Endpoints of φ-weak and generalized φ-weak contractive mappings

Sirous Moradi[a], Farshid Khojasteh[b]

[a]*Department of Mathematics, Faculty of Science, Arak University, Arak 38156-8-8349, Iran*
[b]*Department of Mathematics, Islamic Azad University, Arak-Branch, Arak, Iran*

Abstract. Let (X, d) be a complete metric space, and let $T : X \longrightarrow P_{cl,bd}(X)$ be a multi-valued φ-weak or generalized φ-weak contractive mapping. Then T has a unique endpoint if and only if T has the approximation endpoints property. Our results extend previous results given by Ćirić(1974), Nadler (1969), Daffer-Kaneko (1995), Rhoades (2001), Rouhani and Moradi (2010), Amini-Harandi (2010) and Moradi and Khojasteh (2011).

1. Introduction and Preliminaries

Let (X, d) be a metric space and $P(X)$ denotes the class of all subsets of X. Define

$$P_f(X) = \left\{ A \subseteq X : A \neq \emptyset \text{ has property f} \right\}.$$

Thus $P_{bd}(X)$, $P_{cl}(X)$, $P_{cp}(X)$ and $P_{cl,bd}(X)$ denote the classes of bounded, closed, compact and closed bounded subsets of X, respectively. Also $T : X \longrightarrow P_f(X)$ is called a multi-valued mapping on X. A point x is called a *fixed point* of T if $x \in Tx$. Denote $Fix(T) = \{x \in X : x \in Tx\}$. An element $x \in X$ is said to be an *endpoint* of multi-valued mapping T, if $Tx = \{x\}$. The set of all endpoints of T denotes by $End(T)$. Obviously, $End(T) \subseteq Fix(T)$. In recent years many authors studied the existence and uniqueness of endpoints for a multi-valued mappings in metric spaces, see for example [2]-[5] and references therein. A mapping $T : X \to X$ is said to be a *weak contraction* if there exists $0 \leq \alpha < 1$ such that $d(Tx, Ty) \leq \alpha N(x, y)$, for all $x, y \in X$, where

$$N(x, y) := \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

A multi-valued mapping $T : X \to P_{cl,bd}(X)$ is said to be *weak contraction* if there exists $0 \leq \alpha < 1$ such that $H(Tx, Ty) \leq \alpha N(x, y)$, for all $x, y \in X$, where H denotes the Hausdorff metric on $P_{cl,bd}(X)$ induced by d, that is,

$$H(A, B) := \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},$$

for all $A, B \in P_{cl,bd}(X)$, and where

$$N(x,y) := \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2} \right\},$$

such that $d(a, A) = dist(a, A)$ for all $a \in X$ and all $A \in P_{cl,bd}(X)$.

A mapping $T : X \longrightarrow P_{cl,bd}(X)$ is said to be *φ-weak contractive* if there exists a map $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$H(Tx, Ty) \le d(x, y) - \varphi(d(x, y))$$

for all $x, y \in X$.

Also a mapping $T : X \longrightarrow P_{cl,bd}(X)$ is said to be *generalized φ-weak contractive* if there exists a map $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$H(Tx, Ty) \le N(x, y) - \varphi(N(x, y))$$

for all $x, y \in X$.

The concept of weak and φ-weak contractive mappings were defined by Daffer and Kaneko [4] in 1995.

Many authors have studied fixed points for multi-valued mappings. Among many others, see, for example [9]-[7] and the references therein.

Rhoades [10, Theorem 2] proved the following fixed point theorem for φ-weak contractive single valued mappings, giving another generalization of the Banach Contraction Principle.

Theorem 1.1. *Let (X, d) be a complete metric space, and let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y)),$$

for every $x, y \in X$ (i.e. φ-weak contractive), where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

Ćirić [3] extended the Banach contraction principle as follows.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$d(Tx, Ty) \le \alpha N(x, y),$$

for all $x, y \in X$ (i.e., weak contraction), then there exists a point $x \in X$ such that $x = Tx$.

In the following theorem, Nadler [9] extended the Banach contraction principle to multi-valued mappings.

Theorem 1.3. *Let (X, d) be a complete metric space. Suppose that $T : X \to P_{cl,bd}(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$H(Tx, Ty) \le \alpha d(x, y),$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$.

