde{\mathbf{r}}^{-1} * C_0$ and C_0 obviously is a BK space with the norm defined by $\|x\|_{C_\infty} = \sup_n |\sigma_n(x)|$, it suffices to show by [10, Theorem 4.3.6] that C_0 has AK.

First, we observe that $\phi \subset C_0$, since $e^{(n)} \in c_0$ for all n and the $C^{(1)}$ matrix is regular.

Let $\varepsilon > 0$ and $x \in C_0$ be given. Then there exists $N_0 \in \mathbb{N}_0$ such that

$$|\sigma_n(x)| < \frac{\varepsilon}{2} \text{ for all } n \ge N_0. \tag{2}$$

Now let $m \geq N_0$ be given. Then we have for all $n \geq m+1$ by (2)

$$\left|\sigma_n\left(x - x^{[m]}\right)\right| = \left|\frac{1}{n+1}\sum_{k=m+1}^{n} x_k\right| \leq |\sigma_n(x)| + |\sigma_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whence $\|x - x^{[m]}\|_{C_\infty} \leq \varepsilon$ for all $m \geq N_0$. This shows $x = \lim_{m\to\infty} x^{[m]}$. It is clear that this representation is unique.

Let $x \in a_c^r$ be given. Then there is a unique $\xi \in \mathbb{C}$ such that $\lim_{n\to\infty} \sigma_n(x \cdot \tilde{\mathbf{r}}) = \xi$. It follows that

$$\begin{aligned} 0 &\leq |\sigma_n((x - \xi \cdot e) \cdot \tilde{\mathbf{r}})| \leq |\sigma_n(x \cdot \tilde{\mathbf{r}}) - \xi| + |\sigma_n(\xi \cdot (e - \tilde{\mathbf{r}}))| \\ &\leq |\sigma_n(x \cdot \tilde{\mathbf{r}}) - \xi| + \frac{|\xi|}{n+1}\sum_{k=0}^{n} r^k \leq |\sigma_n(x \cdot \tilde{\mathbf{r}}) - \xi| + \frac{|\xi|}{(n+1)(1-r)} \text{ for all } n. \end{aligned}$$

Letting $n \to \infty$, we conclude $\lim_{n\to\infty} \sigma_n((x - \xi \cdot e)\tilde{\mathbf{r}}) = 0$. Thus, if $x \in a_c^r$ then there is a unique $\xi \in \mathbb{C}$ such that $x^{(0)} = x - \xi \cdot e \in a_0^r$. Since a_0^r has AK, as we have just shown, it follows that $x^{(0)}$ has a unique representation $x^{(0)} = \sum_{k=0}^{\infty} x_k^{(0)} e^{(k)} = \sum_{k=0}^{\infty}(x_k - \xi)e^{(k)}$, hence $x = \xi \cdot e + x^{(0)}$ has the unique representation in (1). □

Remark 2.2. We put $d^{(n)} = e - \sum_{k=0}^{n-1} e^{(k)}$ for $n = 0, 1, \ldots$ and define the sequence $d^{(-1)}$ by $d_k^{(-1)} = (k+1)$ $(k = 0, 1, \ldots)$. Then it follows from [7, Corollary 2.3 (a)] and Theorem 2.1 that every sequence $x \in a_0^r(\Delta)$ has a unique representation $x = \sum_{n=0}^{\infty}(x_n - x_{n+1})d^{(n)}$. Since $a_c^r(\Delta) = (a_0^r \oplus e)_\Delta$, it follows from (1) and [7, Corollary 2.3 (c)] that every sequence $x \in a_c^r(\Delta)$ has a unique representation $x = \xi \cdot d^{(-1)} + \sum_{n=0}^{\infty}(x_n - x_{n+1} - \xi)d^{(n)}$, where ξ is the uniquely determined complex number such that $\Delta x - \xi e \in a_0^r$. These results will be simplified in Remark 2.6.

We need the following lemma to establish a result which is fundamental in the simplification of the results in [1] and [2].

