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Unique pseudolifting property in digital topology

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Abstract. The generalized universal lifting property plays an important role in classical topology. In digital topology we also have its digital version [5, 6, 14]. More precisely, the paper [6] established the concept of a digital covering (see also [11, 17]). It has substantially contributed to the calculation of digital fundamental groups of some digital spaces, the classification of digital spaces and so forth. The paper [6] also established the unique lifting property of a digital covering which plays an important role in studying both digital covering and digital homotopy theory. Motivated by the unique lifting property, the paper develops a pseudocovering which is weaker than a digital covering and investigates its various properties. Furthermore, the paper proves that a pseudocovering with some hypothesis has the unique pseudolifting property which is weaker than the unique lifting property in digital covering theory.

1. Introduction

Discrete geometry which includes digital topology emerged with the development of both computer and information communications technology. It has played an important role in computer science such as image analysis, pattern recognition, image processing, mathematical morphology and so forth [18, 19, 21, 22]. It has its root in graph theory with the k-adjacency relations of $\mathbf{Z}^n$ (see (2.1) of the present paper), where $\mathbf{Z}^n$ is the set of points in the Euclidean nD space with integer coordinates, $n \in \mathbf{N}$ and $\mathbf{N}$ is the set of natural numbers.

A digital image (or a digital space) (X, k) can be considered to be a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency of $\mathbf{Z}^n$ (or a digital k-graph on $\mathbf{Z}^n$) [21]. It is well known that the notion of a graph covering has strongly contributed to the classification of graphs [4]. Similarly, the notion of a covering space in algebraic topology has been substantially used in classifying topological spaces [20, 23]. By analogy, in relation to the study of digital topological properties of a digital space (X, k) in $\mathbf{Z}^n$, useful tools motivated from both graph theory and algebraic topology include a *digital covering space* [5, 6], a *digital k-fundamental group* [2], an *automorphism group of a digital covering space* [10] and so forth. These have been studied in many papers including [1–3, 5–17].

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In digital covering theory the *unique lifting property* plays an important role in studying digital covering spaces and their applications such as the winding number and so forth. Motivated by the property, the paper develops the notion of a pseudocovering which is weaker than that of a digital covering and proves that a pseudocovering has the unique pseudolifting property.

The paper is organized as follows. Section 2 provides some basic notions. Section 3 establishes the notion of weakly local isomorphism and studies its properties. Section 4 develops the notion of a digital pseudocovering and investigates its properties related to the unique pseudolifting property. Further, it compares a digital pseudocovering with a digital covering. Finally, Section 5 concludes the paper with a summary.

2. Preliminaries

Since a digital image (X, k) in $\mathbf{Z}^n$ can be considered to be a set X in $\mathbf{Z}^n$ with one of the k-adjacency relations of $\mathbf{Z}^n$ [21] (see also (2.1) of the present paper), a digital k-graph or a cell complex [19], the present paper uses the terminology *a digital space* instead of *a digital image*.

In order to study multi-dimensional digital spaces, motivated by the k-adjacency relations of $\mathbf{Z}^2$ and $\mathbf{Z}^3$ in [18, 21], we have often used the k-adjacency relations of $\mathbf{Z}^n$, as follows [6] (for more details in [16]):
For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n$$

are k- (or $k(m, n)$-) adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

By using this operator, we can characterize the k-adjacency relations of $\mathbf{Z}^n$, denoted by $k := k_m$ or $k(m, n)$, where $k := k_m := k(m, n)$ is the number of points q which are k-adjacent to a given point p according to the numbers m and n in $\mathbf{N}$. Consequently, we can represent the k-adjacency relations of $\mathbf{Z}^n$, as follows [13] (for more details, see also [16]).

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\ i!}. \tag{2.1}$$

By using the current k-adjacency relations of $\mathbf{Z}^n$, we can study digital topological properties of multi-dimensional digital spaces in $\mathbf{Z}^n$.