Daffer and Kaneko [4] proved the existence of a fixed point for a multi-valued weak contraction mappings of a complete metric space X into $P_{cl,bd}(X)$.

Theorem 1.4. *Let (X, d) be a complete metric space. Suppose that $T : X \to P_{cl,bd}(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$H(Tx, Ty) \le \alpha N(x, y),$$

for all $x, y \in X$ (i.e., weak contraction). If $x \mapsto d(x, Tx)$ is lower semi-continuous (l.s.c.), then there exists a point $x_0 \in X$ such that $x_0 \in Tx_0$.

Rouhani and Moradi [11] extended Nadler and Daffer-Kanako's Theorems to a coincidence theorem, without assuming $x \mapsto d(x, Tx)$ to be l.s.c.

Theorem 1.5. *Let (X, d) be a complete metric space, and let $T, S : X \to P_{cl,bd}(X)$ be two multi-valued mappings such that for all $x, y \in X$,*

$$H(Tx, Sy) \leq kM(x, y)$$

where $0 \leq k < 1$ and

$$M(x, y) = \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Sx)}{2} \right\}$$

(i.e., multi-valued generalized weak contractions). Then there exists a point $x \in X$ such that $x \in Tx$ and $x \in Sx$ (i.e., T and S have a common fixed point). Moreover, if either T or S is single valued, then this common fixed point is unique.

A mapping $T : X \to P_{cl,bd}(X)$ has the approximation endpoint property [1], if $\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0$.

Let $T : X \to X$ be a single valued mapping. Then T has the approximate endpoint property if and only if T has the approximate fixed point property, i.e., $\inf_{x \in X} d(x, Tx) = 0$.

Recently, Amini-Harandi [1] in 2010 proved the following endpoint result for a multi-valued mappings of a complete metric space X into $P_{cl,bd}(X)$.

Theorem 1.6. *([1], Theorem 2.1) Let (X, d) be a complete metric space. Suppose that $T : X \to P_{cl,bd}(X)$ is a multi-valued mapping that satisfies*

$$H(Tx, Ty) \leq \psi(d(x, y)),$$

for each $x, y \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is upper semi-continuous, $\psi(t) < t$ for all $t > 0$, and satisfies $\liminf_{t \to \infty}(t - \psi(t)) > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.

After that Moradi and Khojasteh [8] in 2011 proved the following endpoint theorem for a multi-valued generalized weak contraction mappings.

Theorem 1.7. *Let (X, d) be a complete metric space. Suppose that $T : X \to P_{cl,bd}(X)$ is a multi-valued mapping that satisfies*

$$H(Tx, Ty) \leq \psi(N(x, y)),$$

for each $x, y \in X$, where $\psi : [0, +\infty) \to [0, +\infty)$ is upper semi-continuous, $\psi(t) < t$ for all $t > 0$, and satisfies $\liminf_{t \to \infty}(t - \psi(t)) > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.

In Section 2 and 3, we prove endpoint theorems for $\varphi-$weak and generalized φ-weak contractive mappings. Our results extend previous results given by Amini-Harandi [1], Moradi and Khojasteh [8], as well as by Ćirić [3] and Rhoades [10].

2. Endpoint of $\varphi-$weak contractive mappings

The following theorem is one of the main results of this paper, that find a new type of endpoint theorem. Also this theorem extends Amini-Harandi's Theorem. In this section also we extend Rhoades' Theorem. At first we introduce the notation Ψ for the class of all mappings $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi^{-1}(0) = \{0\}$ and $\varphi(t_n) \to 0$ implies $t_n \to 0$ as $n \to \infty$. For example every nondecreasing mapping $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi^{-1}(0) = \{0\}$ belong to Ψ. Also every l.s.c. mapping $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi^{-1}(0) = \{0\}$ and $\liminf_{t \to \infty} \varphi(t) > 0$ belong to Ψ.

Theorem 2.1. *Let (X, d) be a complete metric space and let $T : X \to P_{cl,bd}(X)$ be a multi-valued mapping that satisfies*

$$H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1}$$

for each $x, y \in X$, where $\varphi \in \Psi$(i.e., multi-valued φ-weak contractive). Then T has a unique endpoint if and only if T has the approximate endpoint property. Moreover, $End(T) = Fix(T)$.