Lemma 2.3. *Let $a \in \ell_1$ and $b = e + a$. Then we have $X_{C^{(1)}} \subset b^{-1} * X_{C^{(1)}}$ for $X = \ell_\infty, c, c_0$.*

Proof. First we prove the statement for $X = c$.
We assume $x \in C$. Let $\varepsilon > 0$ be given. It is well known that $x \in C$ implies $x_n/(n+1) \to 0$ $(n \to \infty)$ ([9, Theorem I.1]). So there exist a complex number ξ and an integer N_0 such that

$$|\sigma_n(x) - \xi| < \frac{\varepsilon}{3} \text{ and } \left|\frac{x_n}{n+1}\right| < \frac{\varepsilon}{3(\|a\|_1 + 1)} \text{ for all } n \geq N_0, \tag{3}$$

where $\|a\|_1 = \sum_{k=0}^{\infty}|a_k|$ is the natural norm on ℓ_1. Now we choose an integer $N_1 > N_0$ so large that

$$\left|\frac{1}{n+1}\sum_{k=0}^{N_0} a_k x_k\right| < \frac{\varepsilon}{3} \text{ for all } n \geq N_1. \tag{4}$$

Then we obtain for all $n \geq N_1$ by (3) and (4)

$$\begin{aligned} |\sigma_n(x \cdot b) - \xi| &\leq |\sigma_n(x) - \xi| + \left|\frac{1}{n+1}\sum_{k=0}^{N_0} x_k a_k\right| + \left|\frac{1}{n+1}\sum_{k=N_0+1}^{n} x_k a_k\right| \\ &< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3(\|a\|_1 + 1)} \cdot \sum_{k=N_0+1}^{n} |a_k| \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

that is, $x \cdot b \in C$, hence $x \in b^{-1} * C$.
Thus we have shown $C \subset b^{-1} * C$.

The same proof with $\xi = 0$ yields the statement for $X = c_0$.
Finally, if $x \in C_\infty$ then there is a constant M such that $|\sigma_n(x)| \le M$ for all n and we obtain

$$\left|\frac{x_n}{n+1}\right| = \frac{1}{n+1}\,|(n+1)\sigma_n(x) - n\sigma_{n-1}(x)| \le |\sigma_n(x)| + \frac{n}{n+1}\cdot|\sigma_{n-1}(x)| \le 2\cdot M \text{ for all } n,$$

and so

$$|\sigma_n(x\cdot b)| \le |\sigma_n(x)| + |\sigma_n(x\cdot a)| \le M + \sup_k \frac{|x_k|}{k+1}\sum_{k=0}^{n}|a_k| \le (1+2\|a\|_1)M \text{ for all } n,$$

that is, $x\cdot b \in C_\infty$, hence $x \in b^{-1} * C_\infty$.

Thus we have shown $C_\infty \subset b^{-1} * C_\infty$. □

Theorem 2.4. *We have $X_{A^{(r)}} = X_{C^{(1)}}$ for $X = \ell_\infty, c, c_0$.*

Proof. We put $a_k = r^k$ for $k = 0, 1, \dots$, $a = (a_k)_{k=0}^\infty$ and $b = e + a$.
Since clearly $a \in \ell_1$, Lemma 2.3 yields $X_{C^{(1)}} \subset b^{-1} * X_{C^{(1)}} = X_{A^{(r)}}$.
We also have

$$1/b = \left(\frac{1}{1+r^k}\right)_{k=0}^\infty = e + a' \text{ where } a'_k = -\frac{r^k}{1+r^k} \text{ for all } k \text{ and } a' \in \ell_1.$$

Now if $y \in X_{A^{(r)}}$, then $z = b\cdot y \in X_{C^{(1)}}$ and, applying Lemma 2.3 with $1/b$, we obtain $y = (1/b)\cdot z \in X_{C^{(1)}}$. Thus we also have $X_{A^{(r)}} \subset X_{C^{(1)}}$. □

Now we simplify the spaces $a^r_\infty(\Delta)$, $a^r_c(\Delta)$ and $a^r_0(\Delta)$.