A pair (X, k) is usually assumed to be a *(binary) digital space* with k-adjacency in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $k \neq \bar{k}$ except $X \subset \mathbf{Z}$ [21] and the pair $(k, \bar{k})$ depends on the situation. But the current paper is not concerned with the $\bar{k}$-adjacency of $\mathbf{Z}^n \setminus X$. For $\{a, b\} \subset \mathbf{Z}$ with $a \lneq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b | n \in \mathbf{Z}\}$ is considered in $(\mathbf{Z}, 2, 2, [a, b]_{\mathbf{Z}})$ [1]. For a digital space (X, k), two points $x, y \in X$ are k-connected in [18] if there is a k-path from x to y in X and if any two points in X are k-connected, then X is called *k-connected* [18]. For an adjacency relation k of $\mathbf{Z}^n$, a *simple k-path* with l elements in $\mathbf{Z}^n$ is assumed to be an *injective sequence* $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [18]. Besides, we say that the *length* of a simple k-path is the number l. Furthermore, a *simple closed k-curve* with l elements in $\mathbf{Z}^n$ is a sequence $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$ derived from a simple k-curve $(x_i)_{i \in [0, l]_{\mathbf{Z}}}$ with $x_0 = x_l$, where x_i and x_j are k-adjacent if and only if $j = i + 1 (mod\, l)$ or $i = j + 1 (mod\, l)$ [18]. Let $SC_k^{n,l}$ denote a simple closed k-curve with l elements in $\mathbf{Z}^n$ [6].

In order to study various properties of digital spaces, we have often used the following digital k-neighborhood [6]. For a digital space (X, k) in $\mathbf{Z}^n$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X

$$N_k(x_0, \varepsilon) = \{x \in X | \, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.2}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.

To map every k_0-connected subset of (X, k_0) into a k_0-connected subset of (Y, k_1), the papers [1, 21] established the notion of digital continuity. Since every point x of a digital space (X, k) in $\mathbf{Z}^n$ always has an $N_k(x, 1) \subset X$, the digital continuities of [1] and [21] can be represented as the following form.

Proposition 2.1. ([11]) *Let* (X, k_0) *and* (Y, k_1) *be digital spaces in* $\mathbf{Z}^{n_0}$ *and* $\mathbf{Z}^{n_1}$*, respectively. A function* $f : X \to Y$ *is* (k_0, k_1)*-continuous if and only if for every point* $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

As mentioned in the previous part, since a digital space (X, k) can be considered to be a digital k-graph, we have often used a (k_0, k_1)-*isomorphism* as in [11] instead of a (k_0, k_1)-*homeomorphism* as in [1], as follows:

Definition 2.2. ([1]; see also [11]) For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism* and use the notation $X \approx_{k_0} Y$.

3. Weakly local (k_0, k_1)-isomorphism and its properties

In the study of the preservation of local properties of a digital space, this section establishes the notion of weakly local (k_0, k_1)-isomorphism. Since the notion of local (k_0, k_1)-isomorphism is so related to the establishment of that of weakly local (k_0, k_1)-isomorphism, let us now recall the following:

Definition 3.1. ([5]; see also [13]) For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a (k_0, k_1)-continuous map $h : X \to Y$ is called a local (k_0, k_1)-isomorphism if for every $x \in X$ h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a local k_0-isomorphism.

Example 3.2. (1) Let us consider the map

$$f_1 : [0, l-1]_{\mathbf{Z}} \to SC_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbf{Z}}}, l \geq 4 \tag{3.1}$$

given by $f_1(i) = c_i$.
Let us consider the map

$$f_2 : \mathbf{Z}^+ \to SC_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbf{Z}}}, l \geq 4$$

given by $f_2(i) = c_i$, where $\mathbf{Z}^+ := \{n | \, n \geq 0, n \in \mathbf{Z}\}$.
Then, owing to the points 0 and $l - 1$ of $[0, l - 1]_{\mathbf{Z}}$, f_1 cannot be a local $(2, k)$-isomorphism [13]. Owing to the point $0 \in \mathbf{Z}^+$, neither is f_2.

(2) Let us consider the map

$$g : SC_{k_0}^{n_0,ml} := (e_i)_{i\in[0,ml-1]_{\mathbf{Z}}} \to SC_{k_1}^{n_1,l} := (c_j)_{j\in[0,l-1]_{\mathbf{Z}}} \tag{3.2}$$

given by $g(e_i) = c_{i(mod\, l)}$. Then the map g is a local (k_0, k_1)-isomorphism [12].

The notion of local (k_0, k_1)-isomorphism can be often used for studying some local properties such as the (generalized) topological k-number and preserving local properties of a digital space [12]. Besides, for a local (k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ its restriction on a subset of (X, k_0) need not be a local (k_0, k_1)-isomorphism [12]. As a weak form of the local (k_0, k_1)-isomorphism, we establish the following.