Proof. Unicity of the endpoint follows from (1).

It is clear that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property; then there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. For all $m, n \in \mathbb{N}$ we have

$$
\begin{aligned}
d(x_n, x_m) &= H(\{x_n\}, \{x_m\}) \leq H(\{x_n\}, Tx_n) + H(Tx_n, Tx_m) + H(Tx_m, \{x_m\}) \\
&\leq d(x_n, x_m) - \varphi(d(x_n, x_m)) + H(\{x_n\}, Tx_n) + H(\{x_m\}, Tx_m).
\end{aligned}
\tag{2}
$$

So

$$\varphi(d(x_n, x_m)) \leq H(\{x_n\}, Tx_n) + H(\{x_m\}, Tx_m).$$

This shows that $\lim_{m,n\to\infty} \varphi(d(x_n, x_m)) = 0$. Hence $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. Thus $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Also

$$H(\{x_n\}, Tx) - H(\{x_n\}, Tx_n) \leq H(Tx_n, Tx) \leq d(x_n, x) - \varphi(d(x_n, x) \leq d(x_n, x). \tag{3}$$

Letting $n \longrightarrow \infty$ in (3) we get $H(\{x\}, Tx) = 0$. Hence $Tx = \{x\}$ and therefore $End(T) = \{x\}$. Now suppose $y \in Fix(T)$ be arbitrary. We need to show that $y = x$. Suppose that $y \neq x$ then

$$d(x, y) \leq H(\{x\}, Ty) = H(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) < d(x, y)$$

and this is a contradiction. This completes the proof. $\square$

Remark 2.2. Note that we say that the fixed point problem is well posed (see [7]) for T with respect to H if:

(i) $End(T) = \{x\}$,

(ii) if $\{x_n\} \subset X$, $n \in \mathbb{N}$ and $H(\{x_n\}, Tx_n) \to 0$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$.

The proof of Theorem 2.1 shows that the fixed point problem in Theorem 2.1 is well posed for T with respect to H.

Remark 2.3. By define $\varphi(t) = t - \psi(t)$ and use Theorem 2.1 we conclude Amini-Harandi's Theorem (Theorem 1.6).

The following example shows that Theorem 2.1 is an extension of Theorem 1.6.

Example 2.4. Let $X = [0, +\infty)$ endowed with the Euclidean metric. Define $Tx = [0, \frac{1}{2}x]$ and let

$$
\varphi(t) = \begin{cases} t/5 & 0 \leq t < 1 \\ 1/3 & t = 1 \\ t/2 & t > 1, \end{cases}
$$

Obviously, $\varphi \in \Psi$ and $H(Tx, Ty) = \frac{1}{2}|x - y| \leq |x - y| - \varphi(|x - y|)$. Thus, all conditions of Theorem 2.1 hold. If we define

$$
\psi(t) = t - \varphi(t) = \begin{cases} 4t/5 & 0 \leq t < 1 \\ 2/3 & t = 1 \\ t/2 & t > 1 \end{cases}
$$

then $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) = +\infty$ but ψ is not upper semi-continuous. Therefore, T does not satisfies in Theorem 1.6. Note that $T0 = \{0\}$ is unique endpoint of T and $\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0$.

The following corollary is a direct result of Theorem 2.1.

Corollary 2.5. *Let (X, d) be a complete metric space and let $f : X \to X$ be a map satisfies*

$$d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)), \tag{4}$$

for each $x, y \in X$, where $\varphi \in \Psi$. Then f has a unique fixed point if and only if f has the approximate fixed point property.

Proof. Let $Tx = \{f(x)\}$ and apply Theorem 2.1. □

The following theorem shows that for single valued mapping the condition (4) is sufficient for f to have the approximate fixed point property.

Theorem 2.6. *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping satisfies*

$$d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)), \tag{5}$$

for each $x, y \in X$, where $\varphi \in \Psi$. Then f has the approximate fixed point property.