Corollary 2.5. *Let $0 < r < 1$ and $B^{(r)} = A^{(r)}\cdot\Delta$. Then we have*

$$X_{B^{(r)}} = (1/(\mathbf{n}+\mathbf{1}))^{-1} * X \text{ for } X = \ell_\infty, c, c_0. \tag{5}$$

Proof. Since matrix multiplication of triangles is associative, it follows from Theorem 2.4 that $X_{B^{(r)}} = X_{(A^{(r)}\cdot\Delta)} = (X_{A^{(r)}})_\Delta = (X_{C^{(1)}})_\Delta$. We also have for all $x \in \omega$ and all $n \in \mathbb{N}_0$

$$C^{(1)}_n(\Delta x) = \frac{1}{n+1}\sum_{k=0}^{n}(x_k - x_{k-1}) = \frac{x_n}{n+1},$$

which immediately yields (5). □

We observe that, by (5), the spaces $a^r_0(\Delta)$, $a^r_c(\Delta)$ and $a^r_\infty(\Delta)$ are equal to s^0_α, s_α and $s^{(c)}_\alpha$ for $\alpha = (\mathbf{n}+\mathbf{1})$ of [5].
Now we give bases for $a^r_0(\Delta)$ and $a^r_c(\Delta)$.

Remark 2.6. Let $0 < r < 1$. Since c_0 has AK and $a^r_0(\Delta) = (1/(\mathbf{n}+\mathbf{1}))^{-1} * c_0$ by (5), $a^r_0(\Delta)$ has AK by [10, Theorem 4.3.6].
If $x \in a^r_c(\Delta) = (1/(\mathbf{n}+\mathbf{1}))^{-1} * c$ by (5), then there exists a unique $\xi \in \mathbb{C}$ such that

$$\lim_{n\to\infty}\frac{x_n}{n+1} = \xi, \text{ and so } \lim_{n\to\infty}\frac{x_n - (n+1)\xi}{n+1} = 0.$$

Therefore we have $x^{(0)} = x - (\mathbf{n}+\mathbf{1})\xi \in a^r_0(\Delta)$, and since $a^r_0(\Delta)$ has AK, we obtain

$$x = (\mathbf{n}+\mathbf{1})\xi + \sum_{n=0}^{\infty}(x_n - (n+1)\xi)e^{(n)}.$$

Now we determine the α–, β– and γ–duals of the spaces a_0^r, a_c^r and a_∞^r. We need the following known result which we state here for the reader's convenience.

Proposition 2.7. ([6, Theorem 3.1]) *Let $u, v \in \mathcal{U}$. We write $b = (1/u)\Delta^+(a/v)$ for $a \in \omega$ and $S(u,v) = \{a \in \omega : b \in \ell_1\}$. Then we have*
(a) $(W(u,v,X))^\alpha = S(u,v) \cap [(1/(uv))^{-1} * \ell_1]$ for $X = \ell_\infty, c, c_0$;
(b) $(W(u,v,\ell_\infty))^\beta = S(u,v) \cap [(1/(uv))^{-1} * c_0]$, $(W(u,v,c))^\beta = S(u,v) \cap [(1/(uv))^{-1} * c]$ and $(W(u,v,c_0))^\beta = S(u,v) \cap [(1/(uv))^{-1} * \ell_\infty]$;
(c) $(W(u,v,X))^\gamma = S(u,v) \cap [(1/(uv))^{-1} * \ell_\infty]$ for $X = \ell_\infty, c, c_0$.