Definition 3.3. For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a weakly local (k_0, k_1)-isomorphism (shortly, a WL-(k_0, k_1)-isomorphism) if for every $x \in X$ h maps $N_{k_0}(x, 1) \subset X$ (k_0, k_1)-isomorphically onto $h(N_{k_0}(x, 1)) \subset Y$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a weakly local k_0-isomorphism (shortly, a WL-k_0-isomorphism).

For instance, the maps $f_i, i \in \{1, 2\}$ in Example 3.2 are WL-$(2, k)$-isomorphisms. Also, it is clear that a (k_0, k_1)-continuous map need not be a WL-(k_0, k_1)-isomorphism. We can observe a difference between a WL-(k_0, k_1)-isomorphism and a local (k_0, k_1)-isomorphism, as follows.

Theorem 3.4. *While a local* (k_0, k_1)*-isomorphism* $h : (X, k_0) \to (Y, k_1)$ *implies a WL-*(k_0, k_1)*-isomorphism, the converse does not hold.*

Proof. Let us prove that a local (k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ implies a WL-(k_0, k_1)-isomorphism. For any point $x \in X$ since there is an $N_{k_0}(x, 1) \subset X$, according to the local (k_0, k_1)-isomorphism h, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$. Thus, we obtain that

$$h(N_{k_0}(x, 1)) \approx_{k_1} N_{k_1}(h(x), 1). \tag{3.3}$$

By using the restriction of h on $N_{k_0}(x, 1)$, denoted by $h|_{N_{k_0}(x,1)}$, we obtain that

$$N_{k_0}(x, 1) \approx_{(k_0,k_1)} h(N_{k_0}(x, 1)),$$

which makes the map h a WL-(k_0, k_1)-isomorphism.
Consequently, in view of the maps f_1 and g in Example 3.2, the converse does not hold. □

We can observe that a WL-(k_0, k_1)-isomorphism has its own intrinsic properties, as follows.

Theorem 3.5. *(1) A WL-(k_0, k_1)-isomorphism is a (k_0, k_1)-continuous map.*

(2) Consider two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$. Let $h : X \to Y$ be a WL-(k_0, k_1)-isomorphism. Then the restriction of h on a subset (X_0, k_0) is also a WL-(k_0, k_1)-isomorphism, where $(X_0, k_0) \subset (X, k_0)$.

(3) A WL-(k_0, k_1)-isomorphism does not imply an injection.

(4) A WL-(k_0, k_1)-isomorphic bijection does not imply a local (k_0, k_1)-isomorphism.

Proof. (1) For a WL-(k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ and any point $x \in X$ we obtain that $N_{k_0}(x, 1) \approx_{(k_0,k_1)} h(N_{k_0}(x, 1)) \subset N_{k_1}(h(x), 1)$, the proof is completed.

(2) Consider any subset $(X_0, k_0) \subset (X, k_0)$. For every point $x \in X_0$ take $N_{k_0}(x, 1) \subset X_0$. Then, by using the restriction of the given WL-(k_0, k_1)-isomorphism $h : (X, k_0) \to (Y, k_1)$ on $N_{k_0}(x, 1)$, we obtain that $N_{k_0}(x, 1) \simeq_{(k_0,k_1)} h(N_{k_0}(x, 1))$ so that the restriction of h on (X_0, k_0) is a WL-(k_0, k_1)-isomorphism.
For instance, consider the map in (3.2)

$$h : SC_8^{2,8} := (e_i)_{i\in[0,7]_{\mathbf{Z}}} \to SC_8^{2,4} := (c_j)_{j\in[0,3]_{\mathbf{Z}}} \tag{3.4}$$

defined by $h(e_i) = c_{i(mod\, 4)}$. Then we clearly observe that the map h is a WL-$(8, 8)$-isomorphism. Let us now delete an arbitrary point $e_i \in SC_8^{2,8}$. Then the restriction of h on $SC_8^{2,8} - \{e_i\}$ is also a WL-$(8, 8)$-isomorphism.

(3) While the map h in (3.4) is a WL-(k_0, k_1)-isomorphism, it cannot be an injection.

(4) In view of the map f_1 in (3.1), we can prove the assertion. □

Remark 3.6. In view of Theorems 3.4 and 3.5(2), while a WL-(k_0, k_1)-isomorphism is weaker than a local (k_0, k_1)-isomorphism, in relation to the restriction property we can observe that a WL-(k_0, k_1)-isomorphism has a useful property.