Proof. Let $x_0 \in X$ and $x_n = f x_{n-1}$ for all $n \in \mathbb{N}$. From (5), for all $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) = d(f x_n, f x_{n-1})) \leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1})). \tag{6}$$

Therefore the sequence $\{d(x_n, x_{n+1})\}$ is monotone non-increasing and bounded below. So, there exists $r \geq 0$ such that $\lim_{n \to \infty} d(x_{n+1}, x_n) = r$.
From (6) we conclude that

$$0 \leq \varphi(d(x_n, x_{n-1})) \leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n),$$

and so $\varphi(d(x_n, x_{n-1})) \to 0$ as $n \to \infty$. Hence $\lim_{n \to \infty} d(x_n, x_{n-1}) = 0$. Therefore $\inf_{x \in X} d(x, fx) = 0$. So f has the approximate fixed point property. □

As an application of Corollary 2.5 and Theorem 2.6 we obtain the following fixed point result. This result extends Rhoades' Theorem (Theorem 1.1).

Corollary 2.7. *Let (X, d) be a complete metric space, and let $f : X \to X$ be a mapping such that for all $x, y \in X$,*

$$d(f(x), f(y)) \leq d(x, y) - \varphi(d(x, y)),$$

for each $x, y \in X$, where $\varphi \in \Psi$. Then f has a unique fixed point and for every $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.

3. Endpoints of generalized φ−weak contractive mappings

The following theorem is another main results of this paper, that find another new type of endpoint theorem. This theorem extends Moradi and Khojasteh's Theorem. Also using this theorem we can extend Ćirić's Theorem.

Theorem 3.1. *Let (X, d) be a complete metric space and let $T : X \to P_{cl,bd}(X)$ be a multi-valued mapping satisfies*

$$H(Tx, Ty) \leq N(x, y) - \varphi(N(x, y)), \tag{7}$$

for each $x, y \in X$, where $\varphi \in \Psi$(i.e., multi-valued generalized φ-weak contractive). Then T has a unique endpoint if and only if T has the approximate endpoint property. Moreover, $Fix(T) = End(T)$. Also the fixed point problem is well posed for T.

Proof. It is clear that if T has an endpoint, then T has the approximate endpoint property. Conversely, suppose that T has the approximate endpoint property, then there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} H(\{x_n\}, Tx_n) = 0$. For all $m, n \in \mathbb{N}$ we have

$$
\begin{aligned}
N(x_n, x_m) &= \max\left\{d(x_n, x_m), d(x_n, Tx_n), d(x_m, Tx_m), \frac{d(x_n, Tx_m) + d(x_m, Tx_n)}{2}\right\} \\
&\leq \max\left\{d(x_n, x_m), H(\{x_n\}, Tx_n), H(\{x_m\}, Tx_m), \frac{H(\{x_n\}, Tx_m) + H(\{x_m\}, Tx_n)}{2}\right\} \\
&\leq d(x_n, x_m) + H(\{x_n\}, Tx_n) + H(\{x_m\}, Tx_m) \\
&\leq H(Tx_n, Tx_m) + 2H(\{x_n\}, Tx_n) + 2H(\{x_m\}, Tx_m) \\
&\leq N(x_n, x_m) - \varphi(N(x_n, x_m)) + 2H(\{x_n\}, Tx_n) + 2H(\{x_m\}, Tx_m).
\end{aligned}
$$

So

$$\varphi(N(x_n, x_m)) \leq 2H(\{x_n\}, Tx_n) + 2H(\{x_m\}, Tx_m).$$

This shows that $\lim_{m,n\to\infty} \varphi(N(x_n, x_m)) = 0$. Hence $\lim_{m,n\to\infty} N(x_n, x_m) = 0$. Therefore $\lim_{m,n\to\infty} d(x_n, x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Now we show that $x \in Tx$. If $x \notin Tx$ then $d(x, Tx) > 0$. For all $n \in \mathbb{N}$,

$$\frac{d(x_n, Tx) + d(x, Tx_n)}{2} \leq \frac{d(x_n, x) + d(x, Tx) + d(x_n, x) + d(x_n, Tx_n)}{2}. \tag{8}$$