Corollary 2.8. *Let $0 < r < 1$. We put*

$$S_1 = S(1/(\mathbf{n}+\mathbf{1}), e) = \left\{a \in \omega : ((n+1)\Delta_n^+ a)_{n=0}^\infty \in \ell_1\right\}$$
$$= \left\{a \in \omega : \sum_{n=0}^\infty (n+1)|a_n - a_{n+1}| < \infty\right\}.$$

Then we have
(a) $(a_\infty^r)^\alpha = (a_c^r)^\alpha = (a_0^r)^\alpha = (\mathbf{n}+\mathbf{1})^{-1} * \ell_1$;
(b) $(a_\infty^r)^\beta = S_1 \cap [(\mathbf{n}+\mathbf{1})^{-1} * c_0]$, $(a_c^r)^\beta = S_1 \cap [(\mathbf{n}+\mathbf{1})^{-1} * c]$, $(a_0^r)^\beta = S_1 \cap [(\mathbf{n}+\mathbf{1})^{-1} * \ell_\infty$;
(c) $(a_\infty^r)^\gamma = (a_c^r)^\gamma = (a_0^r)^\gamma = S_1 \cap [(\mathbf{n}+\mathbf{1})^{-1} * \ell_\infty]$.

Proof. Since $X_{A^{(r)}} = X_{C^{(1)}}$ for $X = \ell_\infty, c, c_0$ by Theorem 2.4, we apply Proposition 2.7 (b) and (c) with $u = 1/(\mathbf{n}+\mathbf{1})$ and $v = e$, and immediately obtain Parts (b) and (c).

(a) Proposition 2.7 (a) yields that $(X_{C^{(1)}})^\alpha = S_1 \cap [(\mathbf{n}+\mathbf{1})^{-1} * \ell_1]$. Since

$$\sum_{n=0}^\infty (n+1)|a_n - a_{n+1}| \le \sum_{n=0}^\infty (n+1)|a_n| + \sum_{n=0}^\infty (n+2)|a_{n+1}|,$$

we have $S_1 \supset (\mathbf{n}+\mathbf{1})^{-1} * \ell_1$. □

Remark 2.9. We obviously have $e \in S_1 \setminus [(\mathbf{n}+\mathbf{1})^{-1} * \ell_\infty]$. Let $a_n = (-1)^n/(n+1)^{3/2}$ for $n = 0, 1, \dots$. Then $(n+1)a_n \to 0$ $(n \to \infty)$, that is, $a \in (\mathbf{n}+\mathbf{1})^{-1} * c_0$, but

$$\sum_{n=0}^\infty (n+1)|a_n - a_{n+1}| = \sum_{n=0}^\infty (n+1)\left(\frac{1}{(n+1)^{3/2}} + \frac{1}{(n+2)^{3/2}}\right) \ge \sum_{n=0}^\infty \frac{1}{\sqrt{n+1}} = \infty,$$

that is, $a \notin S_1$. Therefore, the sets in Corollary 2.8 (b) and (c) cannot be reduced in a similar way as those in Part (a).

Remark 2.10. Let $u \in \mathcal{U}$, X be an arbitrary subset of ω and $\dagger$ denote any of the symbols α, β or γ. Then we obviously have $(u^{-1} * X)^\dagger = (1/u)^{-1} * X^\dagger$. Since $X^\dagger = \ell_1$ for $X = \ell_\infty, c, c_0$, it follows from Corollary 2.5 that $(a_\infty^r(\Delta))^\dagger = (a_c^r(\Delta))^\dagger = (a_0^r(\Delta))^\dagger = (\mathbf{n}+\mathbf{1})^{-1} * \ell_1$.

We now retrieve [1, Theorems 4.4 and 4.5].

Remark 2.11. In view of Theorem 2.4, it suffices to compare the conditions in Corollary 2.8 (b) and (c) with those of [1, Theorems 4.4 and 4.5] with r replaced by 0. Then the conditions are obviously identical in each case except for C^β.