Even though both a WL-(k_0, k_1)-isomorphism and a local (k_0, k_1)-isomorphism can be used for preserving local properties of a digital space, let us now discuss their limitations.

Remark 3.7. (1) In a digital 3-cube (see Figure 1), let us consider the map (see Figure 1)

$$h : (A := (a_i)_{i\in[0,5]_{\mathbf{Z}}}, 26) \to (B := (b_i)_{i\in[0,5]_{\mathbf{Z}}}, 26) \tag{3.5}$$

given by $h(a_i) = b_i, i \in [0, 5]_{\mathbf{Z}}$. The map h is both a WL-26-isomorphism and a local 26-isomorphism because for every point $a_i \in A$ and $b_i \in B, i \in [0, 5]_{\mathbf{Z}}$ we observe that $N_{26}(a_i, 1) = A$ and $N_{26}(b_i, 1) = B$.

(2) In Figure 1 let us consider the spaces A and B with 18-adjacency instead of 26-adjacency. Then for the two spaces $(A, 18)$ and $(B, 18)$ the map $g : (A := (a_i)_{i\in[0,5]_{\mathbf{Z}}}, 18) \to (B := (b_i)_{i\in[0,5]_{\mathbf{Z}}}, 18)$ given by $g(a_i) = b_i, i \in [0, 5]_{\mathbf{Z}}$ cannot be a WL-18-isomorphism.

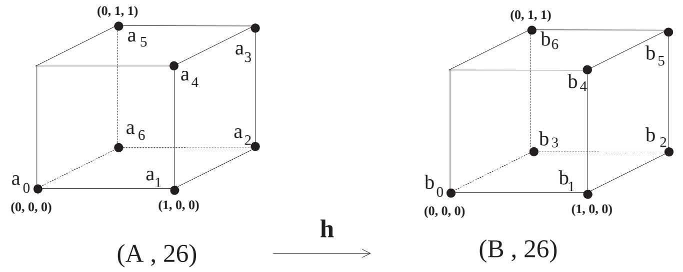

Figure 1: Comparison between a WL-(k_0, k_1)-isomorphism and a local (k_0, k_1)-isomorphism in a 3-cube, $k_i \in \{26, 18\}$ and $i \in \{0, 1\}$.

4. Digital pseudocovering space and unique pseudolifting property

Motivated by the notion of a digital covering space which has been often used in classifying digital spaces [6, 11, 17], by using the notion of *WL-(k_0, k_1)-isomorphism*, this section establishes a pseudo-(k_0, k_1)-covering which is weaker than a (k_0, k_1)-covering. Indeed, a (k_0, k_1)-covering map in Definition 4.4 satisfies the unique lifting property which plays an important role in digital homotopy theory (see Theorem 4.7). Since a pseudo-(k_0, k_1)-covering is weaker than a (k_0, k_1)-covering (Remark 4.6), we can prove that a pseudo-(k_0, k_1)-covering with some hypothesis also has the unique pseudolifting property instead of the unique lifting property (see Theorem 4.9 and Corollary 4.11).

Definition 4.1. Let (E, k_0) and (B, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : (E, k_0) \to (B, k_1)$ be a surjection. Suppose that for any $b \in B$ the map p has the following properties.

(1) for some index set M $p^{-1}(N_{k_1}(b, 1)) = \cup_{i \in M} N_{k_0}(e_i, 1)$ with $e_i \in p^{-1}(b)$

(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1)$ is an empty set; and

(3) the restriction of p on $N_{k_0}(e_i, 1)$ is a WL-(k_0, k_1)-isomorphism for all $i \in M$.

Then the map p is called a pseudo-(k_0, k_1)-covering map, (E, p, B) is said to be a pseudo-(k_0, k_1)-covering and (E, k_0) is called a pseudo-(k_0, k_1)-covering space over (B, k_1).

According to Definition 4.1, consider a pseudo-(k_0, k_1)-covering map, $((E, e_0), p, (B, b_0))$ such that $p(e_0) = b_0$. Then we say that the map p is a *pointed* pseudo-(k_0, k_1)-covering map.

Remark 4.2. In the property (2) of Definition 4.1, for $i, j \in M$ and $i \neq j$ $N_{k_0}(e_i, 1)$ need not be k_0-isomorphic to $N_{k_0}(e_j, 1)$.