Since $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, Tx_n) = 0$, $d(x, Tx) > 0$ and (8) holds, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$

$$\frac{d(x_n, Tx) + d(x, Tx_n)}{2} \leq d(x, Tx).$$

So for all $n \geq N_0$, $N(x_n, x) = d(x, Tx)$ and hence for all $n \geq N_0$,

$$
\begin{aligned}
H(\{x_n\}, Tx) - H(\{x_n\}, Tx_n) &\leq H(Tx_n, Tx) \\
&\leq N(x_n, x) - \varphi(N(x_n, x)) \\
&= d(x, Tx) - \varphi(d(x, Tx)). \tag{9}
\end{aligned}
$$

Letting $n \to \infty$ in (10), we conclude that

$$H(\{x\}, Tx) \leq d(x, Tx) - \varphi(d(x, Tx)) < d(x, Tx), \tag{10}$$

and this is a contradiction, because $d(x, Tx) \leq H(\{x\}, Tx)$. So $x \in Tx$.
Now we prove that $Tx = \{x\}$.
For all $n \in \mathbb{N}$,

$$
\begin{aligned}
H(\{x_n\}, Tx) - H(\{x_n\}, Tx_n) &\leq H(Tx_n, Tx) \\
&\leq N(x_n, x) - \varphi(N(x_n, x) \leq N(x_n, x), \tag{11}
\end{aligned}
$$

where $\lim_{n\to\infty} N(x_n, x) = 0$. So letting $n \to \infty$ in (11) we get $H(\{x\}, Tx) = 0$. Hence $Tx = \{x\}$. So $x \in End(T)$. Form (7) one can conclude that this endpoint is unique. So $End(T) = \{x\}$. Let $y \in Fix(T)$ be arbitrary. We need to show that $y = x$. If $y \neq x$ then

$$d(x, y) \leq H(\{x\}, Ty) = H(Tx, Ty) \leq N(x, y) - \varphi(N(x, y)) < N(x, y), \tag{12}$$

where $N(x, y) = \max\{d(x, y), \frac{d(x, Ty) + d(y, x)}{2}\}$. Since $y \in Ty$, $d(x, Ty) \leq d(x, y)$ and hence $N(x, y) = d(x, y)$. From (12) we conclude that $d(x, y) < d(x, y)$ and this is a contradiction. Therefore, $End(T) = Fix(T)$ and this completes the proof. $\square$

Remark 3.2. By define $\varphi(t) = t - \psi(t)$ we can show that Theorem 3.1 is an extension of Moradi and Khojasteh's Theorem (Theorem 1.7).

Corollary 3.3. *Let (X, d) be a complete metric space and let $T : X \to P_{cl,bd}(X)$ be a mapping such that, for all $x, y \in X$,*

$$H(Tx, Ty) \leq kN(x, y),$$

for some $0 \leq k < 1$ (i.e., contraction). Then T has unique endpoint if and only if T has the approximate endpoint property.

Proof. Let $\varphi(t) = (1 - k)t$ and apply Theorem 3.1. $\square$

The following corollary extends Nadler and Daffer-Kaneko's Theorems.

Corollary 3.4. *Let (X, d) be a complete metric space and let $T : X \to P_{cl,bd}(X)$ be a mapping such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq kN(x, y),$$

for some $0 \leq k < 1$. Then there exists a point $x \in X$ such that $x \in Tx$. Also if T has an approximate endpoint property then $Fix(T) = End(T) = \{x\}$ (so the fixed point is unique).

Proof. Using Theorem 1.5, there exists $x \in X$ such that $x \in Tx$. If T has the approximate endpoint property, by Corollary 3.1, we conclude that T has the unique endpoint and $End(T) = Fix(T) = \{x\}$. $\square$

Corollary 3.5. *Let (X, d) be a complete metric space, and let $f : X \to X$ be a map satisfies*

$$d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)), \tag{13}$$

for each $x, y \in X$, where $\varphi \in \Psi$. Then f has a unique fixed point if and only if f has the approximate fixed point property.

Proof. Let $Tx = \{f(x)\}$ and apply Theorem 3.1. $\square$

The following theorem shows that for single valued mapping the condition (13) is sufficient for f to have the approximate fixed point property.