By Corollary 2.8, we have $a \in C^\beta$ if and only if

$$a \in S_1 \text{ and } \lim_{n\to\infty}(n+1)a_n \text{ exists,} \tag{6}$$

whereas the corresponding conditions in [1, Theorems 4.4] are

$$a \in S_1 \text{ and } a \in cs. \tag{7}$$

Applying Abel's summation by parts

$$\sum_{k=0}^{n} x_k y_k = x_n Y_n + \sum_{k=0}^{n-1} Y_k (x_k - x_{k+1}) \text{ where } Y_k = \sum_{j=0}^{k} y_j \ (k = 0, 1, \ldots, n)$$

with $x = a$ and $y = e$, we see that the conditions in (6) and (7) are equivalent.

The following remark concerns the α–duals of a_0^r and a_c^r given in [1, Theorem 4.3]; as before, we may replace r by 0.

Remark 2.12. As would be logical from the proof given there, the correct condition in [1, Theorem 4.3] for $a \in (X_{C^{(1)}})^\alpha$ seems to be

$$\sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \left(\sum_{n=0}^{\infty} \left| a_n \sum_{k \in K \cap \{n-1, n\}} (-1)^{n-k} (k+1) \right| \right) < \infty \tag{8}$$

instead of

$$\sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \left(\sum_{n=0}^{\infty} \left| a_n \sum_{k \in K} (-1)^{n-k} (k+1) \right| \right) < \infty. \tag{9}$$

It is easy to see that the condition in (8) is equivalent to that in Corollary 2.8 (a) for $a \in (X_{C^{(1)}})^\alpha$ when $X = c_0, c, \ell_\infty$.

On the other hand, if we define the sequence $a = (a_n)_{n=0}^\infty$ by $a_n = (n+1)^{-5/2}$ $(n = 0, 1, \ldots)$, then clearly $a \in (\mathbf{n} + \mathbf{1})^{-1} * \ell_1$, but, for each given $m \in \mathbb{N}_0$ and $K_m = \{0, 2, \cdots, 2m\}$, we obtain

$$\sum_{n=0}^{2m} |a_n| \cdot \left| \sum_{k=0}^{m} (-1)^{n-2k} (2k+1) \right| = \sum_{n=0}^{2m} |a_n| (m+1)^2 \geq \frac{1}{4} \cdot \sum_{n=0}^{2m} |a_n| (n+1)^2 = \frac{1}{4} \sum_{n=0}^{2m} \frac{1}{\sqrt{n+1}},$$

and so the sequence a does not satisfy the condition in (9).

Finally, we retrieve [2, Theorems 4.5, 4.6 and 4.7].

Remark 2.13. In view of Theorem 2.4 and Remark 2.10, it suffices to compare the conditions in Remark 2.10 (b) and (c) with those of [2, Theorems 4.5, 4.6 and 4.7] with r replaced by 0.
By [2, Theorem 4.5], we have $a \in (a_0^0(\Delta))^\alpha = (a_c^0(\Delta))^\alpha$ if and only if

$$\sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \sum_{n=0}^{\infty} \left| a_n \sum_{k \in K} c_{nk}^0 \right| < \infty \text{ where } C^0 \text{ is the diagonal matrix with } c_{nn}^0 = (n+1);$$

this condition obviously is equivalent to $a \in (\mathbf{n} + \mathbf{1})^{-1} * \ell_1$.

Furthermore, by [2, Theorem 4.6], we have $a \in (a_0^0(\Delta))^\beta$ if and only if

$$\sup_n \sum_{k=0}^{\infty} |e_{nk}^0| < \infty, \text{ where } E^0 \text{ is the triangle with } e_{nk}^0 = (k+1)a_k \text{ for } 0 \leq k \leq n, \tag{10}$$

and

$$\sum_{j=k}^{\infty} a_j \text{ exists for all } j. \tag{11}$$

Obviously the condition in (10) implies that in (11) and the condition in (10) is equivalent to $a \in (\mathbf{n}+\mathbf{1})^{-1} * \ell_1$. We also have by [2, Theorem 4.6] that $a \in (a_c^0(\Delta))^\beta$ if and only if $a \in (a_c^0)^\beta$ and

$$\lim_{n\to\infty} \sum_{k=0}^{n} (k+1)a_k \text{ exists.} \tag{12}$$

Since the condition in (10) obviously implies that in (12), the conditions in [2, Theorem 4.6] for $a \in (a_c^0(\Delta))^\beta$ are again equivalent to $a \in (\mathbf{n}+\mathbf{1})^{-1} * \ell_1$.