Example 4.3. (1) The maps $f_i, i \in \{1, 2\}$ in Example 3.2(1) are clearly pseudo-(2, 8)-covering maps and the map g in (3.2) is also a pseudo-(k_0, k_1)-covering map.

(2) The map h of (3.5) is a pseudo-26-covering map

(3) The map $p : (C, 8) \to (D, 8)$ given by $p(c_i) = d_i, i \in [0, 3]_{\mathbf{Z}}$ in Figure 2 is clearly a pseudo-8-covering map.

(4) In Figure 2 let us consider the two sets C and D with 4-adjacency instead of 8-adjacency. Then the map $q : (C, 4) \to (D, 4)$ given by $q(c_i) = d_i, i \in [0, 3]_{\mathbf{Z}}$ cannot be a pseudo-4-covering map (see the point $c_1 \in C$).

Since the notion of a digital covering space has been often used in calculating the digital fundamental group [2, 6, 11, 12, 14, 15, 17] and classifying digital covering spaces [2, 8, 10, 11, 14–16]. Let us now recall the axiom of a digital covering space which is equivalent to the previous version in [5].

Let us now recall the typical axioms of a digital covering space.

Definition 4.4. ([5]; see also [11, 17]) Let (E, k_0) and (B, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a ((k_0, k_1)-continuous) surjection. Suppose, for any $b \in B$ there exists $\varepsilon \in \mathbf{N}$ such that

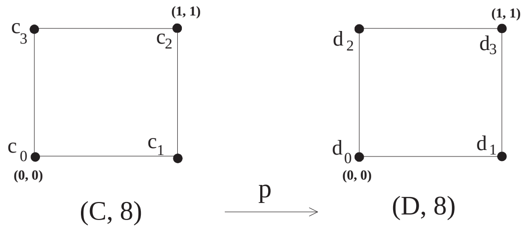

Figure 2: Pseudo-$(8,8)$-covering and a non-pseudo-$(4,4)$-covering

(1) for some index set M, $p^{-1}(N_{k_1}(b,\varepsilon)) = \cup_{i\in M} N_{k_0}(e_i,\varepsilon)$ with $e_i \in p^{-1}(b)$;
(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i,\varepsilon) \cap N_{k_0}(e_j,\varepsilon)$ is an empty set; and
(3) the restriction map p on $N_{k_0}(e_i,\varepsilon)$ is a (k_0,k_1)-isomorphism for all $i \in M$.
Then the map p is called a (k_0,k_1)-covering map, (E,p,B) is said to be a (k_0,k_1)-covering and (E,k_0) is called a digital (k_0,k_1)-covering space over (B,k_1).

The k_1-neighborhood $N_{k_1}(b,\varepsilon)$ of Definition 4.4 is called an elementary k_1-neighborhood of b with some radius ε. While in Definition 4.4 we may take $\varepsilon = 1$ [8], the paper [17] established a simpler form of the axioms of a digital covering space, as follows.

Remark 4.5. ([17]) For the (k_0,k_1)-covering of Definition 4.4 we can replace "(k_0,k_1)-continuous surjection" with "surjection" because the surjection of p with the properties (1) and (3) of Definition 4.4 implies that p is (k_0,k_1)-continuous. Furthermore, we may take $\varepsilon = 1$.

By comparing between a (k_0,k_1)-covering and a pseudo-(k_0,k_1)-covering (see Definitions 4.1 and 4.4, and Theorem 3.4), we obtain the following:

Remark 4.6. While a (k_0,k_1)-covering implies a pseudo-(k_0,k_1)-covering, the converse does not hold.

Proof. A (k_0,k_1)-covering clearly implies a pseudo-(k_0,k_1)-covering. In view of the maps f_1 and f_2 in Example 3.2(1), the converse need not be successful. □

Let us now review the unique lifting property of a digital covering. For three digital spaces (E,k_0) in $\mathbf{Z}^{n_0}$, (B,k_1) in $\mathbf{Z}^{n_1}$, and (X,k_2) in $\mathbf{Z}^{n_2}$, let $p: E \to B$ be a (k_0,k_1)-covering map. For a (k_2,k_1)-continuous map $f: (X,k_2) \to (B,k_1)$ we say that a *digital lifting* of f is a (k_2,k_0)-continuous map $\tilde{f}: X \to E$ such that $p \circ \tilde{f} = f$ [6]. We now recall a theorem related to the *unique lifting property* in [6] as follows:

Theorem 4.7. ([5]) *For pointed digital spaces $((E,e_0),k_0)$ in $\mathbf{Z}^{n_0}$ and $((B,b_0),k_1)$ in $\mathbf{Z}^{n_1}$, let $p:(E,e_0) \to (B,b_0)$ be a pointed (k_0,k_1)-covering map. Any k_1-path $f:[0,m]_{\mathbf{Z}} \to B$ beginning at b_0 has a unique lifting to a k_0-path $\tilde{f}$ in E beginning at e_0.*

Unlike the unique lifting property of a digital covering space in [6], we can observe that a pseudo-(k_0,k_1)-covering map does not have the unique lifting property, as follows.

Remark 4.8. Consider the map

$$p: SC_8^{2,8} - \{e_2,e_7\} := A \to SC_8^{2,4} := (c_j)_{j\in[0,3]_{\mathbf{Z}}}$$

given by $p(e_i) = c_{i(mod\,4)}$, where $SC_8^{2,8} := (e_i)_{i\in[0,7]_{\mathbf{Z}}}$ which is a sequence with 8-connectivity. Furthermore, assume an 8-path $f:[0,2]_{\mathbf{Z}} \to SC_8^{2,4}$ defined by $f(i) = c_i, i \in [0,2]_{\mathbf{Z}}$. While the map p is a pseudo-8-covering map, it cannot have the unique lifting property owing to the point $2 \in [0,2]_{\mathbf{Z}}$. More precisely, we cannot have a digital lifting of f, such as $\tilde{f}:[0,2]_{\mathbf{Z}} \to A$ such that $p \circ \tilde{f} = f$ because the given space $SC_8^{2,8} - \{e_2,e_7\}$ is not 8-connected.

Let us now prove that a pseudo-(k_0, k_1)-covering map with some hypothesis has the following property.

Theorem 4.9. *Consider digital spaces $(\tilde{X}, k_0), (X, k_1)$ and (Y, k) such that Y is k-connected and $\tilde{X}$ is k_0-connected. Let $(\tilde{X}, p, X)$ be a pseudo-(k_0, k_1)-covering. Let $g : (Y, k) \to (X, k_1)$ be a (k, k_1)-continuous map. If there are two (k, k_0)-continuous maps $f_0, f_1 : Y \to \tilde{X}$ both coinciding at one point $y_0 \in Y$ and satisfying $p \circ f_0 = p \circ f_1 = g$, then $f_0 = f_1$.*

Before proving the assertion, we need to point out that a pseudo-(k_0, k_1)-covering $(\tilde{X}, p, X)$ such that $\tilde{X}$ is k_0-connected need not be a (k_0, k_1)-covering (see the maps f_1 and f_2 in Example 3.2(1)). More precisely, the maps f_1 and f_2 are pseudo-$(2, k)$-covering maps such that $Dom(f_i)$ is 2-connected, $i \in \{1, 2\}$. Owing to the point $0 \in [0, l-1]_{\mathbf{Z}}$ and $\mathbf{Z}^+$ each of these two pseudo-$(2, k)$-covering maps cannot be a $(2, k)$-covering map.

Proof. Suppose that there is a point $y \in Y$ such that $f_0(y) \neq f_1(y)$. Since the digital space (Y, k) is k-connected, there is a k-path s on Y, denoted by $s : [0, l]_{\mathbf{Z}} \to Y$, such that $s(0) = y_0$ and $s(l) = y$. Since there is $y \in Y$ such that $f_0(y) \neq f_1(y)$, we may assume that $l \gneq 1$ is the smallest $t \in [0, l]_{\mathbf{Z}}$ such that $f_0(s(t)) \neq f_1(s(t))$. Thus we have the following:

$$\left\{ \begin{array}{l} f_0(s(l)) \neq f_1(s(l)); \text{and} \\ p \circ f_0(s(t)) = p \circ f_1(s(t)) = g(s(t)), t \in [0, l-1]_{\mathbf{Z}}. \end{array} \right\}$$

Take the following four digital neighborhoods of the points $l-1$, $s(l-1)$, $g(s(l-1))$ and $f_i(s(l-1))$

$$\left\{ \begin{array}{l} N_2(l-1, 1) \subset [0, l]_{\mathbf{Z}}, N_k(s(l-1), 1) \subset Y, \\ N_{k_1}(g(s(l-1)), 1) \subset X \text{ and } N_{k_0}(f_i(s(l-1)), 1) \subset \tilde{X}, i \in \{0, 1\}. \end{array} \right\}$$