Theorem 3.6. *Let (X, d) be a complete metric space and let $f : X \to X$ be a mapping satisfies*

$$d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)), \tag{14}$$

for each $x, y \in X$, where $\varphi \in \Psi$. Then f has the approximate fixed point property.

Proof. Let $x_0 \in X$ and $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. From (14), for all $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) \leq N(x_n, x_{n-1}) - \varphi(N(x_n, x_{n-1})), \tag{15}$$

where

$$
\begin{aligned}
N(x_n, x_{n-1}) &= \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2}\right\} \\
&= \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\
&= \max\left\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\right\}. \tag{16}
\end{aligned}
$$

By (15) and (16) we have $d(x_n, x_{n+1}) \leq d(x_n, x_{n-1})$. Hence,

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) - \varphi(d(x_n, x_{n-1})). \tag{17}$$

Therefore the sequence $\{d(x_n, x_{n+1})\}$ is monotone non-increasing and bounded below. So, there exists $r \geq 0$ such that $\lim\limits_{n \to \infty} d(x_{n+1}, x_n) = r$.

From (17) we conclude that

$$0 \leq \varphi(d(x_n, x_{n-1})) \leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n),$$

and so $\varphi(d(x_n, x_{n-1})) \to 0$ as $n \to \infty$. Hence $\lim\limits_{n \to \infty} d(x_n, x_{n-1}) = 0$. Therefore $\inf\limits_{x \in X} d(x, fx) = 0$. So f has the approximate fixed point property. $\square$

As an application of Corollary 3.5 and Theorem 3.6 we obtain the following fixed point result. This result extends Ćirić's Theorem (Theorem 1.2).

Corollary 3.7. *Let (X, d) be a complete metric space, and let $f : X \to X$ be a mapping satisfies*

$$d(f(x), f(y)) \leq N(x, y) - \varphi(N(x, y)),$$

for each $x, y \in X$, where $\varphi \in \Psi$. Then f has a unique fixed point. Also for every $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.

References

[1] A. Amini-Harandi, Endpoints of set-valued contractions in metric spaces, Nonlinear Anal. (TMA) 72 (2010) 132–134.
[2] D. W. Boyd, J. S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969) 458–464.
[3] Lj. B. Ćirić, Generalized contraction and fixed point theorems, Publ. Inst. Math (Beograd) 12 (1971) 19–26.
[4] P. Z. Daffer, H. Kaneko, Fixed points of generalized contractive multi-valued mappings, J. Math. Anal. Appl. 192 (1995) 655-666.
[5] M. Fakhar, Endpoints of set-valued asymptotic contractions in metric spaces, Appl. Math. Lett. 24 (2011) 428-431.
[6] N. Hussain, A. Amini-Harandi, Y. J. Cho, Approximate endpoints for set-valued contractions in metric spaces, Fixed Point Theory and Appl. DOI:10.1155/2010/614867 (2010).
[7] G. Mo, A. Petrusel, Fixed point theory for a new type of contractive multi-valued operators, Nonlinear Anal. (TMA) 70 (2009) 3371-3377.
[8] S. Moradi, F. Khojasteh, Endpoints of multi-valued generalized weak contraction mappings, Nonlinear Anal. (TMA) 74 (2011) 2170-2174.
[9] S. B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30 (1969) 475-488.
[10] B. E. Rhoades, Some theorems on weakly contractive maps, Proceedings of the Third World Congress of Nonlinear Analysis, Part 4 (Catania 2000), Nonlinear Anal. 47 (2001) 2683-2693.
[11] B. D. Rouhani, S. Moradi, Common Fixed Point of Multi-valued Generalized ϕ-Weak Contractive Mappings, Fixed Point Theory and Appl. DOI:10.1155/2010/708984 (2010).
[12] N. Shahzad, A. Lone, Fixed points of multimaps which are not necessarily nonexpansive, Fixed Point Theory and Appl. 2 (2005) 169-176.
[13] K. Włodarczyk, D. Klim, R. Plebaniak, Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces, J. Math. Anal. Appl. 328 (2007) 46–57.
[14] K. Włodarczyk, D. Klim, R. Plebaniak, Endpoint theory for set-valued nonlinear asymptotic contractions with respect to generalized psedodistances in uniform spaces, J. Math. Anal. Appl. 339 (2008) 344–358.