Finally, by [2, Theorem 4.6], we have $a \in (a_0^0(\Delta))^\gamma = (a_c^0(\Delta))^\gamma$ if and only if the condition in (10) holds which is equivalent to $a \in (\mathbf{n}+\mathbf{1})^{-1} * \ell_1$, as we have seen above.

We close with a few remarks on some characterisations of matrix transformations.

Remark 2.14. (a) The necessary and sufficient conditions for $A \in (W(u,v,X);Y)$ for arbitrary sequences $u,v \in \mathcal{U}$ were given in [6, Theorem 3.3] when $X = \ell_p$ $(1 \leq p \leq \infty)$, $X = c_0$ or $X = c$ and $Y = \ell_\infty, c_0, c$ or $Y = \ell_r$ $(1 < r < \infty;$ only for $X = \ell_1)$. In particular, putting $u = 1/\mathbf{n}+\mathbf{1}$ and $v = e$ and applying Theorem 2.4, we observe that the characterisations of the classes (a_c^r, ℓ_1), (a_c^r, ℓ_p) $(1 < p < \infty)$, (a_c^r, ℓ_∞) *([1, Theorem 5.3])* and (a_c^r, c) *([1, Theorem 5.4])* would be special cases of [6, Theorem 3.3 (12.), (19.), (9.), (11.)]. In the same way we would obtain the characterisations of the classes (a_c^r, c_0) (a_0^r, c_0), (a_0^r, c), (a_0^r, ℓ_∞), (a_∞^r, c_0), (a_∞^r, c) and $(a_\infty^r, \ell_\infty)$ from [6, Theorem 3.3, (10.), (6.), (7.), (5.), (14.), (15.) and (13.)]. Furthermore, by [6, Remark 3.1] the necessary and sufficient conditions for C to map any of the above spaces into a_0^r, a_c^r or a_∞^r can immediately be obtained from the respective one for A mapping into c_0, c or ℓ_∞ by replacing the entries a_{nk} of A by $c_{nk} = (1/(n+1)) \sum_{j=0}^{k} a_{nk}$ $(n,k = 0,1,\dots)$ in the corresponding conditions.

(b) Let $u,v \in \mathcal{U}$. Then it is clear that

$$A \in (u^{-1} * X, v^{-1} * Y) \text{ if and only if } B \in (X,Y) \text{ where } b_{nk} = \frac{v_n a_{nk}}{u_k} \text{ for all } n,k.$$

Applying this result with $u = 1/\mathbf{n}+\mathbf{1}$ and $v = e$, and Corollary 2.5, we immediately obtain the characterisations of the classes $(a_c^r(\Delta), \ell_1)$, $(a_c^r(\Delta), \ell_p)$ $(1 < p < \infty)$, $(a_c^r(\Delta), \ell_\infty)$ and $(a_c^r(\Delta), c)$ *([2, Theorems 5.3 and 5.4])* from the well–known characterisations of the classical classes (c, ℓ_1), (c, ℓ_p) $(1 < p < \infty)$, (c, ℓ_∞) and (c,c) in [10, 8.4.9A, 8.4.8A, 8.4.5A, 8.4.5A]. Similarly, putting $u = v = 1/\mathbf{n}+\mathbf{1}$ the characterisations of the classes (X,Y) where X and Y are any of the spaces $a_0^r(\Delta)$, $a_c^r(\Delta)$ or $a_\infty^r(\Delta)$ can easily be obtained from the well-known characterisations of the classes of matrix transformations between the spaces c_0, c and ℓ_∞ that can be found in [10].

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