Due to the hypothesis of $p \circ f_0 = p \circ f_1 = g$ and the (k, k_0)-continuity of $f_i, i \in \{0, 1\}$, we obtain that

$$\left\{ \begin{array}{l} f_i(N_k(s(l-1), 1)) \subset N_{k_0}(f_i(s(l-1)), 1), i \in \{0, 1\}; \text{and} \\ f_i(s(l)) \in N_{k_0}(f(s(l-1)), 1), \end{array} \right\}$$

and further,

$$p \circ f_0(N_k(s(l-1), 1)) = p \circ f_1(N_k(s(l-1), 1)) = g(N_k(s(l-1), 1)). \tag{4.1}$$

Furthermore, by using the pseudo-(k_0, k_1)-covering map p at the points $f_i(s(l-1)), i \in \{0, 1\}$, we obtain that

$$N_{k_0}(f_i(s(l-1)), 1) \simeq_{(k_0, k_1)} p(N_{k_0}(f_i(s(l-1)), 1)) \subset X, i \in \{0, 1\}. \tag{4.2}$$

Since

$$\left\{ \begin{array}{l} f_i(y) \in N_{k_0}(f_i(s(l-1)), 1), i \in \{0, 1\}; \text{and} \\ g(s(l)) \in N_{k_1}(g(s(l-1)), 1), \end{array} \right\}$$

by (4.1) and (4.2), $f_0(y)$ should be equal to $f_1(y)$, which contradicts to the assumption of $f_0(y) \neq f_1(y)$. □

Corollary 4.10. *For pointed digital spaces $((E, e_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((B, b_0), k_1)$ in $\mathbf{Z}^{n_1}$ let $((\tilde{X}, \tilde{x}_0), p, (X, x_0))$ be a pointed pseudo-(k_0, k_1)-covering such that $\tilde{X}$ is k_0-connected. Let $g : [0, m]_{\mathbf{Z}} \to X$ be a $(2, k_1)$-continuous map beginning at x_0. If there are two $(2, k_0)$-continuous maps $f_0, f_1 : [0, m]_{\mathbf{Z}} \to \tilde{X}$ both coinciding at one point $y_0 \in Y$ and satisfying $p \circ f_0 = p \circ f_1 = g$, then $f_0 = f_1$.*

As proved in Theorem 4.9, for digital spaces $(\tilde{X}, k_0), (X, k_1)$ and (Y, k) such that Y is k-connected and $\tilde{X}$ is k_0-connected, let $(\tilde{X}, p, X)$ be a (k_0, k_1)-continuous map. Let $g : (Y, k) \to (X, k_1)$ be a (k, k_1)-continuous map. Assume that there are two (k, k_0)-continuous maps $f_0, f_1 : Y \to \tilde{X}$ both coinciding at one point $y_0 \in Y$ such that $p \circ f_0 = p \circ f_1 = g$. If we obtain that $f_0 = f_1$, then we say that the map p has the *unique pseudolifting property*.

Motivated by Corollary 4.10, we can point out the following:

Corollary 4.11. *Consider pointed digital spaces* $(\tilde{X}, \tilde{x}_0), k_0)$ *in* $\mathbf{Z}^{n_0}$ *and* $((X, x_0), k_1)$ *in* $\mathbf{Z}^{n_1}$ *such that* $\tilde{X}$ *is* k_0*-connected. Then a pointed pseudo-*(k_0, k_1)*-covering* $((\tilde{X}, \tilde{x}_0), p, (X, x_0))$ *has the unique pseudolifting property.*

Remark 4.12. Since a pointed (k_0, k_1)-covering map implies a pointed pseudo-(k_0, k_1)-covering map, a pointed (k_0, k_1)-covering has the unique pseudolifting property.

Example 4.13. The maps f_1 and f_2 in Example 3.2(1) have the unique pseudolifting property.

5. Summary and concluding remark

We have established the notion of a pseudo-(k_0, k_1)-covering space and have proved that a pseudo-(k_0, k_1)-covering map has the unique pseudolifting property instead of the unique lifting property. As a further work, we can study a pseudo-(k_0, k_1)-covering map satisfying the unique lifting property and can investigate some properties of a pseudo-(k_0, k_1)-covering map related to digital homotopic properties.

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